Energy I: Week 1 Recitation Problems

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15-2. We are given the mass m, amplitude A, and period T of a body undergoing Simple Harmonic Motion. Newton's Second Law tells us that

$$F = -kx,$$

so that F is maximum when x is maximum, that is, when we have $x = A$. Thus,

$$\begin{aligned} F_{max} &= kA \\ &= (\omega^2 m)A, \end{aligned}$$

since we have a spring system. We can obtain ω from T by

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

This yeilds a value for k of

$$k = (2\pi f)^2 m = 118 \text{ N/m},$$

and a value of the maximum force of 10 N.

15-7. Consider, for concreteness, the system to be a mass on a horizontal spring. The mass then takes $t = .25$ s to go from the far right to the far left, since $v = 0$ at the end points of the motion. Thus the mass has moved through *half* of it's periodic motion. Thus the period will be twice this amount of time or $T = .5$ s. The frequency is the reciprocal of the period, or 2 Hz.

Finally, the amplitude of the motion is the distance from equilibrium to on of the end points. Since the distance between end points is 36 cm, the amplitude is half this amount, or 18 cm.

15-8. We want to find the spring constant for the spring in each wheel. Its relationship to other known quantities is $k = m\omega^2$. Since each spring in the car is supporting a quarter of the total car weight, we have $m = M/4$. The angular frequency has the value

$$\omega = 2\pi f = 18.85 \text{ s}^{-1},$$

which gives k a value of 129 kN/m.

Next, we're going to add an amount of 5×73.0 kg to the total weight. Then each spring supports $1/4$ of this new total weight and, since the spring constant doesn't change when the weight changes, we have

$$\omega' = \sqrt{\frac{k}{M'/4}} = 16.84\ \mathrm{s}^{-1}.$$

15-26.This one is a bit tricky since it's not obvious how to apply the results we have for spring systems to this case. There are a few different ways to look at this, but consider the following. Both springs have the same spring constant, k. If we move the mass to the left, symmetry demand than both springs stretch by an equal amount. So if the distance of the mass from equilibrium is x, and of the two springs x_1 and x_2, then $x_1 = x_2 = x/2$. Now, the potential energy of the two spring system is given by:

$$\begin{aligned} U &= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 \\ &= \frac{k}{2}\left(x_1^2 + x_2^2\right) \\ &= k(x/2)^2 \\ &= kx^2/4. \end{aligned}$$

Thus the force is

$$F = -\frac{dU}{dx} = -\frac{1}{2}kx,$$

So that the effective spring constant is cut in half. Thus the new frequency of oscillations will be given by

$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{2m}} = \frac{f}{\sqrt{2}} = 18.23\ \mathrm{Hz},$$

where f would be the frequency if only one spring were present.

This result should make sense: if a spring is made longer, then a given stretch of that spring is spead out over the the spring more. Thus there is less stress on each part of the spring and less force. Thus the overall spring constant is decreased. From the above formula we see that this descrease is linear, that is

$$k \propto \frac{1}{L}$$

for a given material composition, where L is the spring length.

15-28. In general we have the energy relationship

$$E = K + U$$

where K is the kinetic, U the potential, and E the total energy. Kinetic energy is always non-negative ($K \propto v^2$), and for a spring so too is the potential ($U \propto x^2$).

Thus, if we set $K = 0$ we get

$$E = U_{max} = \frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2,$$

while if we set $U = 0$ we get

$$E = K_{max} = \frac{1}{2}mv_{max}^2.$$

Solving the first eqation for k yeilds

$$k = \frac{2E}{A^2} = 200\ \text{N/m},$$

and solving the second for m yields

$$m = \frac{2E}{v_{max}^2} = 1.39\ \text{kg}.$$

15-35. The period $T = 1/f$, while $f = \omega/2\pi$, and $\omega^2 = k/m$. Thus we need to find k. We know a_{max}, which can be utilized in a few different ways. The most staightforward is the use the force equation:

$$F = ma = -kx \rightarrow F_{max} = ma_{max} = kA.$$

Thus we have

$$k = \frac{m}{A}a_{max} = 4 \times 10^4\ \text{N/m}.$$

Doing all of the back substitutions yields $T = 3.1$ ms.

Next, we want to find the maximum velocity. Since velocity is the derivative of position we have

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi).$$

So, for our maximum value we have

$$v_{max} = A\omega = 4\ \text{m/s}.$$

We know from 28 that we can use the maximum speed or displacement to calculate the total energy. Choosing displacement, we have

$$E = \frac{1}{2}kA^2 = 80\ \text{mJ}.$$

15-36. We know from 28 that the energy and amplitude are directly related: $E = kA^2/2$. Thus, to know the amplitude after the collision we need to know the energy after the collision. That energy has both kinetic and potential contributions in general, so we need to know the position of the spring at the collision (potential) as well as the velocities after the collision (kinetic). Thus the need to find the position and velocity of mass 2 at the given time, and then solve the collision.

However, in the present case we can simplify the analysis because the numbers are nice. First, there are relationship between sine and cosine when the arguments differ by certain phase factors. Whenever you have a phase which is some integer multiple of $\pi/2$, there may be a possible simplification. In this case we have the relationship

$$\cos(\omega t + \pi/2) = -\sin(\omega t),$$

so that mass 2 starts out at the equilibrium position at $t = 0$.

Next, notice that the period is 20 ms, while the time of impact is 5 ms, or $T/4$. So, the collision happens when mass 2 is at the end of the first quarter of its motion. Since the equation is $-\sin$, the mass will have no speed and will be at position $x = -A = -1$ cm.

First we'll calculate the spring energy, $U = kA^2/2$, but first we need k which is given by

$$k = m_2\omega^2 = 197 \text{ kN/m}.$$

Thus we have $U = 9.85$ J

Next we need to solve the collision, which is inelastic and the initial velocity of the second mass is 0. Thus we have from conservation of momentum:

$$\begin{aligned} p_i &= p_f \\ m_1 v_{1i} + m_2 v_{2i} &= (m_1 + m_2) v_f \\ m_1 v_{1i} &= (m_1 + m_2) v_f. \end{aligned}$$

This yeilds the value

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i} = 4 \text{ m/s}.$$

This gives a kinetic energy of

$$K = \frac{1}{2}(m_1 + m_2) v_f^2 = 48 \text{ J}.$$

Thus the total energy is

$$E = K + U = 57.85 \text{ J},$$

and then, finally, the amplitude is

$$A = \frac{2E}{k} = 2.4 \text{ cm}.$$

12-42. Since this problem doesn't involve springs, we'll have to start from a free body diagram before we can apply the equations we know. Newtons equations will give:

$$\begin{aligned} T\sin(\theta) &= ma_x \\ T\cos(\theta) &= mg. \end{aligned}$$

In the x and y directions respectively. Dividing the top equation by the bottom gives

$$\tan(\theta) = a_x/g.$$

Next, since we will only discuss small oscillations (larger oscillations are still harmonic, but are not simple), we will approximate the tangent by sine (remember this procedure - it is often used). Thus we have

$$\frac{a_x}{g} \approx \sin(\theta) = \frac{x}{l},$$

where l is the length of the pendulum. So if we compare this equations with the spring equations from before we see that we have

$$\omega^2 = g/l.$$

From this expression we can find l as

$$l = g/\omega^2 = .50 \text{ m}.$$

To find the maximum kinetic energy we will use the velocity if we remember an equation from circular motion:

$$v = l\frac{d\theta}{dt} = -l\omega(.08 \text{ r}ad)\sin(\omega t + \varphi),$$

so that

$$v_{max} = -l\omega(.08 \text{ r}ad) = .177 \text{ m}/s,$$

and finally

$$K = 94 \text{ m}J.$$