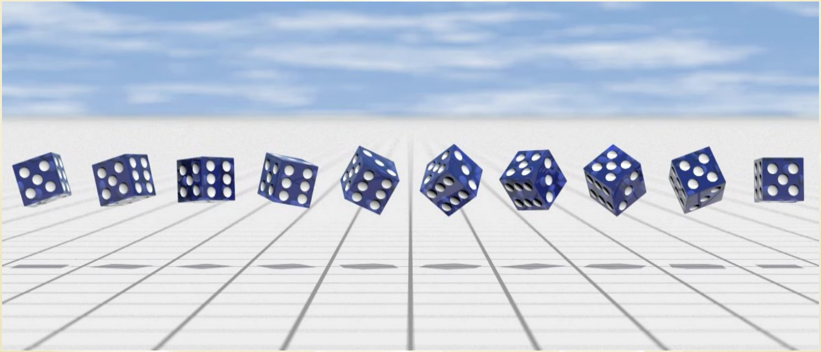
Why Spinors?



Daniel J. Cross
PGSA
9 April 2010

Outline

- Rotations in Q.M.
- Topology of the Rotation Group
- Pauli Spinors
- Anyons
- Weyl & Dirac Spinors

Rotations in Q.M.

Rotational Invariance, $x \to Rx$

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Projective Representation $U(R_1)U(R_2) = e^{i\theta(R_1,R_2)}U(R_1 R_2)$

Roots and Riemann

Weyl: "gauge" transformation $U(R) \to \frac{U(R)}{\sqrt[N]{\det U(R)}}$

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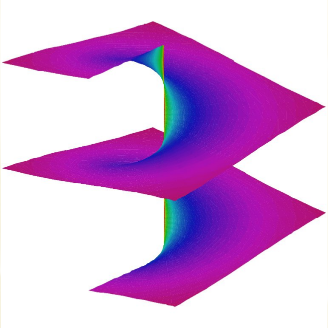
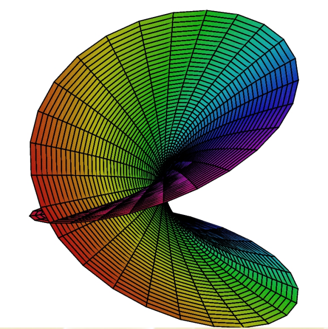
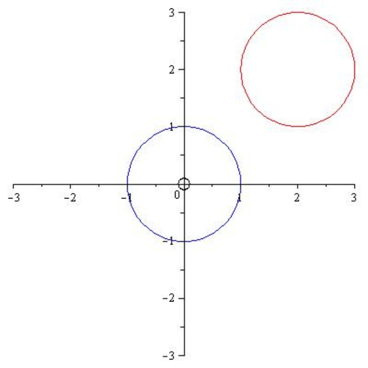




Topology of $\mathbb{C}\setminus\{0\}$



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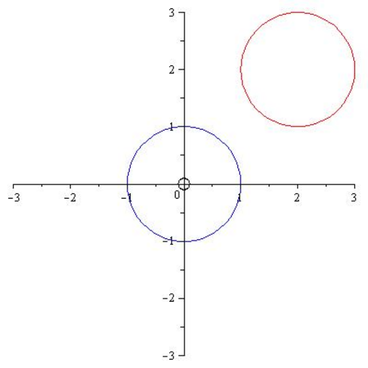

Red circle is contractible

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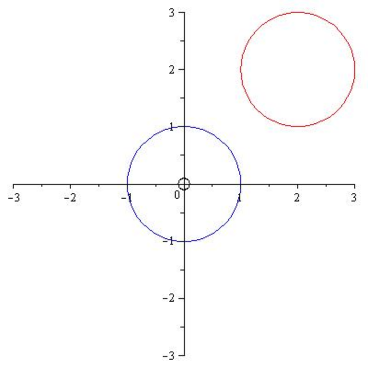

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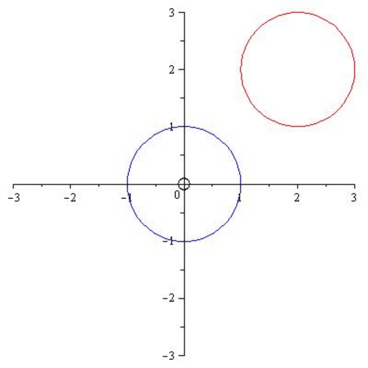

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$\gamma \in SO(3) \rightarrow ?$

Topology of $SO(3)$

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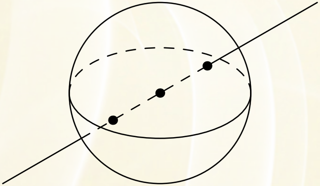

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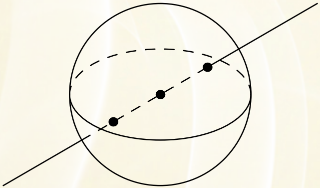

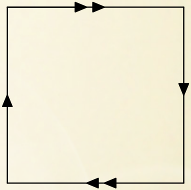

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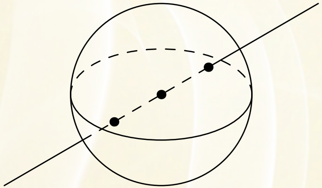

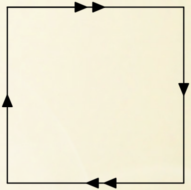

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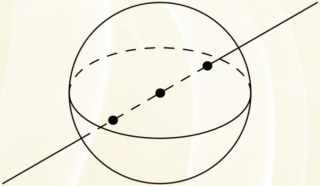

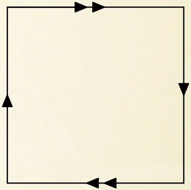

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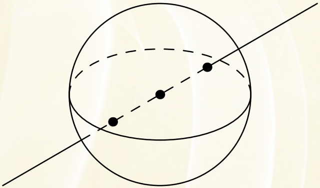

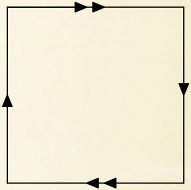

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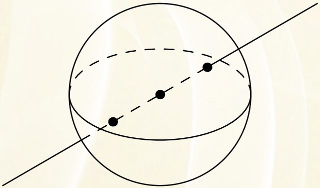

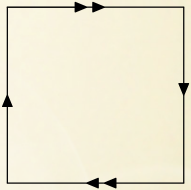

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Two kinds of closed curves in $SO(3)$:
1 rotation $\neq$ 0 rotations
2 rotations $=$ 0 rotations!

Observable?

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- [insert physicist] [insert object] Trick

Observable?

- Feynman Plate Trick

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- Dirac Scissors Trick

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- Ball in the Jell-O®

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Sends V and $-V$ in $SU(2)$ to same matrix R in $SO(3)$.

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$$V = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \to R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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θ	V	R
0	I_2	I_3
2π	$-I_2$	I_3
4π	I_2	I_3

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Equivalent to a single-valued spinor representation of $SU(2)$.

$$\begin{array}{ccccc} SU(2) & \longrightarrow & SO(3) & \longrightarrow & U(N) \\ \pm V & \longrightarrow & R & \longrightarrow & \pm U \end{array}$$

$\Longrightarrow$ Why we have only Fermions (spinor) and Bosons (true)!

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$$\left| \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\rangle \rightarrow \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \left| \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\rangle$$

SO(2) and Anyons

$Spin(n) \to SO(n)$ is $2:1$ and $Spin(n)$ simply connected $n \geq 3$.

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$\Longrightarrow$ 2-space has Anyons!

$R(2\pi) : |\psi\rangle \rightarrow e^{is\theta} |\psi\rangle \rightarrow$ spin s.

$SO(3,1)$ and Weyl Spinors

Symmetry of Relativity: Lorenz Group, $SO(3,1)$

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$$\begin{tikzcd}
Sl(2,\mathbb{C}) \arrow[r, "\text{retract}"] \arrow[d, "2:1"'] & SU(2) \arrow[d, "2:1"'] \\
SO(3,1) \arrow[r, "\text{retract}"] & SO(3)
\end{tikzcd}$$

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Symmetry of Relativity: Lorenz Group, $SO(3,1)$

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Thus relativistic Q.M. has only Fermions and Bosons!

Fermions = Weyl Spinors

$$U \in Sl(2,\mathbb{C}), \left| \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\rangle \rightarrow \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \left| \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right\rangle$$

Dirac Equation

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$$\begin{aligned}
\eta^{ij}\partial_i\partial_j &= (\gamma^i\partial_i)(\gamma^j\partial_j) \\
&= \gamma^i\gamma^j\partial_i\partial_j \\
&= \frac{1}{2}\left(\gamma^i\gamma^j + \gamma^j\gamma^i\right)\partial_i\partial_j \\
2\eta^{ij} &= \left\{\gamma^i, \gamma^j\right\}
\end{aligned}$$

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$$\gamma^0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

Dirac Spinors

$$\mathcal{D}^{\frac{1}{2},0} : \; U \in Sl(2,\mathbb{C}) \to U \qquad |\Psi_L\rangle \to U\,|\Psi_L\rangle$$

$$\mathcal{D}^{0,\frac{1}{2}} : \; U \in Sl(2,\mathbb{C}) \to U^{\dagger -1} \qquad |\Psi_R\rangle \to U^{\dagger -1}\,|\Psi_R\rangle$$

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$$\not{\partial}\,|\Psi\rangle = m\,|\Psi\rangle \to \begin{array}{ll} (\boldsymbol{\sigma}\cdot\boldsymbol{\partial} - \partial_t)\Psi_R & = m\Psi_L \\ (\boldsymbol{\sigma}\cdot\boldsymbol{\partial} + \partial_t)\Psi_L & = m\Psi_R \end{array}$$

"No one fully understands spinors. Their algebra is formally understood but their geometrical significance is mysterious. In some sense they describe the 'square-root' of geometry and, just as understanding the concept of the square root of -1 took centuries, the same might be true of spinors."

Sir Michael Atiyah

Fin

Special thanks to the PovRay wizard, Tim Jones.