Comments on the Cooperstock-Tieu Galaxy Model

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ABSTRACT

The recently proposed Cooperstock-Tieu galaxy model claims to explain the flat rotation curves without dark matter. The purpose of this note is to show that this model is internally inconsistent and thus cannot be considered a valid solution. Moreover, by making the solution consistent the ability to explain the flat rotation curves is lost.

Subject headings: galaxies: kinematics and dynamics-gravitation-relativity-dark matter

1. Introduction

Cooperstock and Tieu model a galaxy in general relativity as an axially symmetric, pressure free dust cloud with metric

$$ds^2 = -e^w(cdt - Nd\varphi)^2 + r^2e^{-w}d\varphi^2 + e^{v-w}(dr^2 + dz^2) \tag{1}$$

where w, v, and N are functions only of r and z, thus the metric is stationary (Cooperstock & Tieu 2005). They further assume their coordinates to be co-moving with the galactic dust, thus

$$u^\mu = e^{-w/2}\delta^\mu_t \tag{2}$$

is the four-velocity.

By performing a local diagonalization of the metric they obtain a relation between the metric and local angular velocity as

$$\omega = \frac{cNe^w}{r^2e^{-w} - N^2e^w} \approx \frac{cN}{r^2} \tag{3}$$

where the approximate value is appropriate to the order they consider. The tt and $t\varphi$ Einstein equations are respectively

$$r^{-2}\left(N_{,r}^2 + N_{,z}^2\right) = \frac{8\pi G\rho}{c^2} \tag{4a}$$

$$N_{,rr} - \frac{1}{r}N_{,r} + N_{,zz} = 0. \tag{4b}$$

The second equation is equivalent to the formula

$$\nabla^2\Phi = 0 \tag{5}$$

where they have defined[1]

$$\Phi = \int \frac{N}{r}dr. \tag{6}$$

Thus the observed rotation curve becomes a boundary condition for the solution to Laplace's equation (5) which they take in the form

$$\Phi = \sum_n C_n e^{-k_n|z|} J_0(k_n r) \tag{7}$$

where the k_n are chosen for orthogonality over the radius of the galaxy. Once N is found by fitting this function to the obsersved rotation curve, they derive the density by (4a) and in this way they obtain an excellent fit to the data while obtaining a density profile that accords with observation.

However, it has been pointed out (Korzynski 2005; Vogt & Letelier 2005) that, since the solution depends on $|z|$, equation (4b) is not satisfied, but rather yields a singular contribution to the $z = 0$ plane, which has the properties of an exotic form of matter. It may be wondered whether this singular disk can be removed by choosing a different solution form or by increasing the complexity of the model. However, in the following analysis we will show that this is not possible and that model fails to accord with general relativity.

2. Analysis

Assuming this form of the metric, and without making any approximations, the scalar of volume expansion $\Theta \equiv u^{\mu}{}_{;\mu}$ vanishes (a semicolon denotes covariant differentiation and a comma denotes partial differentiation). Defining the space-projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{8}$$

[1] It should be noted that this Φ is not the Newtonian gravitational potential.

the shear tensor is given by[2]

$$\sigma_{\mu\nu} = u_{(\alpha;\beta)} h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \tag{9}$$

and this vanishes as well[3]. However, the vorticity tensor, given by

$$\omega_{\mu\nu} = u_{[\alpha;\beta]} h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \tag{10}$$

does not vanish, but has nonzero components

$$\omega_{\varphi r} = -\omega_{r\varphi} = \tfrac{1}{2} e^{w/2} N_{,r} \tag{11a}$$
$$\omega_{\varphi z} = -\omega_{z\varphi} = \tfrac{1}{2} e^{w/2} N_{,z}. \tag{11b}$$

Though the matter in this model does indeed rotate, the rotation is rigid and thus cannot characterize a galaxy which is differentially rotating. It should be emphasized that since $\sigma_{\mu\nu}$ is a tensor, this cannot be a coordinate effect.

With the present solution Raychaudhuri's equation (Ciufolini & Wheeler 1955) simplifies to

$$-\omega_{\mu\nu}\omega^{\mu\nu} = R_{tt}/g_{tt} \tag{12}$$

and in fact reduces to the condition $\nabla^2 w = 0$. This condition is demanded in Cooperstock & Tieu (2005) on the grounds that the geodesic equation be satisfied. This is equivalent to saying that the geodesics must be circular orbits about the z-axis, which should not hold in general. Orbits in the $z = 0$ plane should indeed be azimuthal, but we cannot expect this behavior off that plane. That Raychaudhuri's equation demands this condition again reveals the rigidity of the rotation[4].

Now, if we seek solutions to (4b) that are symmetric about the plane, and singularity free, then must require N to be independent[5] of z. Thus (4b) has the trivial solutions

$$N = A \quad \text{or} \quad N = Br^2 \tag{13}$$

where A and B are constants. The first solution leads to zero density and the second to a constant density under rigid rotation, according to equations (3) and (4a). Thus it

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[2] (μ, ν) and $[\mu, \nu]$ denote symmetrization and antisymmetrizaion with respect to the enclosed indices, respectively.

[3] The vanishing of shear was also found by Bonner in his solution (Bonner 1977).

[4] Actually, any co-moving coordinate system requires $g_{tt} = -1$ as the coordinate points are in free fall and thus keep proper time (Weinberg 1972), which here requires $w \equiv 0$.

[5] We could choose $\cosh(z)$, but this would lead to an exponentially increasing matter density.

appears that the physical origin of the singularity is in attempting to describe, in co-moving coordinates, a non-rigidly rotating dust cloud, which the metric (1) cannot.

Next, in order to solve the Einstein Equations Cooperstock and Tieu perform an expansion of the metric in $\sqrt{G}$ and conclude that the functions w and v are of second order, but the function N, which couples to the rotation, is of first order. Strictly speaking, this expansion is not well defined as the expansion parameter has dimensions. We can form the dimensionless parameter

$$\lambda = \sqrt{\frac{GM}{Lc^2}} \tag{14}$$

where M is some characteristic mass of the system and L some characteristic length (for example, the mass and radius of the galactic core). We then compare equations (4b) and (3):

$$\begin{aligned} \lambda^2 r^{-2} \left(N_{,r}^2 + N_{,z}^2\right) &= \frac{8\pi G\rho}{c^2} \\ \lambda \frac{Nc}{r} &= v \end{aligned}$$

where the order has been shown explicitly. Substituting v for N in the first equation yields the relation

$$v = \mathcal{O}\left(\sqrt{8\pi G\rho L^2}\right) \tag{15}$$

where we have taken derivatives to be of order $1/L$. This can be rewritten as

$$v = \mathcal{O}\left(c\lambda\sqrt{\frac{\rho L^3}{M}}\right) \tag{16}$$

and since the quantity under the square root is of order unity, we have

$$\frac{v}{c} = \mathcal{O}(\lambda) = \mathcal{O}\left(\sqrt{\frac{GM}{Lc^2}}\right) \tag{17}$$

which is expected from Newtonian theory and is the basis of the PPN expansion.

Now, suppose we choose a coordinate system in which the galactic dust has coordinate velocity ω/c, so that the stress-energy tensor has the form

$$T^{\mu\nu} \propto \begin{pmatrix} 1 & \dfrac{\omega}{c} \\ \dfrac{\omega}{c} & \left(\dfrac{\omega}{c}\right)^2 \end{pmatrix} \tag{18}$$

in the $t\varphi$-subspace. The $t\varphi$-Einstein equation then has the form

$$G^{t\varphi} = \frac{8\pi G}{c^2} T^{t\varphi} \propto \frac{8\pi G}{c^2} \frac{\omega}{c} \tag{19}$$

up to a factor of order unity due to the constraint condition $u_\mu u^\mu = -1$. Thus, the right hand side of this equation begins at order λ^3/L^2 according to (17), whereas the left hand side is of order λ/L^2 according to (4b), since N is assumed to be of order λ. Thus the assumption that N is of first order is inconsistent, while consistency requires that N be of at least third order[6].

Moreover, given the form of the stress-energy tensor above, suppose we make a *global* transformation $\varphi \to \varphi + \omega(r, z)t$ to the co-moving frame, so that all components of the new stress-energy tensor vanish except the density. The new metric will have the same form as the old metric, but for the differential $d\varphi'$ we have

$$d\varphi' = \omega dt + d\varphi + t(\omega_{,r} dr + \omega_{,z} dz) \tag{20}$$

which necessarily introduces time-dependence into the new metric unless ω is spatially-independent, that is, unless the rotation of the matter is rigid. Thus, contrary to the assumption of Coopertock and Tieu, the metric (1) cannot both be co-moving and time-independent. This accords with zero value of the shear tensor above.

It can be seen in the following figure that a co-moving metric of a differentially rotating system is time-dependent and possesses shear. Here, r-coordinate lines "twist" up in time relative to observers at spatial infinity. $\varphi = \omega t$ has been plotted, where $\omega(r)$ is the fit to the Milky Way from Cooperstock & Tieu (2005).

Fig. 1.— The twisting of r-coordinate lines for various φ at two different times as seen by observers at spatial infinity.

3. Non-co-moving Expansion

In this section we carry out an expansion of the metric (1) in a system of reference in which the galactic dust has coordinate velocity

$$\frac{u^2}{u^0} = \frac{d\varphi}{dct} = \frac{\omega}{c} \tag{21}$$

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[6]This will be demonstrated more explicitly in the next section.

which is physically the angular velocity measured by observers at spatial infinity (Bardeen & Wagoner 1971). In this direct approach we will show that the angular momentum coupling is too weak to account for the flat rotation curves.

We expand the metric (1) as

$$\begin{aligned}
g_{tt} &= -1 - \overset{2}{w} - \overset{4}{w} + \mathcal{O}(\lambda^6) & \text{(22a)}\\
g_{t\varphi} &= \overset{1}{N} + \overset{2}{w}\overset{1}{N} + \overset{3}{N} + \mathcal{O}(\lambda^5) & \text{(22b)}\\
g_{\varphi\varphi} &= r^2 - \overset{1}{N}{}^2 - r^2\,\overset{2}{w} + \mathcal{O}(\lambda^4) & \text{(22c)}\\
g_{rr} = g_{zz} &= 1 + \overset{2}{w} - \overset{2}{v} + \mathcal{O}(\lambda^4) & \text{(22d)}
\end{aligned}$$

where the over-script indicates the order of the term in λ, which is the same as defined in (14). The presence of only even terms in the $g_{\mu\mu}$ and odd in $g_{t\varphi}$ are for the appropriate behavior under time-reversal (Weinberg 1972).

The constraint condition $u_\mu u^\mu = -1$ requires that

$$u^t = (g_{tt} + 2\frac{\omega}{c} g_{t\varphi} + \frac{\omega^2}{c^2} g_{\varphi\varphi})^{-1/2} \tag{23}$$

and thus the stress energy tensor has the expansion

$$\begin{aligned}
T^{tt} &= \rho c^2 \left(1 - \overset{2}{w} + \frac{r^2\omega^2}{c^2}\right) + \mathcal{O}(\lambda^4) & \text{(24a)}\\
T^{t\varphi} &= \rho c^2 \frac{\omega}{c} + \mathcal{O}(\lambda^3) & \text{(24b)}\\
T^{\varphi\varphi} &= \rho c^2 \frac{\omega^2}{c^2} + \mathcal{O}(\lambda^4) & \text{(24c)}
\end{aligned}$$

As seen in the previous section, $8\pi G/c^2$ increases the order by two so that the right-hand side of the Einstein equations are of second order and higher, thus the $t\varphi$ equation to first order is

$$\overset{1}{N}_{,rr} - \frac{1}{r}\,\overset{1}{N}_{,r} + \overset{1}{N}_{,zz} = 0 \tag{25}$$

so that the lowest order term of N is sourceless. We are free to choose $\overset{1}{N}$ however we wish to make the solution simplest, thus we set $\overset{1}{N} = 0$. With this selection the Einstein equations through third order become

$$\begin{aligned}
-\nabla^2\,\overset{2}{w} + \tfrac{1}{2}\left(\overset{2}{v}_{,zz} + \overset{2}{v}_{,rr}\right) &= \frac{8\pi G\rho}{c^2} & \text{(26a)}\\
\overset{2}{v}_{,r} &= 0 & \text{(26b)}
\end{aligned}$$

$$\begin{aligned}
\overset{2}{v}_{,zz} + \overset{2}{v}_{,rr} &= 0 && (26c)\\
\overset{2}{v}_{,z} &= 0 && (26d)\\
-\tfrac{1}{2}r^{-2}\left(\overset{3}{N}_{,rr} - \frac{1}{r}\overset{3}{N}_{,r} + \overset{3}{N}_{,zz}\right) &= \frac{8\pi G\rho r\omega}{c^3}. && (26e)
\end{aligned}$$

We see that $\overset{2}{v}$ must be a constant so that we have

$$\begin{aligned}
\nabla^2 \overset{2}{w} &= -\frac{8\pi G\rho}{c^2} && (27a)\\
\overset{3}{N}_{,rr} - \frac{1}{r}\overset{3}{N}_{,r} + \overset{3}{N}_{,zz} &= -\frac{16\pi G\rho r^3\omega}{c^3}. && (27b)
\end{aligned}$$

Thus we see that the coupling to the angular momentum is of third order, there is no longer a nonlinear term in the mass density[7], and we can identify

$$\overset{2}{w} = -\frac{2\Phi}{c^2} \tag{28}$$

with the Newtonian gravitational potential.

If we analyze circular orbits on the plane the geodesic equation demands that

$$\Gamma^{\mu}_{tt} + 2\Gamma^{\mu}_{t\varphi}\frac{\omega}{c} + \Gamma^{\mu}_{\varphi\varphi}\frac{\omega^2}{c^2} = 0 \tag{29}$$

which to third order can be written for $\mu = r$ as

$$\frac{v^2}{r} = -\Phi_{,r}\left(1 + \frac{v^2}{c^2}\right) - \frac{vc}{r}\overset{3}{N}_{,r} \tag{30}$$

which is recognized as the Newtonian centripetal equation plus second order corrections[8]. Thus the matter essentially moves according to the predictions of Newtonian gravitation with corrections of order v^2/c^2, which cannot account for flattening of the rotation curves without extra non-luminous matter.

Finally, we can compute the shear tensor, which has the non-zero components

$$\begin{aligned}
\sigma_{tr} = \sigma_{rt} &= -\frac{(u^t)^3 r^2}{2c^2}\omega\omega_{,r} && (31a)\\
\sigma_{tz} = \sigma_{zt} &= -\frac{(u^t)^3 r^2}{2c^2}\omega\omega_{,z} && (31b)\\
\sigma_{\varphi r} = \sigma_{r\varphi} &= \frac{(u^t)^3 r^2}{2c}\omega_{,r} && (31c)\\
\sigma_{\varphi z} = \sigma_{z\varphi} &= \frac{(u^t)^3 r^2}{2c}\omega_{,r} && (31d)
\end{aligned}$$

[7]Even when G^{tt} is written to fourth order the nonlinearity due to N is not present.

[8]The presence of the c in the last term effectively lowers the order by one.

all of which vanish exactly when ω is constant.

4. Conclusion

It has been shown that the Cooperstock-Tieu galaxy model is inconsistent as a general relativistic model and that a proper model fails to account for the flatness of the rotation curves without the dark-matter hypothesis. This failure is due to the weakness of the metric coupling to the angular momentum of the galaxy.

However, the flat rotation curves seem to imply a large inertial induction effect, where the rotating inner matter boosts the rotation of the outer matter, leveling off the rotation curve, which is what the Cooperstock-Tieu model attempts to describe within general relativity. Since their solution predicts a matter density well within visible limits it is quite possible that their solution represents an alternative, more Machian, gravitational theory where inertial induction effects are much larger than in General Relativity.

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