Energy I: Week 2 Recitation Problems

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Vertical Springs

Vertical Spring problems are more complicated than their horizontal counterparts, but they're really just a more general case. In fact, we can look at springs on any angle of an incline for a continuous range of problems. We begin our analysis with a spring hanging without any mass. The spring length (now unstretched) is l. Now, suppose we hang a mass on the spring and then gently let it stretch the spring until equilibrium is achieved, and suppose this stretching to have length Δl. Then Newton's Second Law says that:

$$\Sigma F_y = -k\Delta l - mg = 0,$$

(Δl is negative) which gives us our expression for the stretch of the spring from the old equilibrium position $\Delta l = -mg/k$. Notice that the new equilibrium position of the system (no force) is *different* from the old one (no stretch in the spring). The two are the same only when a spring is horizontal.

Now, suppose the spring is further stretched from the new equilibrium position, so that the total stretching is y. Then we have

$$\begin{aligned} a &= F/m \\ \frac{d^2y}{dt^2} &= -\frac{k}{m}y - g \\ &= -\omega_0^2 y - g \\ &= -\omega_0^2\left(y + \frac{g}{\omega_o^2}\right) \\ &= -\omega_0^2(y - \Delta l) \end{aligned}$$

where we have set $\omega_0^2 = k/m$. Note that we don't know the angular frequency yet - this was simply an algebraic replacement. This last equation suggests that we define a new quantity $\bar{y} = y - \Delta l$, since derivatives annihilate constants. A quick calculation gives the equation

$$\frac{d^2\bar{y}}{dt^2} = -\omega_0^2\bar{y},$$

which is the usual equation for harmonic motion. Now we can conclude that ω_0 is in fact the angular frequency, and the mass executes oscillations with respect to the new equilibrium position.

Thus, the total effect of having a vertical spring is to shift the equilibrium position down by the amount mg/k, so that at equilibrium the spring is under tension and has energy. Make sure you remember this when computing energies.

Finally, note that if we have a spring on an incline, the equilibrium position is determined by

$$\Sigma F_y = -k\Delta l - mg\sin(\theta) = 0,$$

where θ is incline angle. Thus we see that the separation between the new and old equilibrium depends on that angle and goes to zero as the angles does, which is a horizontal spring. Here the two positions are degenerate (the same).

15-59.

The solution for a damped spring can be written in the form

$$x(t) = A(t)\cos(\omega t + \varphi)$$

where we have

$$A(t) = Ae^{-bt/2m}$$

When written in this form it is obvious that the non-damped spring is written the same way with $A(t) = A$ is time-independent. In any case, part a) refers to the amplitude only - the $A(t)$. Thus, since

$$A_0 = A(0) = A$$

we have

$$\begin{aligned} A/3 &= A(t) = Ae^{-bt/2m} \\ 3^{-1} &= e^{-bt/2m} \\ -\ln 3 &= -\frac{bt}{2m} \\ t &= 2\ln 3\frac{m}{b} \\ &= 14.33\text{s}. \end{aligned}$$

There are a few ways to consider the next part. If there are n oscillations then the angle has changed by an mount $2\pi n$, that is

$$\Delta\theta = 2\pi n = \omega_f - \omega_i = (\omega t + \varphi) - (\varphi) = \omega t.$$

Which yields

$$n = \frac{\omega t}{2\pi} = ft = \frac{t}{T},$$

expressed in various ways (note that we don't need φ). Thus all we need is ω for a damped system, which is given by

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

15-79.

What we have to note here is that the condition of the cars before the bottom one breaks away enables us to determine the spring constant of the rope since the system was in static equilibrium. The component of weight along the incline of the 3 cars balances the spring force, so Newton's Second Law gives us

$$3mg\sin(\theta) = F_s = kx$$

where m is the mass of one car and x the stretch of the spring, so solving for k yields

$$k = 3mg\sin(\theta)/x = 9.81 \times 10^5 \text{ N/m},$$

which allows us to determine the frequency. But, be careful, only two of the cars oscillate, so the mass now is $2m$!

Now, for the amplitude we note that the breaking free of the bottom car set our initial conditions: the string is stretched by $x = 15$ cm and the initial velocity is zero. When the bottom one breaks free our forces are unbalanced so the net force up the incline is still $f_s = 3mg\sin(\theta)$, but the gravitation part downward is only $2mg\sin(\theta)$. The net force is then kA (since we oscillate about the new equilibrium position), so

$$A = \frac{F_{net}}{k} = \frac{mg\sin(\theta)}{3mg\sin(\theta)/x} = \frac{x}{3}.$$

Another way to do this part is to to write $A = x - \Delta l$, and find the stretch of the spring from its unstretched length as we did before. Both expressions should agree.

15-87.

The point of this problem is to give some practice in evaluating the two constants A and φ given an initial condition. We have the equations

$$\begin{array}{rcl|rcl} x(t) & = & A\cos(\omega t + \varphi) & v(t) & = & -\omega A\sin(\omega t + \varphi) \\ x(0) & = & A\cos(\varphi) & v(0) & = & -\omega A\sin(\varphi), \end{array}$$

from which we can evaluate the initial conditions. Now, one condition is that $V(0) = 0$, which yields

$$0 = \omega A\sin(\varphi),$$

so either ω, A, or $\sin(\varphi)$ is zero. Our oscillator would be boring if either the first two were true, so we go with the last one, which means either $\varphi = 0$ OR $\varphi = \pi$. Make sure you realize the angle can have multiple values!

Next, we use the position condition

$$x(0) = .37 = A\cos(\varphi).$$

Since we (usually) take A to be a positive quantity, we need $\varphi = 0$, since otherwise cos comes out negative. But, $\cos(0) = 1$, so $A = .37$.

Note, we could have used $\varphi = \pi$ and $A = -.37$. Either is fine.

15-96

Like all problems that change the basic setup, we need to go back to FBD's before we can apply our spring results. The forces acting are gravity, spring, normal, and friction (rolls without slipping!). So, taking our center of torque to be the center of the wheel, we get

$$\begin{aligned} \Sigma F_x &= N - mg = 0 \\ \Sigma F_y &= -kx - F_f = ma \\ \Sigma\tau &= -F_f R = I\alpha \end{aligned}$$

where I is the moment of inertia and R the radius. Note that acceleration to the right means clockwise rolling, or negative angular acceleration, so $a = -\alpha R$. Eliminating F_f from the second two equations yields

$$\begin{aligned} -kx + \frac{I\alpha}{R} &= ma \\ -kx - \frac{Ia}{R^2} &= ma, \end{aligned}$$

or, after collection of terms and rearranging

$$a\left(m + \frac{I}{R^2}\right) = -kx$$

Thus we have an equation of the form

$$a = \omega^2 x,$$

but with

$$\omega^2 = \frac{k}{m + I/R^2},$$

that is, we still have simple harmonic motion, but the rolling inertia makes our 'effective mass' larger. In this case $I = mR^2/2$, so

$$\omega^2 = \frac{k}{3m/2} = \frac{2}{3}\frac{k}{m}.$$

It is straightforward to show the period is as advertised at this point.

The total energy any point is the sum of three terms:

$$E = K_{rot} + K_{trans} + U_s.$$

Notice that $\omega = v/R$, so that

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2 = K_{trans}/2.$$

Now,

$$E = \frac{3}{4}mv^2 + \frac{1}{2}kx^2,$$

and the time derivative of E is

$$\begin{aligned}\frac{dE}{dt} &= \frac{3}{2}mv\frac{dv}{dt} + kx\frac{dx}{dt} \\ &= \frac{3}{2}mva + kxv \\ &= v\left(\frac{3}{2}ma + kx\right) = 0,\end{aligned}$$

where the last follows since E is constant. This equation is identical to the one derived above.