Solutions to *General Relativity* by Wald

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Chapter 2 Solutions

1. *a) Show that the overlap functions $f_i^\pm \circ (f_j^\pm)^{-1}$ are C^∞, thus completing the demonstration given in section 2.1 that S^2 is a manifold.*

b) Show by explicit construction that two coordinate systems (as opposed to the six used in the text) suffice to cover S^2. (It is impossible to cover S^2 with a single chart, as follows from the fact that S^2 is compact, but very open subset of $\mathbb{R}^2$ is noncompact.)

a) The maps are defined as acting on the standard embedding of the sphere into $\mathbb{R}^3$. $f_i^\pm$ is a projection of upper (lower) hemisphere into the plane, for example

$$f_2^\pm(x^1, x^2, x^3) = (x^1, x^3),$$

and the others are similar. The inverse map stretches the plane up or down into the corresponding hemisphere,

$$(f_2^\pm)^{-1}(y^1, y^2) = (y^1, \pm\sqrt{1-((y^1)^2+(y^2)^2)}, y^2),$$

and so on. The overlap map then projects out two of these coordinates, so up to exchanging coordinates it is either of the two maps

$$\begin{aligned} (y^1, y^2) &\mapsto (y^1, y^2) \\ (y^1, y^2) &\mapsto (y^i, \pm\sqrt{1-((y^1)^2+(y^2)^2)}), \end{aligned}$$

both of which are C^∞ in their domain of definition, $(y^1)^2+(y^2)^2<1$.

b) Construct a chart by drawing a line from the north pole N through any given point $p \neq N$ on the sphere. The point where this line hits the plane $x^3 = 0$ is the image of p under the chart. This map is defined for all points except N and is called stereographic projection from N. A similar chart defined at S (or any other chart about S) then together with the first form an atlas for S^2. An explicit form of the map can be found geometrically. The line through p and N (thought of as vector in $\mathbb{R}^3$) is (with $N = (0,0,1)$)

$$p + t(p-N) = (t(1+p^1), t(1+p^2), t(1+p^3)-t).$$

Solving for when $x^3 = 0$ we get

$$t = \frac{p^3}{1-p^3},$$

which then gives

$$x^i = \frac{p^i}{1-p^3}.$$

The inverse maps are found similarly by starting with a point (y^1, y^2) in the plane, constructing the line $N + t(N-y)$ through N and y and finding where this hits the sphere:

$$1 = |x(t)|^2 = (y^1)^2t^2 + (y^2)^2t^2 + (t+1)^2,$$

which has solution

$$t = \frac{-2}{1+|y|^2},$$

and gives the mapping

$$(y^1, y^2) \mapsto \frac{1}{1+|y|^2}(2y^1, 2y^2, |y|^2-1).$$

2. *Prove that any smooth function $F:\mathbb{R}^n\to\mathbb{R}$ can be written in the form equation (2.2.2).*

Start with the fundamental theorem of calculus

$$F(x)-F(a)=\int_a^x F'(s)ds,$$

and make the substitution $s=t(x-a)+a$, which linearly rescales the interval $[a,x]$ to $[0,1]$. Then $ds=dt(x-a)$ and we get

$$F(x)-F(a)=(x-a)\int_0^1 F'(t(x-a)+a)dt,$$

which is the result for $n=1$. If $n>1$ then can write $F:\mathbb{R}^n\to\mathbb{R}$ as $F=(F^1,\dots,F^n)$, where each $F^i:\mathbb{R}\to\mathbb{R}$. Then apply the above to each F^i. To check the derivative condition compute

$$\begin{aligned}\left.\frac{\partial F}{\partial x}\right|_a &= (x-a)\frac{\partial}{\partial x}\int_0^1 F'(t(x-a)+a)dt+\left.\int_0^1 F'(t(x-a)+a)dt\right|_a\\ &= \int_0^1 F'(a)dt\\ &= F'(a),\end{aligned}$$

which is the result.

3. *a) Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field.*

b) Let X,Y,Z be smooth vector fields on a manifold M. Verify that their commutator satisfies the Jacobi identity:

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

c) Let $Y_{(1)},\dots,Y_{(n)}$ be smooth vector fields on an n-dimensional manifold M such that at each point $p\in M$ they form a basis of the tangent space T_pM. Then, at each point, we may expand each commutator $[Y_{(\alpha)},Y_{(\beta)}]$ in this basis, thereby defining the functions $C^\gamma{}_{\alpha\beta}=-C^\gamma{}_{\beta\alpha}$ by

$$[Y_{(\alpha)},Y_{(\beta)}]=C^\gamma{}_{\alpha\beta}Y_{(\gamma)}.$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$. (This equation is a useful algebraic relation if the $C^\gamma{}_{\alpha\beta}$ are constants, which will be the case if $Y_{(1)},\dots,Y_{(n)}$ are left (or right) invariant vector fields on a Lie group.)

a) First, linearity:

$$v(w(f+g))=v(w(f)+w(g))=vw(f)+vw(g),$$

similarly,

$$w(v(f+g))=wv(f)+wv(g),$$

so that

$$[v,w](f+g)=(vw-wv)(f)+(vw-wv)(g)=[v,w](f)+[v,w](g).$$

And for Leibnitz we have

$$\begin{aligned}v(w(fg)) &= v(fw(g)+gw(f))\\ &= v(f)w(f)+fv(w(g))+v(g)w(f)+gv(w(f)),\end{aligned}$$

likewise

$$\begin{aligned}w(v(fg)) &= w(fv(g)+gv(f))\\ &= w(f)v(f)+fw(v(g))+w(g)v(f)+gw(v(f)).\end{aligned}$$

when put together the $v(f)w(f)$ and like terms cancel, giving

$$\begin{aligned}[v,w](fg) &= fv(w(g))-fw(v(g))+gv(w(f))-gw(v(f))\\ &= f[v,w](g)+g[v,w](f).\end{aligned}$$

b) Consider the first term

$$\begin{aligned}
[[x,y],z] &= [xy - yx, z] \\
&= (xy - yx)z - z(xy - yx) \\
&= xyz - yxz - (zxy - zyx).
\end{aligned}$$

It's then easy to see that writing all the terms out they will cancel pairwise.

c) That the commutator can be written as such just says that since the commutator is vector, it can be written out as a linear combination of basis vectors and the combination you get depends on the basis vectors you're commuting. Taking a third basis vector Y_γ we get

$$\begin{aligned}
[[Y_{(\alpha)}, Y_{(\beta)}], Y_{(\gamma)}] &= [C^\delta{}_{\alpha\beta} Y_{(\delta)}, Y_{(\gamma)}] \\
&= C^\delta{}_{\alpha\beta}[Y_{(\delta)}, Y_{(\gamma)}] \\
&= C^\delta{}_{\alpha\beta} C^\sigma{}_{\delta\gamma} Y_{(\sigma)}.
\end{aligned}$$

The cyclic sum over (α, β, γ) then gives

$$C^\delta{}_{\alpha\beta} C^\sigma{}_{\delta\gamma} + C^\delta{}_{\beta\gamma} C^\sigma{}_{\delta\alpha} + C^\delta{}_{\gamma\alpha} C^\sigma{}_{\delta\beta} = 0.$$

4. *a) Show that in any coordinate basis, the components of the commutator of two vector fields v, w are given by*

$$[v,w]^\mu = v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}.$$

b) Let $Y_{(1)}, \ldots, Y_{(n)}$ be as in problem 3(c). Let $Y^{(1)}, \ldots, Y^{(n)}$ be the dual basis. Show that the components $Y^{(\gamma)}_\mu$ of $Y^{(\gamma)}$ in any coordinate basis satisfy

$$\frac{\partial Y^{(\gamma)}_\mu}{\partial x^\nu} - \frac{\partial Y^{(\gamma)}_\nu}{\partial x^\mu} = C^\gamma{}_{\alpha\beta} Y^{(\alpha)}_\mu Y^{(\beta)}_\nu.$$

a) We have

$$\begin{aligned}
[v,w](f) &= v^\nu \frac{\partial}{\partial x^\nu}\left(w^\mu \frac{\partial f}{\partial x^\mu}\right) - w^\nu \frac{\partial}{\partial x^\nu}\left(v^\mu \frac{\partial f}{\partial x^\mu}\right) \\
&= v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} + v^\nu \frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + \cdots \\
&+ w^\nu v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} + w^\nu \frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} \\
&= \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\right)\frac{\partial f}{\partial x^\mu} \\
&= [v,w]^\mu \frac{\partial}{\partial x^\mu}(f)
\end{aligned}$$

b) Since the commutator is a vector field, let it act on a dual basis vector $Y^{(\sigma)}$. On the one hand we get

$$C^\gamma{}_{\alpha\beta} Y_{(\gamma)}(Y^{(\sigma)}) = C^\sigma{}_{\alpha\beta}$$

by definition of the dual. On the other hand we get

$$\begin{aligned}
[Y_{(\alpha)}, Y_{(\beta)}] Y^{(\sigma)} &= \left(Y^\nu_{(\alpha)} \frac{\partial Y^\mu_{(\beta)}}{\partial x^\nu} - Y^\nu_{(\beta)} \frac{\partial Y^\nu_{(\alpha)}}{\partial x^\nu}\right) \frac{\partial}{\partial x^\mu}(Y^{(\sigma)}_\rho dx^\rho) \\
&= Y^\nu_{(\alpha)} \frac{\partial Y^\mu_{(\beta)}}{\partial x^\nu} Y^{(\sigma)}_\mu - Y^\nu_{(\beta)} \frac{\partial Y^\nu_{(\alpha)}}{\partial x^\nu} Y^{(\sigma)}_\mu \\
C^\sigma{}_{\alpha\beta} &= Y^\nu_{(\alpha)} Y^\mu_{(\beta)} \frac{\partial Y^{(\sigma)}_\mu}{\partial x^\nu} - Y^\nu_{(\beta)} Y^\nu_{(\alpha)} \frac{\partial Y^{(\sigma)}_\mu}{\partial x^\nu},
\end{aligned}$$

where in line three we use $\partial_\nu (Y^\mu_{(\beta)} Y^{(\sigma)}_\mu) = \partial_\nu \delta^\sigma_\beta = 0$. The result follows by multiplying both sides by $Y^{(\alpha)}_\gamma Y^{(\beta)}_\rho$ and contracting.

5. *Let $Y_{(1)}, \ldots, Y_{(n)}$ be smooth vector fields on an n-dimensional manifold M which form a basis of T_pM for each $p \in M$. Suppose that $[Y_{(\alpha)}, Y_{(\beta)}] = 0$ for all α, β. Prove that in a neighborhood of each $p \in M$ there exists coordinate $y^1, \ldots, y^n$ such that $Y_{(1)}, \ldots, Y_{(n)}$ are the coordinate vector fields, $Y_{(\mu)} = \frac{\partial}{\partial y^\mu}$.*

First select an arbitrary chart ψ about p with coordinates x^μ. We wish to construct coordinates y^μ with the indicated properties. We can create a differential relation between y and x by taking the differential of the functions y^j, which is then

$$dy^j = \frac{\partial y^j}{\partial x^i} dx^i.$$

Now, if $Y_{(\mu)} = \frac{\partial}{\partial y^\mu}$, then the dual satisfies $Y^{(\mu)} = dy^\mu$, hence the Jacobian matrix in the above equation expresses the components of the dual basis vectors in the dx^i basis, i.e.

$$dy^j = Y^{(j)} = Y_i^{(j)} dx^i,$$

or, comparing the two equations,

$$Y_i^{(j)} = \frac{\partial y^j}{\partial x^i}$$

Now, for each j we have an equation of the form

$$\frac{\partial f}{\partial x^\mu} = F_\mu,$$

with the necessary integrability condition (regarding f and F as differential forms)

$$d^2 f = dF = \frac{\partial F_\mu}{\partial x^\nu} - \frac{\partial F_\nu}{\partial x^\mu} = 0,$$

which is also sufficient when the first cohomology group H^1 vanishes. Whatever the domain of the chart we may restrict to an open star-shaped subset where this condition holds, hence these equations have solutions when that condition is met. By substituting in $Y^{(j)}$ for F we obtain the differential equation from the previous problem. Since the Y's commute, $C \equiv 0$, and the integrability condition is met.

6. *a) Verify that the dual vectors $\{v^\mu\}$ defined by equation (2.3.1) constitute a basis of V^*.*

b) Let $e_1, \ldots, e_n$ be a basis of V and $e^1, \ldots e^n$ be its dual basis. Let $w \in V$ and $\omega \in V^*$. Show that

$$\begin{aligned} w &= e^\alpha(w) e_\alpha \\ \omega &= \omega(e_\alpha) e^\alpha. \end{aligned}$$

c) Prove that the operation of contraction, equation (2.3.2), is independent of the choice of basis.

a) Let $v \in V$ and let e_i be a basis of V and e^j be defined by $e^j(e_i) = \delta_i^j$. Let $f = f_j e^j$ be a dual vector, then

$$f_j e^j(v^i e_i) = f_j v^i e^j(e_i) = f_j v^j,$$

which is zero for arbitrary v only if $f_j = 0$, hence the e^j are linearly independent. On the other hand, for an arbitrary dual f we have

$$f(v) = f(v^i e_i) = v^i f(e_i) = f_i$$

so, it is determined by it's action on the n basis vectors of V. Then we can write $f = f_j e^j$, since $f_j e^j(e_i) = f_j \delta_i^j = f_i$, and the e^j span V^*.

b) We have $w = w^i e_i$ and $e^j(w) = e^j(w^i e_j) = w^j$. The second equation comes out similarly.

c) It is sufficient to look at a (1,1) tensor. Then we have for any isomorphism M

$$\begin{aligned} CT &= T(e^j, e_j) \\ &= T({M^j}_a e^a, e_b {(M^{-1})^b}_j) \\ &= {M^j}_a {(M^{-1})^b}_j T(e^a, e_b) \\ &= \delta_a^b T(e^a, e_b) \\ &= T(e^a, e_a) \\ &= CT \end{aligned}$$

7. *Let V be an n-dimensional vector space and let g be a metric on V.*

a) Show that one always can find an orthonormal basis $v_{(1)}, \ldots, v_{(n)}$ of V, i.e, a basis such that $g(v_{(\alpha)}, v_{(\beta)}) = \pm\delta_{\alpha\beta}$.

b) Show that the signature of g is independent of the choice of orthonormal basis.

a) Suppose $n = 1$, then by non-degeneracy there is a vector v such that $g(v, v) = l \neq 0$, then the vector $v' = v/\sqrt{|l|}$ satisfies $g(v', v') = \pm 1$. Now suppose the condition holds for $n - 1$. Choose a vector $v \in V$. It may happen that v is null, $g(v, v) = 0$. We cannot choose the orthogonal complement of v because the induced metric would be degenerate. By non-degeneracyof V, there must be at least one other vector w such that $g(v, w) \neq 0$. If w is not null use w, otherwise use $v' = v + w$, then

$$\begin{aligned} g(v', v') &= g(v + w, v + w) \\ &= g(v, v) + 2g(v, w) + g(w, w) \\ &= 2g(v, w). \end{aligned}$$

Normalize v' and then consider the orthogonal complement $v'^{\perp}$ of v'. We claim the induced metric $g' = g|_{v'^{\perp}}$ is non-degenerate. If $v \in v'^{\perp}$ satisfies $g'(v, w) = 0$ for all $w \in v'^{\perp}$, then $g(v, v') = 0$ by definition of orthogonal complement, but then g is degenerate. Let the dimension of $v'^{\perp}$ be m. By induction we have an orthonormal basis $\{e_i\}$. We need to show that $\{e_i, v'\}$ form a basis of V. First we need to show that v' is linearly independent of the $\{e_i\}$, but this follows from the definition of orthogonal complement. Finally, take a vector v and remove its projection onto v', that is let $w = v - ||v'||g(v, v')v'$. Then we have $g(w, v') = g(v, v') - ||v'||^2 g(v, v') = 0$, and w is in $v'^{\perp}$, hence these elements span V.

b) Let $\{e_a\}$ be a basis as in a) so that $g(e_a, e_b) = \pm\delta_{ab}$ and let M be an orthonormal transformation. Then

$$\begin{aligned} g(M_a{}^r e_r, M_b{}^s e_s) &= M_a{}^r M_b{}^s g(e_r, e_s) \\ &= \pm\delta_{rs} M_a{}^r M_b{}^s \\ &= \pm\delta_{ab}, \end{aligned}$$

where the last line follows by definition of orthonormal transformation (i.e. the matrix is orthogonal, hence the norm of the row vectors is one).

8. *a) The metric of flat, three-dimensional Euclidean space is*

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Show that the metric components $g_{\mu\nu}$ in spherical polar coordinates r, θ, ϕ defined by

$$\begin{aligned} r &= (x^2 + y^2 + z^2)^{1/2} \\ \cos\theta &= z/r \\ \tan\phi &= y/x \end{aligned}$$

is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

b) The spacetime metric of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates" defined by

$$\begin{aligned} t' &= t \\ x' &= (x^2 + y^2)^{1/2} \cos(\phi - \omega t) \\ y' &= (x^2 + y^2)^{1/2} \sin(\phi - \omega t) \\ z' &= z, \end{aligned}$$

where $\tan\phi = y/x$

a) The metric is determined by the transformation rule

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}.$$

The derivatives determine the inverse Jacobian matrix:

$$\frac{\partial x^\mu}{\partial x'^\alpha} = (J^{-1})^\mu{}_\alpha$$

(since as a matrix the unprimed variables label rows and primed columns). So we have

$$g'_{\alpha\beta} = (J^{-1})^\mu{}_\alpha g_{\mu\nu}(J^{-1})^\nu{}_\beta.$$

and can write the matrix equation

$$g' = (J^{-1})^t g(J^{-1}).$$

The inverse transformation is easy to obtain. We know $z = r\cos\theta$. We then have

$$\begin{aligned}
y &= x\tan\phi \\
y^2 &= x^2(\sec^2\phi - 1) \\
y^2 + x^2 &= x^2\sec^2\phi \\
r^2 - z^2 &= x^2\sec^2\phi \\
r^2(1-\cos^2\theta) &= x^2\sec^2\phi \\
r^2\sin^2\theta &= x^2\sec^2\phi,
\end{aligned}$$

which gives $x = r\cos\phi\sin\theta$ and $y = r\sin\phi\sin\theta$.
It is straightforward to calculate the partial derivatives to obtain

$$J^{-1} = \begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\sin\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

Since $g = I$,we have

$$(J^{-1})(J^{-1})^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

or

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

b) We will first change the metric to stationary cylindrical coordinates. This is similar to the transformation in part a) and gives the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2,$$

with inverse

$$-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial}{\partial \phi}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

The Jacobian is

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega y & x/r & -y & 0 \\ -\omega x & y/r & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse metric transforms as

$$g'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g^{\mu\nu} = J^\alpha{}_\mu g^{\mu\nu} J^\beta{}_\nu,$$

or

$$(g^{-1})' = J(g^{-1})J^t,$$

which gives

$$(g^{-1})' = \begin{pmatrix} -1 & -\omega y & \omega x & 0 \\ -\omega y & 1-\omega^2 y^2 & \omega^2 xy & 0 \\ \omega x & \omega^2 xy & 1-\omega^2 x^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The metric itself can be obtain from inverting the 3×3 submatrix. The Jacobian determinant is r, and $\det(g^{-1}) = -r^{-2}$, so $\det(g^{-1})' = -1$ and we get (computing minors)

$$g = \begin{pmatrix} -1+\omega^2 r^2 & -\omega y & \omega x & 0 \\ -\omega y & 1 & 0 & 0 \\ \omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chapter 3 Solutions

1. *Let property (5) (the "torsion free" condition) be dropped from the definition of derivative operator ∇ in section 3.1*

a) Show that there exists a tensor $T^c{}_{ab}$ (called the torsion tensor) such that for all smooth functions f, we have $(\nabla_a\nabla_b - \nabla_b\nabla_a)f = -T^c{}_{ab}\nabla_c f$.

b) Show that for any smooth vector fields X, Y we have

$$T^c{}_{ab}X^aY^b = \nabla_X Y^c - \nabla_Y X^c - [X,Y]^c.$$

c) Given a metric, g, show that there exists a unique derivative operator ∇ with torsion T such that $\nabla g = 0$. Derive the analog of equation 3.1.29, expressing this derivative operator in terms of an ordinary derivative ∂ and T.

a) We note that (3.1.7): $\nabla_a\omega_b = \tilde\nabla_a\omega_b - C^c{}_{ab}\omega_c$ is still valid with torsion, so setting $\omega_b = \nabla_b f = \tilde\nabla_b f$, we get $\nabla_a\nabla_b = \tilde\nabla_a\tilde\nabla_b f - C^c{}_{ab}\nabla_c f$. Since $\tilde\nabla$ is torsion free we get

$$\begin{aligned}(\nabla_a\nabla_b - \nabla_b\nabla_a)f &= (\tilde\nabla_a\tilde\nabla_b - \tilde\nabla_b\tilde\nabla_a)f - (C^c{}_{ab} - C^c{}_{ba})\nabla_c f \\ &= -2C^c{}_{[ab]}\nabla_c f,\end{aligned}$$

Thus T is essentially the anti-symmetric part of C.

b) We compute

$$\begin{aligned}[v,w]f &= v^a\nabla_a(w^b\nabla_b f) - w^a\nabla_a(v^b\nabla_b f) \\ &= v^a w^b(\nabla_a\nabla_b - \nabla_b\nabla_a f) + (v^a\nabla_a w^c - w^a\nabla_a v^c)\nabla f \\ &= -T^c{}_{ab}v^a w^b\nabla_c f + (v^a\nabla_a w^c - w^a\nabla_a v^c)\nabla_c f,\end{aligned}$$

or, dropping f and basis vector $\nabla_c = \partial_c$, $[v,w]^c = -T^c{}_{ab}v^a w^b + (v^a\nabla_a w^c - w^a\nabla_a v^c)$.

c) The condition $\nabla g = 0$ becomes, again

$$\nabla_a g_{bc} = \tilde\nabla_a g_{bc} - C^d{}_{ab}g_{dc} - C^d{}_{ac}g_{bd} = \tilde\nabla_a g_{bc} - C_{cab} - C_{bac}.$$

If we add the permutation (ab) and subtract the permutation (cab) as before we get

$$\tilde\nabla_a g_{bc} + \tilde\nabla_b g_{ac} - \tilde\nabla_c g_{ab} = T_{bac} + T_{abc} + 2C_{c(ab)},$$

so that we nay solve for the symmetric part of C given the antisymmetric part, the torsion T. Thus we have

$$C_{c(ab)} = \frac{1}{2}\left(\partial_a\, g_{bc} + \partial_b\, g_{ac} - \partial_c\, g_{ab}\right) - \frac{1}{2}\left(T_{bac} + T_{abc}\right),$$

or since $C_{abc} = C_{c(ab)} + C_{c[ab]} = C_{c(ab)} + T_{cab}/2$, we get

$$C_{abc} = \frac{1}{2}\left(\partial_a\, g_{bc} + \partial_b\, g_{ac} - \partial_c\, g_{ab}\right) - \frac{1}{2}\left(T_{bac} + T_{abc} - T_{cab}\right).$$

2. *Let M be a manifold with metric g and associated derivative operator ∇. A solution of the equation $\nabla^a\nabla_a\alpha = 0$ is called a harmonic function. In the case where M is 2-dimensional, let α be harmonic and let ϵ_{ab} be an antisymmetric tensor field satisfying $\epsilon_{ab}\epsilon^{ab} = 2(-1)^s$, where s is the number of minuses occurring in the signature of the metric. Consider the equation $\nabla_b\beta = \epsilon_{ab}\nabla^b\alpha$.*

a) Show that the integrability conditions for this equation are satisfied, and thus, locally, there exists a solution, β. Show that β also is harmonic.

b) By choosing α and β as coordinates, show that the metric takes the form

$$ds^2 = \pm\Omega(\alpha,\beta)\left[d\alpha^2 + (-1)^s d\beta^2\right].$$

a) Since $\epsilon_{ab}\epsilon^{ab}$ is a constant, we have $\nabla_c \epsilon_{ab}\epsilon^{ab} = 0$, but we can also write

$$\nabla_e \epsilon_{ab}\epsilon^{ab} = \epsilon_{ab}\nabla_e \epsilon^{ab} + \epsilon^{ab}\nabla_e \epsilon_{ab} = 2\epsilon^{ab}\nabla_e \epsilon_{ab},$$

where we use $\nabla g = 0$. This then requires that $\nabla_e \epsilon_{ac} = 0$. Since $\beta = d\alpha$, the integrability condition is again $d^2\alpha = d\beta = 0$, which is $(\nabla_b\nabla_a - \nabla_a\nabla_b)\beta = 0$. We have

$$\begin{aligned}(\nabla_b\nabla_a - \nabla_a\nabla_b)\beta &= \nabla_a(\epsilon_{bc}\nabla^c\alpha) - \nabla_b(\epsilon_{ac}\nabla^c\alpha) \\ &= (\epsilon_{bc}\nabla_a\nabla^c - \epsilon_{ac}\nabla_b\nabla^c)\alpha.\end{aligned}$$

Now we can write $\epsilon_{bc}\nabla^c$ as the covector

$$\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\begin{pmatrix} \nabla^1 \\ \nabla^2 \end{pmatrix} = \epsilon\left(\nabla^2, -\nabla^1\right),$$

and then consider $(\epsilon_{bc}\nabla^c)\nabla_a$ as an outer product

$$\epsilon\left(\nabla^2, -\nabla^1\right) \otimes \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} = \epsilon\begin{pmatrix} \nabla_1\nabla^2 & -\nabla_1\nabla^1 \\ \nabla_2\nabla^2 & -\nabla_2\nabla^1 \end{pmatrix} = M,$$

and the equation becomes

$$(M - M^t)\alpha = \epsilon\begin{pmatrix} 0 & -\nabla_1\nabla^1 - \nabla_2\nabla^2 \\ \nabla_2\nabla^2 + \nabla_1\nabla^1 & 0 \end{pmatrix}\alpha,$$

and both non-trivial terms vanish since α is harmonic and thus the integrability condition is satisfied.

Now we want to compute

$$g^{ab}\nabla_a\nabla_b\beta = \nabla^a\nabla_a\beta = \nabla^a\epsilon_{ab}\nabla^b\alpha = \epsilon_{ab}\nabla^a\nabla^b\alpha,$$

where

$$\epsilon_{ab}\nabla^a\nabla^b = \epsilon\left(\nabla^2, -\nabla^1\right)\begin{pmatrix} \nabla^1 \\ \nabla^2 \end{pmatrix} = \epsilon(\nabla^2\nabla^1 - \nabla^1\nabla^2),$$

which vanishes acting on α (no torsion), and so β is harmonic.

b) Given an arbitrary system of coordinates x^μ, the inverse metric transforms as

$$\begin{aligned} g'^{\alpha\beta} &= \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g^{\mu\nu} \\ &= \nabla_\mu x'^\alpha \nabla_\nu x'^\beta g^{\mu\nu}.\end{aligned}$$

with $x'^1 = \alpha$ and $x'^2 = \beta$, we have

$$\begin{aligned} g' &= g^{\mu\nu}\begin{pmatrix} \nabla_\mu\alpha\nabla_\nu\alpha & (\nabla_\mu\alpha)\,\epsilon_{\nu\sigma}\nabla^\sigma\alpha \\ (\nabla_\nu\alpha)\,\epsilon_{\mu\sigma}\nabla^\sigma\alpha & (\epsilon_{\mu\sigma}\nabla^\sigma\alpha)\,(\epsilon_{\nu\rho}\nabla^\rho\alpha) \end{pmatrix} \\ &= \begin{pmatrix} g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha & \epsilon_{\mu\nu}\nabla^\mu\alpha\nabla^\nu\alpha \\ \epsilon_{\mu\nu}\nabla^\mu\alpha\nabla^\nu\alpha & g^{\mu\nu}\epsilon_{\mu\sigma}\epsilon_{\nu\rho}\nabla^\sigma\alpha\nabla^\rho\alpha \end{pmatrix}.\end{aligned}$$

Analogously to part *a*) the off-diagonal terms become

$$\epsilon_{\mu\nu}\nabla^\mu\alpha\nabla^\nu\alpha = \epsilon(\nabla^2\alpha\nabla^1\alpha - \nabla^1\alpha\nabla^2\alpha) = 0.$$

To evaluate the 22-component we will write it as $g^{\mu\nu}\epsilon_\mu{}^\sigma\epsilon_\nu{}^\rho\nabla_\sigma\nabla_\rho$ and consider the tensor contracted against the derivatives. This can be written as the matrix equation

$$\epsilon^t g^{-1}\epsilon,$$

with $\epsilon = \epsilon_\alpha{}^\beta = g^{\beta\sigma}\epsilon_{\alpha\sigma}$. So we have

$$\begin{aligned} \epsilon_\alpha{}^\beta &= g^{\beta\sigma}\epsilon_{\alpha\sigma} \\ &= \epsilon\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \\ &= \epsilon\begin{pmatrix} g^{21} & g^{22} \\ -g^{11} & -g^{12} \end{pmatrix},\end{aligned}$$

and thus

$$\begin{aligned}
\epsilon^t g^{-1}\epsilon &= \epsilon^2 \begin{pmatrix} g^{21} & g^{22} \\ -g^{11} & -g^{12} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} g^{21} & -g^{11} \\ g^{22} & -g^{12} \end{pmatrix} \\
&= \epsilon^2 \det g \begin{pmatrix} g^{21} & g^{22} \\ -g^{11} & -g^{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \epsilon^2 \det g \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \\
&= \epsilon^2 \det g (g^{-1}),
\end{aligned}$$

so that the 22-element simplifies to $(\epsilon^2 \det g) g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha$, and the inverse metric becomes

$$\left(\frac{\partial}{\partial s}\right)^2 = (g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha)\left[\left(\frac{\partial}{\partial \alpha}\right)^2 + \epsilon^2 \det g \left(\frac{\partial}{\partial \beta}\right)^2\right].$$

Finally, since $\epsilon^t g^{-1}\epsilon = \epsilon_\mu{}^\rho \epsilon^{\mu\sigma}$, we have

$$\begin{aligned}
\epsilon_{\mu\rho}\epsilon^{\mu\rho} &= g_{\rho\sigma}\epsilon_\mu{}^\rho\epsilon^{\mu\sigma} \\
&= (\epsilon^2 \det g) g_{\rho\sigma} g^{\rho\sigma} \\
&= \epsilon^2 \det g \,\mathrm{tr}(\delta^\rho_\sigma) \\
&= 2\epsilon^2 \det g,
\end{aligned}$$

since $n = 2$. Now this must equal $2(-1)^s$, so $\epsilon^2 \det g = (-1)^s$, and the metric becomes

$$\left(\frac{\partial}{\partial s}\right)^2 = \pm\Omega^{-1}(\alpha,\beta)\left[\left(\frac{\partial}{\partial \alpha}\right)^2 + (-1)^s \left(\frac{\partial}{\partial \beta}\right)^2\right],$$

(where we set $\Omega^{-1} = ||\nabla_\mu\alpha||^2$), or

$$ds^2 = \pm\Omega(\alpha,\beta)\left[d\alpha^2 + (-1)^s d\beta^2\right].$$

3. *a) Show that $R_{abcd} = R_{cdab}$.*

b) In n dimensions, the Riemann tensor has n^4 components. However, on account of the symmetries (3.2.13), (3.2.14), and (3.2.15), not all of these components are independent. Show that the number of independent components is $n^2(n^2-1)/12$.

a) The first two identities imply that cyclic sum on the first three indices vanishes, $R_{abcd} + R_{bcad} + R_{cabd} = 0$. If we then add four copies of this equation using the four cyclic permutations on all four indices then all terms pairwise cancel except for four, leaving $2(R_{acdb} + R_{bdac}) = 0$, or $R_{acbd} = R_{bdac}$.

b) Start with R_{abcd} with n^4 unconstrained components, and consider the various identities as imposing constraints on these components. The first symmetry is $ab = -ba$, which imposes n constraints when $a = b$ and $\binom{n}{2}$ constraints when $a \neq b$, which gives a total of

$$\binom{n}{2} + n = \frac{n!}{2(n-2)!} + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

constraints on the first two indices. At the same time it leaves $n^2 - n(n+1)/2 = n(n-1)/2$ terms in a, b to be freely specified. Now, since there are choices of a, b for all c, d, that actually makes $n^2 * n(n+1)/2$ total constraints.

Now the condition $cd = -dc$ imposes $n(n+1)/2$ constraints on c, d for all choices of a, b. But the first two indices are already constraints and only $n(n-1)/2$ are independent, so the total number of imposed constraints is

$$\frac{n(n+1)}{2}\frac{n(n-1)}{2} = \frac{1}{4}n^2(n^2-1).$$

The final symmetry is $abc + bca + cab = 0$. This gives $\binom{n}{3}$ constraints for $a \neq b \neq c$ for each choice of d. No new constraints are introduced if any two or all three of a, b, c are equal, since these cases reduce to the antisymmetry relations considered already. Thus the number of introduced constraints is

$$n\binom{n}{3} = \frac{nn!}{3!(n-3)!} = \frac{1}{6}n^2(n-1)(n-2).$$

The total number of constraints is then

$$\begin{aligned} & \frac{1}{2}n^2(n^2+n)+\frac{1}{4}n^2(n^2-1)+\frac{1}{6}n^2(n-1)(n-2) \\ = & \frac{1}{12}n^2(6n^2+6n+3n^2-3+2n^2-6n+4) \\ = & \frac{1}{12}n^2(11n^2+1). \end{aligned}$$

Thus there remains

$$n^4-\frac{1}{12}n^2(11n^2+1)=\frac{1}{12}n^2(n^2-1)$$

independent components. So for $n=1\ldots5$ we get

n	tot	indep
1	1	0
2	16	1
3	81	6
4	256	20
5	625	50

so the savings by using the symmetries is tremendous. We note that when $n=1$ there are no free curvature terms: no 1-dimensional manifolds have curvature. When $n=2$ there is only one free components, which is essentially the Gaussian curvature of the surface.

4. *a) Show that in two dimensions, the Riemann tensor takes the from* $R_{abcd}=Rg_{a[c}g_{d]b}$.

b) By similar arguments, show that in three dimensions the Weyl tensor vanishes identically; i.e., for $n=3$, *equation (2.2.28) holds with* $C_{abcd}=0$.

a) Consider the tensor $\xi_{abcd}=g_{a[c}g_{d]b}=(g_{ac}g_{bd}-g_{ad}g_{cb})/2$. We have

$$\begin{aligned} 2\xi_{bacd} &= g_{bc}g_{da}-g_{bd}g_{ca} \\ &= -(g_{ac}g_{db}-g_{ad}g_{cb}) \\ &= -2\xi_{abcd}, \end{aligned}$$

so that ξ has the first Riemann symmetry. In the same way $\xi_{abdc}=-\xi_{abcd}$ and $\xi_{cdab}=\xi_{abcd}$, so that ξ has all the symmetries of the Riemann tensor. Thus from 3b) both tensors have one free component, and thus must be proportional: $R_{abcd}=\alpha\xi_{abcd}$. We can establish α by contracting:

$$\begin{aligned} g^{bd}g^{ac}R_{abcd} &= g^{bd}g^{ac}\xi_{abcd} \\ R &= \alpha g^{bd}g^{ac}(g_{ac}g_{bd}-g_{ad}g_{cb})/2 \\ R &= \alpha((\operatorname{tr} g)^2-\operatorname{tr} g)/2 \\ R &= \alpha, \end{aligned}$$

since $\operatorname{tr} g=\delta^a_a=2$. We note that $2K=R$,where K is the Gaussian curvature of the surface. Further, by taking only one contraction in the above we get

$$\begin{aligned} g^{bd}R_{abcd} &= g^{bd}\xi_{abcd} \\ R_{ac} &= Rg^{bd}(g_{ac}g_{bd}-g_{ad}g_{cb})/2 \\ R_{ac} &= R(g_{ac}\operatorname{tr} g-g_{ac})/2 \\ R_{ac} &= Rg_{ac}/2 \\ R_{ac} &= Kg_{ac}. \end{aligned}$$

b) Write $R_{abcd}=C_{abcd}+\xi_{abcd}$. ξ has the same symmetries as R, and so we can write

$$R_{abcd}=\alpha(a,b,c,d)\xi_{abcd},$$

where $\alpha(a,b,c,d)$ is a collection of proportionality coefficients. Since C is traceless, we have

$$R_{ac}=g^{bd}\xi_{abcd}=g^{bd}\alpha(a,b,c,d)\xi_{abcd}=\sum_b\alpha(a,b,c,b)\xi_{abc}{}^{b},$$

which can only hold if in fact $\alpha(a,b,c,b) = 1$. This determines all the coefficients unless all four indices are different, but this is impossible when $n = 3$, so all normalization constants are $+1$, so that $R_{abcd} = \xi_{abcd}$ and $C_{abcd} = 0$. This also demonstrates why when $n > 3$ no reductions are possible and we must consider the full Riemann tensor.

5. *a) Show that any curve whose tangent satisfies equation (3.3.2) can be reparametrized so that equation (3.3.1) is satisfied.*

b) Let t be an affine parameter of a geodesic γ. Show that all other affine parameters of γ take the form $at + b$, where a and b are constants.

a) In coordinates the generalized geodesic equation is

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\sigma}{dt}\frac{dx^\nu}{dt} = \alpha\frac{dx^\mu}{dt}.$$

Introduce a new parameter $s = s(t)$ so that $d/dt = (ds/dt)d/ds$. Then

$$\begin{aligned}\frac{d}{dt}\frac{dx^\mu}{dt} &= \frac{d}{dt}\left(\frac{ds}{dt}\frac{dx^\mu}{ds}\right)\\ &= \left(\frac{ds}{dt}\right)^2\frac{d^2x^\mu}{ds^2} + \frac{dx^\mu}{ds}\frac{d}{dt}\left(\frac{ds}{dt}\right).\end{aligned}$$

Now we rewrite the geodesic equation as

$$\left(\frac{ds}{dt}\right)^2\frac{d^2x^\mu}{ds^2} + \left(\frac{ds}{dt}\right)^2\Gamma^\mu{}_{\nu\sigma}\frac{dx^\sigma}{ds}\frac{dx^\nu}{ds} = \frac{dx^\mu}{ds}\left(\alpha\frac{ds}{dt} - \frac{d}{dt}\left(\frac{ds}{dt}\right)\right),$$

so that to make the r.h.s zero we need to solve the equation(setting $ds/dt = \lambda$)

$$\alpha\lambda = \frac{d\lambda}{dt} \rightarrow \alpha dt = \frac{d\lambda}{\lambda}$$

which integrates to

$$\lambda = \lambda_0 e^{\alpha(t-t_0)}.$$

b) Let s_1 and s_2 be affine parameters with $\lambda_i = ds_i/dt$ in terms of some arbitrary parameter t. From part a) we have

$$\frac{ds_1}{ds_2} = \frac{\lambda_1}{\lambda_2} = \frac{(\lambda_1)_0}{(\lambda_2)_0} = \text{const.}$$

Thus $ds_1/ds_2 = \text{const}$ and $s_1 = as_2 + b$, for constants a, b.

6. *The metric of Euclidean $\mathbb{R}^3$ in spherical coordinates is $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.*

a) Calculate the Christoffel components $\Gamma^\sigma{}_{\mu\nu}$ in this coordinate system.

b) Write down the components of the geodesic equation in this coordinate system and verify that the solutions correspond to straight lines in Cartesian coordinates.

a) There are only three non-trivial derivatives of the metric tensor:

$$\frac{\partial\, g_{\theta\theta}}{\partial\, r} = 2r,\ \frac{\partial\, g_{\phi\phi}}{\partial\, r} = 2r\sin^2\theta,\ \frac{\partial\, g_{\phi\phi}}{\partial\, \theta} = 2r^2\sin\theta\cos\theta.$$

Since the metric is diagonal the components of the inverse are the inverse of the components. We will consider the six different combinations for the bottom indices, which gives

$$\begin{aligned}2\Gamma^\lambda{}_{r\theta} &= g^{\lambda\sigma}\left(\partial_r\, g_{\theta\sigma} + \partial_\theta\, g_{r\sigma} - \partial_\sigma\, g_{r\theta}\right) &&= g^{\lambda\theta}\,\partial_r\, g_{\theta\theta}\\ 2\Gamma^\lambda{}_{rr} &= g^{\lambda\sigma}\left(2\,\partial_r\, g_{r\sigma} - \partial_\sigma\, g_{rr}\right) &&= 0\\ 2\Gamma^\lambda{}_{\theta\theta} &= g^{\lambda\sigma}\left(2\,\partial_\theta\, g_{\theta\sigma} - \partial_\sigma\, g_{\theta\theta}\right) &&= -g^{\lambda r}\,\partial_r\, g_{\theta\theta}\\ 2\Gamma^\lambda{}_{\theta\phi} &= g^{\lambda\sigma}\left(\partial_\theta\, g_{\phi\sigma} + \partial_\phi\, g_{\theta\sigma} - \partial_\sigma\, g_{\theta\phi}\right) &&= g^{\lambda\phi}\,\partial_\theta\, g_{\phi\phi}\\ 2\Gamma^\lambda{}_{\phi\phi} &= g^{\lambda\sigma}\left(2\,\partial_\phi\, g_{\phi\sigma} - \partial_\sigma\, g_{\phi\phi}\right) &&= -g^{\lambda r}\,\partial_r\, g_{\phi\phi} - g^{\lambda\theta}\,\partial_\theta\, g_{\phi\phi}\\ 2\Gamma^\lambda{}_{r\phi} &= g^{\lambda\sigma}\left(\partial_r\, g_{\phi\sigma} + \partial_\phi\, g_{r\sigma} - \partial_\sigma\, g_{r\phi}\right) &&= g^{\lambda\phi}\,\partial_r\, g_{\phi\phi},\end{aligned}$$

and the non-zero components come out to be

$$\begin{array}{lclcl}
\Gamma^{\theta}{}_{r\theta} & = & g^{\theta\theta}\,\partial_r\, g_{\theta\theta}/2 & = & \dfrac{1}{r} \\
\Gamma^{r}{}_{\theta\theta} & = & -g^{rr}\,\partial_r\, g_{\theta\theta}/2 & = & -r \\
\Gamma^{\phi}{}_{\theta\phi} & = & g^{\phi\phi}\,\partial_\theta\, g_{\phi\phi}/2 & = & \cot\theta \\
\Gamma^{r}{}_{\phi\phi} & = & -g^{rr}\,\partial_r\, g_{\phi\phi}/2 & = & -r\sin^2\theta \\
\Gamma^{\theta}{}_{\phi\phi} & = & -g^{\theta\theta}\,\partial_\theta\, g_{\phi\phi}/2 & = & -\sin\theta\cos\theta \\
\Gamma^{\phi}{}_{r\phi} & = & g^{\phi\phi}\,\partial_r\, g_{\phi\phi}/2 & = & \dfrac{1}{r}
\end{array}$$

b) If we write $dx^\mu/dt = \dot{x}^\mu = (v^r, v^\theta, v^\phi)$ we have

$$\begin{array}{lclcl}
0 & = & \dot{v}^r + \Gamma^{r}{}_{\theta\theta} v^\theta v^\theta + \Gamma^{r}{}_{\phi\phi} v^\phi v^\phi & = & \dot{v}^r - r v^\theta v^\theta - r\sin^2\theta v^\phi v^\phi \\
0 & = & \dot{v}^\theta + 2\Gamma^{\theta}{}_{r\theta} v^r v^\theta + \Gamma^{\theta}{}_{\phi\phi} v^\phi v^\phi & = & \dot{v}^\theta + \frac{2}{r} v^r v^\theta - \sin\theta\cos\theta v^\phi v^\phi \\
0 & = & \dot{v}^\phi + 2\Gamma^{\phi}{}_{\theta\phi} v^\theta v^\phi + 2\Gamma^{\phi}{}_{r\phi} v^r v^\phi & = & \dot{v}^\phi + 2\cot\theta v^\theta v^\phi + \frac{2}{r} v^r v^\phi
\end{array}$$

Consider $x = r\cos\phi\sin\theta$. We have

$$\dot{x} = \dot{r}\cos\phi\sin\theta - r\dot{\phi}\sin\phi\sin\theta + r\dot{\theta}\cos\phi\cos\theta,$$

and similarly,

$$\begin{array}{rcl}
\ddot{x} & = & \ddot{r}\cos\phi\sin\theta - 2\dot{r}\dot{\phi}\sin\phi\sin\theta + 2\dot{r}\dot{\theta}\cos\phi\cos\theta \\
& & -r\ddot{\phi}\sin\phi\sin\theta - r\dot{\phi}^2\cos\phi\sin\theta - 2r\dot{\phi}\dot{\theta}\sin\phi\cos\theta \\
& & +r\ddot{\theta}\cos\phi\cos\theta - r\dot{\theta}^2\cos\phi\sin\theta.
\end{array}$$

If we then substitute in for $\ddot{r} = \dot{v}^r$, etc. all the terms cancel and $\ddot{x} = 0$. The other axis are worked out similarly. (Is there a better way to do this?)

7. *As shown in problem 2, an arbitrary Lorentz metric on a two-dimensional manifold locally always can be put in the form $ds^2 = \Omega^2(x,t)[-dt^2 + dx^2]$. Calculate the Riemann curvature tensor of this metric (a) by the coordinate basis methods of section 3.4a and (b) by the tetrad methods of section 3.4b.*

a) We know that when $n = 2$ there is only one free component of R which we take to be R_{1212}. To calculate this we need only $R_{121}{}^2 g_{22}$, since g is diagonal. Thus we need

$$R_{121}{}^2 = \partial_2\,\Gamma^2{}_{11} - \partial_1\,\Gamma^2{}_{21} + \Gamma^e{}_{11}\Gamma^2{}_{e2} - \Gamma^e{}_{21}\Gamma^2{}_{e1}.$$

Now we note that $g_{ab} = \Omega^2\eta_{ab}$, so that

$$\begin{array}{rcl}
\partial_c\, g_{ab} & = & 2\Omega\eta_{ab}\,\partial_c\,\Omega \\
& = & \dfrac{2}{\Omega} g_{ab}\,\partial_c\,\Omega \\
& = & 2g_{ab}\,\partial_c \ln\Omega.
\end{array}$$

Likewise $g^{ab} = \Omega^{-2}\eta^{ab}$ and $\partial_c\, g^{ab} = -2g^{ab}\,\partial_c\ln\Omega$. Since the metric is diagonal we have $g_{11}g^{11} = g_{22}g^{22} = 1$. Moreover, $g^{22}g_{11} = -1$. We compute the Christoffel symbols using these properties to get

$$\begin{array}{lclclcl}
2\Gamma^1{}_{11} & = & g^{1r}(2\,\partial_1\, g_{1r} - \partial_r\, g_{11}) & = & g^{11}\,\partial_1\, g_{11} & = & 2\,\partial_1\ln\Omega \\
2\Gamma^1{}_{12} & = & g^{1r}(\partial_1\, g_{2r} + \partial_2\, g_{1r} - \partial_r\, g_{12}) & = & g^{11}\,\partial_2\, g_{11} & = & 2\,\partial_2\ln\Omega \\
2\Gamma^1{}_{22} & = & g^{1r}(2\,\partial_2\, g_{2r} - \partial_r\, g_{22}) & = & -g^{11}\,\partial_1\, g_{22} & = & 2\,\partial_1\ln\Omega \\
2\Gamma^2{}_{11} & = & g^{2r}(2\,\partial_1\, g_{1r} - \partial_r\, g_{11}) & = & -g^{22}\,\partial_2\, g_{11} & = & 2\,\partial_2\ln\Omega \\
2\Gamma^2{}_{12} & = & g^{2r}(\partial_1\, g_{2r} + \partial_2\, g_{1r} - \partial_r\, g_{12}) & = & g^{22}\,\partial_1\, g_{22} & = & 2\,\partial_1\ln\Omega \\
2\Gamma^2{}_{22} & = & g^{2r}(2\,\partial_2\, g_{2r} - \partial_r\, g_{22}) & = & g^{22}\,\partial_2\, g_{22} & = & 2\,\partial_2\ln\Omega.
\end{array}$$

Summarizing, we have

$$\begin{array}{rcl}
\Gamma^1{}_{11} = \Gamma^1{}_{22} = \Gamma^2{}_{12} & = & \partial_1\ln\Omega \\
\Gamma^1{}_{12} = \Gamma^2{}_{11} = \Gamma^2{}_{22} & = & \partial_2\ln\Omega.
\end{array}$$

The first term in $R_{121}{}^2$ becomes

$$\partial_2\,\Gamma^2{}_{11} = (\partial_2)^2\ln\Omega.$$

The second term becomes

$$-\partial_1 \Gamma^2{}_{21} = -(\partial_1)^2 \ln\Omega.$$

The third term becomes

$$\Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^2{}_{11}\Gamma^2{}_{22} = (\partial_1 \ln\Omega)^2 + (\partial_2 \ln\Omega)^2.$$

The final term is

$$-\Gamma^1{}_{21}\Gamma^2{}_{11} - \Gamma^2{}_{21}\Gamma^2{}_{21} = -(\partial_2 \ln\Omega)^2 - (\partial_1 \ln\Omega)^2,$$

which will cancel the third term. All together, we get

$$\begin{aligned} R_{121}{}^2 &= \left[-(\partial_1)^2 + (\partial_2)^2\right] \ln\Omega \\ &= \eta^{\mu\nu}\,\partial_\mu\,\partial_\nu \ln\Omega \\ &= \Box^2 \ln\Omega. \end{aligned}$$

Finally, lowering the index with $g_{22} = \Omega^2$ gives

$$R_{1212} = \Omega^2\Box^2 \ln\Omega.$$

b) Next we calculate using the tetrad method. The only non-trivial connection one-forms are

$$\begin{aligned} -\omega_{121} &= \omega_{112} \\ -\omega_{221} &= \omega_{212}. \end{aligned}$$

We can construct an orthonormal basis by

$$\begin{aligned} e_1 &= \frac{1}{\Omega}\frac{\partial}{\partial t} \\ e_2 &= \frac{1}{\Omega}\frac{\partial}{\partial x}, \end{aligned}$$

since $g(e_i, e_j) = \Omega^2\eta_{ab}(e_i)^a(e_j)^b = \Omega^2\eta_{ab}(\Omega^{-1}\delta^a_i)(\Omega^{-1}\delta^b_j) = \eta_{ij}$. The components in the coordinate basis are

$$\begin{aligned} (e_1)^i &= \Omega^{-1}(1,0) \\ (e_2)^i &= \Omega^{-1}(0,1) \\ (e_1)_i &= \Omega(-1,0) \\ (e_2)_i &= \Omega(0,1) \end{aligned}$$

This simplifies equation (3.4.21) for Riemann to

$$R_{(1212)} = \nabla_{e_1}\omega_{212} - \nabla_{e_2}\omega_{112} + \omega^2_{112} - \omega^2_{212},$$

where the ω's here are $\omega_{\sigma\mu\nu} = (e_\sigma)^a\omega_{a\mu\nu}$, and these latter components are determined by equation (3.4.25)

$$\partial_a(e_\sigma)_b - \partial_b(e_\sigma)_a = \eta^{\mu\nu}((e_\mu)_a\omega_{b\sigma\nu} - (e_\mu)_b\omega_{a\sigma\nu}).$$

Due to the anti-symmetry we need only calculate for $a, b = 1, 2$ and $\sigma = 1, 2$:

$$\begin{aligned} \partial_1(e_1)_2 - \partial_2(e_1)_1 &= \eta^{\mu\nu}((e_\mu)_1\omega_{21\nu} - (e_\mu)_2\omega_{11\nu}) \\ \partial_2\,\Omega &= -\Omega\omega_{112} \\ \omega_{112} &= -\partial_2 \ln\Omega, \end{aligned}$$

and

$$\begin{aligned} \partial_1(e_2)_2 - \partial_2(e_2)_1 &= \eta^{\mu\nu}((e_\mu)_1\omega_{22\nu} - (e_\mu)_2\omega_{12\nu}) \\ \partial_1\,\Omega &= \Omega\omega_{221} \\ \omega_{212} &= -\partial_1 \ln\Omega. \end{aligned}$$

The ω's in Riemann are then related to the above by

$$\begin{aligned} \omega_{212} &= (e_2)^2\omega_{212} = -\Omega^{-1}\,\partial_1 \ln\Omega \\ \omega_{112} &= (e_1)^1\omega_{112} = -\Omega^{-1}\,\partial_2 \ln\Omega. \end{aligned}$$

Now we can calculate the derivative terms in Riemann:

$$\begin{aligned}
\nabla_{e_1}\omega_{212} &= (e_1)^a\,\partial_a(-\Omega^{-1}\,\partial_1\ln\Omega) \\
&= -\Omega^{-1}\,\partial_1(\Omega^{-1}\,\partial_1\ln\Omega) \\
&= -\Omega^{-1}\left(-\Omega^{-2}\,\partial_1\,\Omega\,\partial_1\ln\Omega + \Omega^{-1}(\partial_1)^2\ln\Omega\right) \\
&= \Omega^{-2}((\partial_1\ln\Omega)^2 - (\partial_1)^2\ln\Omega)
\end{aligned}$$

and similarly

$$\begin{aligned}
-\nabla_{e_2}\omega_{112} &= -(e_2)^a\,\partial_a(-\Omega^{-1}\,\partial_2\ln\Omega) \\
&= \Omega^{-1}\,\partial_2(\Omega^{-1}\,\partial_2\ln\Omega) \\
&= \Omega^{-1}\left(-\Omega^{-2}\,\partial_2\,\Omega\,\partial_2\ln\Omega + \Omega^{-1}(\partial_2)^2\ln\Omega\right) \\
&= -\Omega^{-2}((\partial_2\ln\Omega)^2 - (\partial_2)^2\ln\Omega).
\end{aligned}$$

The final two terms are

$$\begin{aligned}
\omega_{112}^2 &= \Omega^{-2}(\partial_2\ln\Omega)^2 \\
-\omega_{212}^2 &= -\Omega^{-2}(\partial_1\ln\Omega)^2.
\end{aligned}$$

These two terms will cancel the corresponding terms in the derivative terms, leaving

$$\begin{aligned}
R_{(1212)} &= \Omega^{-2}\left[-(\partial_1)^2 + (\partial_2)^2\right]\ln\Omega \\
&= \Omega^{-2}\Box\ln\Omega.
\end{aligned}$$

Finally, as a sanity check, the frame and coordinate versions are related through the tetrad by

$$R_{(abcd)} = R_{\mu\nu\sigma\rho}(e_a)^\mu(e_b)^\nu(e_c)^\sigma(e_d)^\rho,$$

or

$$\begin{aligned}
R_{(1212)} &= R_{1212}(e_1)^1(e_2)^2(e_1)^1(e_2)^2 \\
&= \Omega^2\Box\ln\Omega(\Omega^{-4}) \\
&= \Omega^{-2}\Box\ln\Omega.
\end{aligned}$$

8. *Using the antisymmetry of $\omega_{a\mu\nu}$ in μ and ν, equation (3.4.15), show that*

$$\omega_{\lambda\mu\nu} = 3\omega_{[\lambda\mu\nu]} - 2\omega_{[\mu\nu]\lambda}.$$

Use this formula together with equation (3.4.23) to solve for $\omega_{\lambda\mu\nu}$ in terms of commutators (or antisymmetrized derivatives) of the orthonormal basis vectors.

Due to the anti-symmetry in the last two indices, the full anti-symmetrization reduces to a double cyclic sum:

$$\begin{aligned}
3\omega_{[\lambda\mu\nu]} &= \frac{1}{2}\left(\omega_{\lambda\mu\nu} - \omega_{\lambda\nu\mu} + \omega_{\mu\nu\lambda} - \omega_{\mu\lambda\nu} + \omega_{\nu\lambda\mu} - \omega_{\nu\mu\lambda}\right) \\
&= \omega_{\lambda\mu\nu} + \omega_{\mu\nu\lambda} + \omega_{\nu\lambda\mu}.
\end{aligned}$$

The second term is

$$-2\omega_{[\mu\nu]\lambda} = -\omega_{\mu\nu\lambda} + \omega_{\nu\mu\lambda},$$

which cancels the last two terms, leaving the first, which is the identity. Let $\Sigma_{\sigma\mu\nu} \equiv (e_\sigma)_a[e_\mu, e_\nu]^a = 2\omega_{[\mu|\sigma|\nu]}$ from equation (3.4.23). By anti-symmetrizing over all indices we get

$$\frac{1}{2}\Sigma_{[\sigma\mu\nu]} = \omega_{[\sigma\mu\nu]}.$$

Now, we can write

$$\begin{aligned}
3\omega_{[\sigma\mu\nu]} &= \omega_{\sigma[\mu\nu]} + \omega_{[\sigma\mu]\nu} + \omega_{[\sigma|\mu|\nu]} \\
&= \omega_{\sigma\mu\nu} + \omega_{[\sigma\mu]\nu} + \omega_{[\sigma|\mu|\nu]}.
\end{aligned}$$

Comparing this against the equation from the first part of the problem we see that

$$-3\omega_{[\mu\sigma]\nu} = \omega_{\mu|\sigma|\nu]} = \frac{1}{2}\Sigma_{\sigma\mu\nu},$$

so that

$$\begin{aligned}
\omega_{\mu\sigma\nu} &= 3\omega_{[\mu\sigma\nu]} - 2\omega_{[\mu\sigma]\nu} \\
&= \frac{3}{2}\Sigma_{[\mu\sigma\nu]} - \frac{1}{3}\Sigma_{\mu\sigma\nu}.
\end{aligned}$$