

# Anisotropy of Inertia from the CMB Anisotropy

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## ABSTRACT

Mach's principle states that the local inertial properties of matter are determined by the global matter distribution in the universe. In 1958 Cocconi and Salpeter suggested that due to the quadrupolar asymmetry of matter in the local galaxy about the earth, inertia on earth would be slightly anisotropic, leading to unequal level splittings of nuclei in a magnetic field [1,2]. Hughes, *et al.*, Drever, and more recently Prestage, *et al.* found no such quadrupole splitting [3–5]. However, recent cosmological observations show an anisotropy in the Cosmic Microwave Background, indicating anisotropy of the matter at much greater distances. Since the inertial interaction acts as a power law of order unity, the effect of this matter would far outweigh the relatively local contribution from the galaxy [1,6]. Thus, the present article extends the work of Cocconi and Salpeter to higher multipoles leading to unequal level splittings that should be measurable by magnetic resonance experiments on nuclei of appropriate spin.

## 1. A Brief History of Inertia

The problem of inertia has been and continues to be just that - a problem. This problem must be faced in any attempt to formulate a system of dynamics since the inertia of a body describes its response to external forces as well as the motion of that body in the absence of forces (and, perhaps more importantly, its motion in the presence and in the absence of other bodies). Here the history of inertia in physical theory will be briefly considered with an emphasis on its relation to Mach's principle.

Mach's principle, loosely stated, is "matter there determines inertia here." Though imprecise, this statement embodies the essence of Mach's thought on the problem of inertia. We will examine this principle in detail later, but first we will search for a motivation for this idea.

As previously stated, inertia is fundamental to dynamics. To understand inertia we must understand the behavior of bodies in the presence and absence of applied forces. However, a practical problem is that the motion of an object is observable only in relation to other objects. This being the case, it seems inevitable that a proper description of inertia must be a relation among different objects. This observational phenomenon was perhaps the impetus of Mach's conviction regarding the origin of inertia<sup>1</sup> - that the inertial properties of matter arise from an interaction with all the matter in the universe. Moreover, this entails a fully relative dynamics where all motions are defined in a relative way<sup>2</sup> [6], which can be regarded as the complete symmetry of all motions [7], implications to which we will later return.

However, another significant problem is whether one is able, even in principle, to distinguish between forced and force-free motions. This problem is especially significant when viewed against Mach's idea of inertia's origin. If a body never escapes the inertia determining power of the universe, then finding a suitable example of force-free motion becomes difficult, if not impossible. This leads one to consider an equivalence between inertial and accelerated motion. This is a statement of Mach's principle of the nature of

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<sup>1</sup> Mach was an extreme positivist and even rejected the atomistic theory of matter on the basis of its unobservability. find citation

<sup>2</sup> Bishop Berkeley, contemporary critic of Newton, maintained that the motion of a single particle in space-time is inconsistent: "there cannot be any relation, if there are no terms to be related." [8].

inertia. Both aspects of Mach's principle (the relation of inertia to other physical objects and the equivalence of inertial and accelerated motion) are seen in his own words:

[The] investigator must feel the need of ... knowledge of the immediate connections, say, of the masses of the universe. There will hover before him as an ideal an insight into the principles of the whole matter, from which accelerated and inertial motions result in the same way [9].

Both aspects of Mach's principle had a profound influence on Einstein's thinking and will become apparent in the principle of equivalence and general covariance in General Relativity [7]. With Mach's Principle introduced, we will here turn to the historical development of inertia and its relation to Mach's principle.

### 1.1. Aristotelian Dynamics

The struggle to understand inertia can be seen even in the Greek philosophers. Aristotle's dynamics was built on the first law:<sup>3</sup>

A body experiencing no force retains its state of absolute rest<sup>4</sup>.

This law was evident to Aristotle who observed that objects came to rest on the earth unless made to move by applying a force (lacking the concept of friction). Consistent with this description of force-free motion is Aristotle's second law which can be written as

$$f = mv \tag{1.1}$$

that is, the velocity of an object is proportional to the force acting upon it, the proportionality defined as inertial mass.

Now, this system implies an absolute meaning for location as defined by objects at rest. By these objects we can absolutely distinguish 'here' from 'there'. This is the truest example of absolute space. So, inertial mass arises when attempting to move an object away from an absolute location. Moreover, for an observer moving with respect to absolute rest with velocity  $u$  the second law becomes

$$f + mu = mv' \tag{1.2}$$

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<sup>3</sup> Never mind that this statement is incorrect.

<sup>4</sup> For a fuller analysis of Aristotelian mechanics and its relation to space-time structure, see reference [7].

where  $v = v' - u$ . If we desire to describe dynamics in this frame we must introduce an 'inertial force'  $mu$  as in Newtonian dynamics. The immediate origin of this inertial force is the absolute motion of the observer with respect to the absolute space. But, what is actually observed is motion with respect to other bodies, so the inertial forces should arise from this relative motion of bodies. Mach's principle can then be satisfied only if the privileged frame is determined by matter. So how do we identify this privileged frame?

In practice the earth would be used as a standard of rest to distinguish between objects truly at absolute rest and objects appearing to be at rest because  $f + mu = 0$ . But why should the earth represent such a standard? In Aristotelian 'gravity' earthly objects fall radially toward the earth which is fixed at the universe's center, and once they stick on the earth they are at absolute rest. But then, by hypothesis, the earth is in no way moved by infalling matter. It is therefore seen that the matter constituting the earth merely responds to the geometry of space-time and does not generate it. Therefore Aristotelian dynamics does not satisfy Mach's Principle [7].

## 1.2. Newtonian Dynamics

Here we will consider the far more familiar (and far more important) Newtonian system of dynamics. One of the principle conceptual advances was the identification of frictional forces that bring moving objects to rest<sup>5</sup>. Therefore, bodies experiencing no forces are not necessarily at rest. The laws of Newtonian dynamics are:

1. In the absence of applied forces a body moves with constant velocity (could be zero).
2. The acceleration of a body is proportional to the applied force ( $F = ma$ ).
3. Action and reaction are equal and opposite.

However, the first law is not so much a general law of dynamics, but the definition of an 'inertial frame,' for there is nothing here preventing one from observing such a free body from frames of reference in arbitrary motions relative to that body. Clearly the body cannot move with constant velocity in all such frames. One either has to supplement this law with another axiom stating the existence of a standard of rest relative to which the law holds [6] or take the law itself as the definition of such a standard.<sup>6</sup>

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<sup>5</sup> In rolling friction, though relative translation motion of the wheel is maintained, the point of contact is brought to instantaneous rest with the ground.

<sup>6</sup> This same comment applies to the first Aristotelian law as well.

This being the case, the laws do not hold in general, but only in these inertial frames defined through the first law. Supposing that one such frame can be found, any frame moving relative to that frame with constant velocity (related by a Galilean transformation) is also inertial, the acceleration being preserved by the transformation<sup>7</sup>.

However, to determine such a standard in practice we again must face the dilemma of having to establish whether bodies are experiencing forces. However, as Einstein pointed out, [10] this involves an argument in a circle: a body is unaccelerated if it experiences no force; we know that it experiences no force only by the fact that it remains unaccelerated. In other words, there exists no standard by which to judge which objects are under no forces independently of the laws themselves, and therefore no independent standard of finding the inertial frames. Einstein in fact used this dilemma to justify abandoning the preference of inertial frames of reference over all others [10].

Now, despite that Newton's laws are only valid in inertial frames, in practice it is often convenient to use non-inertial frames for doing physics, for example on the surface of the rotating earth. Suppose we have an inertial system  $S$  and a non-inertial system  $S'$  with origins displaced at time  $t$  by  $X^{i8}$  with respect to  $S$ . If a particle  $P$  at time  $t$  has coordinates  $x^i$  in  $S$ , then the coordinates in  $S'$  are  $x'^i = x^i - X^i$ . The velocity and acceleration in  $S'$  are given by  $v'^i = v^i - V^i$  and  $a'^i = a^i - A^i$  respectively, where  $v'^i = dx'^i/dt$ ,  $v^i = dx^i/dt$ ,  $V^i = dX^i/dt$ , etc...

If  $A^i = 0$  then  $a^i = a'^i$  and therefore  $F^i = ma^i = ma'^i$  and Newton's laws hold - the new frame is inertial. Otherwise we have

$$F^i = mA^i + ma'^i \quad (1.3)$$

in the new frame, and Newton's laws no longer apply. However, we can rewrite the equation in the form:

$$F'^i = ma'^i \quad (1.4)$$

where we have defined  $F'^i = F^i - mA^i$ . Though a trivial mathematical rearrangement, the physics is profound. For now Newton's laws hold, provided that we regard  $-mA^i$  as

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<sup>7</sup> This is probably a good place to note in passing that Newton's dynamics does not feature an absolute space, at least not in the Aristotelian sense where the same point can be defined at different times. It is conventional, however, to speak of it as absolute, meaning its properties are fixed and remain completely unchanged by the matter placed in it.

<sup>8</sup> latin indices run from 1 to 3

one of the forces acting on the body. For example, if the earth rotates around its axis with time-independent angular velocity<sup>9</sup>  $\omega^i$  then Newton's Law in the rotated frame becomes

$$F^i - 2m(\omega \times v')^i - m(\omega \times (\omega \times x'))^i = ma'^i \quad (1.5)$$

where the two terms new terms on the left are called the Coriolis and centrifugal forces, respectively.

In classical mechanics, any such term that arises in an accelerated reference frame is generally called a fictitious or inertial force. As the name implies, these forces were not regarded as real forces arising from some actual interaction, but as a sort of by-product of the reference system. However, for an observer in the reference system, these forces certainly appear real. For example, consider a geosynchronous satellite above the earth's equator. To an observer on the equator the satellite seems to defy gravity as it retains its position directly overhead at height  $r$  without visible support. But in this rotating frame of the earth there is a centrifugal force of magnitude  $m\omega^2 r$  exactly canceling the gravitational force to keep the satellite stationary.

If we are content restricting the validity of the laws of physics to inertial frames then everything is fine. However, this should make physicists uneasy for two reasons. First, it is quite obvious that physics happens in noninertial frames. The observer of the satellite at the equator is certainly viewing physics. To say that the laws of physics do not hold for this particular observer seems rather strange. Reality is not less reality for one observer than another, regardless of how these observers are related.

Second, we recall Einstein's objection to the preferred role of inertial frames. It is quite disturbing that one cannot, in a non-circular way, conclude that he is in fact in an inertial frame so that the laws of physics hold! Moreover, the earth itself is undergoing circular motion about its axis and about the sun (and the sun about the Milky Way, etc ...) so that the laws of physics truly hold nowhere on earth, since it is not an inertial frame of reference! This extreme restriction of the validity of laws of physics is therefore seen to be quite unsatisfactory.

However, to restore validity we need only regard those fictitious forces and being real. So, for observers on the equator there exists a real centrifugal force holding that satellite

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<sup>9</sup> a time-dependent angular velocity would introduce another term  $-m(d\omega/dt \times x')^i$ , which we here ignore because its effect is less familiar and its inclusion is unnecessary for the main argument here.

overhead. What then is the cause of this force? Similarly to the Aristotelian case, the immediate cause of this force is the rotation relative to an inertial frame. Therefore it appears that Newtonian space-time can exert forces on matter. However, this contradicts Newton's third law since the matter in no way exerts a force back on space-time [6,10].

The significance of this last point cannot be missed. Mach would have objected to unobservable space having any influence on physics based on his positivism, but any physical object that can act without being acted upon is more generally objectionable. Mach therefore postulated that inertial forces should be caused by a direct interaction between physical bodies. In this way mechanics would be causally closed.

But, Mach's hypothesis has a problem of its own to face. Despite the conceptual difficulties of Newtonian mechanics, it is extremely accurate in describing classical physical phenomenon. Therefore, Mach's principle must contend with the *apparent irrelevance* of the distribution of matter in accounting for inertial properties. Newton himself argued in various forms for the absoluteness of acceleration, the most famous of which is the spinning water bucket experiment, which Newton performed himself. Here, a bucket filled with water is hung from a long cord which is then twisted up tightly. The twisted cord is let go so that the bucket begins to rotate. The bucket and water are soon in a state of great relative motion, but with no observable effects - the water surface remaining flat. However, after some time, through friction, the rotation of the bucket is transferred to the water which then begins rotation with the bucket. At this point the surface begins to curve. Newton therefore concluded that it was the absolute rotation of the water with respect to absolute space that caused the recession from the sides, not the relative motion with respect to the bucket.

Mach's response was twofold. First, one may only conclude that the motion of water relative to the 'fixed stars' was responsible for the recession from the sides. That is, in the absence of rotation, the distant stars appear fixed in the sky. If an observer begins to rotate relative to the earth, the stars appear to rotate relative to the observer. Mach argued that there should be a symmetry between fixed observer with rotating universe and fixed universe with rotating observer. It is then this relative rotation which is responsible for the inertial forces acting on the water in the bucket, for example.

In addition, Mach pointed out that the relatively spinning bucket which caused no apparent change in the water surface was of extremely small mass, and therefore should have little effect on the water anyway. It's not just relative motion, but relative motion of masses that is important. Should the bucket have been "many leagues thick" and the experiment reformed, the great mass of the bucket may very well have produced observable effects in the water surface [9]. Put another way, for Mach the bucket was the rest of the universe.

Therefore the apparent irrelevance of matter of the universe is seen in the fact that the rest of the matter is apparently at rest<sup>10</sup> with respect to the earth. This would explain, if inertia is the result of an interaction, why the earth behaves like an inertial frame to a good approximation and why Newtonian physics was so successful.

### 1.3. General Relativity

Mach's convictions that matter in the universe should cause inertia and that inertia and acceleration should have a unified cause greatly impressed Einstein in his development of general relativity [11]. Einstein loosened the rigidity of space-time so that, not only would it affect matter, but it could be affected by matter in return; that is, space-time could bend. Moreover, the symmetry of motion appeared in the principle of general covariance - that the laws of physics should be the same for all observers. Einstein himself noted that this principle is devoid of physical content itself, and that any theory could be made generally covariant<sup>11</sup> [12]. What was significant was Einstein's insistence that a theory be formulated in a generally covariant way, reflecting the conviction that no frame of reference is less valid than another (this does not mean that some frames may be less convenient than others).

The equations of motion for a particle in general relativity [13] are

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad (1.6)$$

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<sup>10</sup> That the distant stars are accelerating away from us is no consequence, since the effect of such an acceleration would average to zero over the whole sky. The important point here is no net acceleration in a particular direction or relative rotation.

<sup>11</sup> In fact, generally covariant formulations of Newtonian gravitation appeared starting in 1923 with the work of E. Cartan. These theories retain an absolute time, but space does curve in response to matter. Note, however, that these formulations are post general relativity [12].



where greek indices range from 0 to 3. The 40 components of the affine connection  $\Gamma$ , which represent the gravitational-inertial field, can be found from the metric  $g$  (which therefore serves as a tensorial gravitational potential and implies this is a metric theory of gravity) as

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\sigma\lambda} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}} \right\} \quad (1.7)$$

where  $g$  has signature  $(-, +, +, +)$  and is found from the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu} + \Lambda g_{\mu\nu} \quad (1.8)$$

where  $G$  is Newton's gravitational constant,  $R_{\mu\nu}$  is the Ricci tensor, which describes the curvature of space-time,  $R = R_{\mu}^{\mu} = R_{\mu\nu}g^{\mu\nu}$  is the curvature scalar,  $\Lambda$  the cosmological constant, and  $T_{\mu\nu}$  is the stress-energy tensor of matter.

Conceptually, the field equations of general relativity are derived by applying general covariance to the strong principle of equivalence [13]. The strong principle of equivalence states that at any space-time point in any gravitational field it is possible to choose coordinates such that in a sufficiently small neighborhood of that point *all* the laws of physics, including gravitation, take their special relativistic form.

Here, mass and energy act as a source of the curvature of space-time which then determines particle trajectories. In this way Einstein hoped to have embodied Mach's principle. To consider explicitly the effect of the surrounding matter distribution, Einstein derived the following weak-field equations of motion [10]

$$\frac{d}{dt} [(1 - \Phi) v^i] = -(\nabla\Phi)^i - \frac{\partial A^i}{\partial t} + (v \times (\nabla \times A))^i \quad (1.9)$$

where  $\Phi$  and  $A^i$  are effective scalar and vector potentials of the gravitational field in this limit, defined as

$$\Phi = -G \int \frac{[\rho]}{r} dV \quad A^i = -4G \int \frac{[\rho u^i]}{r} dV \quad (1.10)$$

where the brackets indicate integrals over the retarded matter and matter current densities in a volume  $dV$  at distance  $r$ . Note the resemblance to the scalar and vector potentials of electromagnetism defined as integrals over charge density and current (hence the symbols  $\Phi$  and  $A^i$ ). From this equation the following phenomenon consistent with Mach's Principle are evident:

1. Inertial mass is proportional to  $1 - \Phi$
2. The local inertial frame is accelerated by
  - a) local mass concentrations ( $-\nabla\Phi$ )
  - b) an induction via linearly accelerated masses ( $-\partial A/\partial t$ )
  - c) and an induction via rotating matter ( $v \times (\nabla \times A)$ )

The effect of the rotational induction was taken up by Lens and Thirring [15] who showed that inertial frames inside of a uniformly rotating shell of matter are dragged around in the sense of the rotating matter (though at a smaller angular velocity). It has also been shown that the angular velocity of the frame-dragging approaches that of the rotating matter shell in the limit that shell encompasses the whole universe [16,17]. Here is Mach's version of Newton's bucket realized in general relativity.

However, Brans [14] demonstrated that the first of these effects is merely an artifact of the choice of coordinates. Inertial mass, in general relativity, is an intrinsic, local, and invariant property of bodies [7]. This is not surprising given the strong principle of equivalence which leads to Einstein's equations [13]. General relativity looks locally like special relativity, where particles masses are certainly intrinsic properties, and matter has definite inertial properties [7].

This last remark raises the next issue in general relativity. Solutions to Einstein's equations can be found which completely violate Mach's principle, the flat space-time of special relativity being such an example, since a single test particle would have inertial properties even though space would be otherwise empty [18] and the curvature tensor  $R^\lambda_{\mu\nu\kappa}$  everywhere zero [13]. This feels like Newton's absolute space all over again [19]. Moreover, this problem appears in any space-time that is asymptotically flat [20].

Aware of this difficulty, Einstein proposed that boundary conditions would have to be properly chosen to ensure a satisfactory solution to his field equations, so that Mach's principle would be implemented, in essence, as a boundary condition [7]. Or, put another way, the entire space-time should be created by matter; no source-free component of the metric would be allowed. This statement is elusive since general relativity is a non-linear theory, and thus without a superposition principle. Thus, how does one determine which contributions are source-free?

This problem was solved through the work of Lynden-Bell [8], Al'tschuler [21], Hoyle and Narlikar [22], and culminated in the work of Sciama *et al.* [23]. In this formulation the contribution to the metric at any point is built up by integrating contributions from all sources along the past light cone by means of a Green's function. Superposition is restored by imagining the effects of sources to be propagated along the actual space-time that results. The general equation, including a cosmological constant, is given by Gilman [20] as

$$g^{\alpha\beta}(x) = 2\kappa \int_{\Omega} \left\{ G_{\mu}^{-\alpha\beta\nu}(x, x') \left[ T_{\nu}^{\mu}(x') - \frac{1}{2} T_{\lambda}^{\lambda}(x') \delta_{\nu}^{\mu} - \frac{\Lambda}{\kappa} \delta_{\mu}^{\nu} \right] [-g(x')]^{1/2} d^4x \right\} + \int_{\partial\Omega} G_{\mu}^{-\alpha\beta\nu;\sigma}(x, x') [-g(x')]^{1/2} dS_{\sigma} \quad (1.11)$$

where the contributions to the metric naturally split between those from inside the volume  $\Omega$  and those on and beyond the surface  $\partial\Omega$  bounding the volume.  $G_{\mu}^{-\alpha\beta\nu}$  is the retarded Green's function, a semicolon (;) denotes covariant differentiation, and  $\kappa = 8\pi G$ . Mach's principle is then implemented by demanding that in the limit that the volume integral fills all past space-time the surface contribution should tend to zero [23]. Moreover, Gilman [20] notes that  $\Lambda$  must be zero as well, regarding it as a non-Machian source term. The author refrains from choosing a side over  $\Lambda$  since exactly what it is remains unknown.

Thus, the metric is determined completely by contributions from mass-energy throughout the universe. According to this criterion, Gilman [20] has shown that probably all nonempty Robertson-Walker solutions are Machian, while all Schwarzschild, vacuum, and  $\Lambda \neq 0$  solutions are non-Machian.

Therefore, if the strong equivalence principle holds, then it appears that Mach's principle should be implemented in the above fashion as a selection criterion for suitable solutions of Einstein's field equations.

#### 1.4. Beyond General Relativity

In Einstein's theory, matter determines the inertial frames. However, once in an inertial frame, the surrounding mass distribution ceases to have any influence, gravitationally or otherwise<sup>12</sup> [13]. From this point of view, because of the strong equivalence principle, it appears that Einstein's theory is quite contrary to Mach's philosophy. It is therefore reasonable to search for violations of, or formulate theories that violate, the strong equivalence principle. For example, Einstein's theory handles the origin of inertial forces, but not of inertial mass. It is natural to ask whether inertial mass has a gravitational origin.

The Brans-Dicke theory [24], which is motivated by Mach's principle, regards inertial masses, not as fundamental constants, but as the results of an interaction with some cosmic field. They assume the existence of a scalar field,  $\varphi$ , in addition to the metric field for this purpose.

However, this field alone is not sufficient to implement Mach's principle beyond general relativity. In the absence of matter  $\varphi \rightarrow 0$ , and the same vacuum solutions of general relativity persist [7], so one still needs to impose Machian boundary conditions. However, particle masses are seen to increase (in an isotropic fashion) when other masses are around through interaction with  $\varphi$ . This is a promising implication. However, the coupling to the  $\varphi$  field in their Lagrangian is through a dimensionless parameter  $\omega$ , which is supposed to be of order unity. Solar-system tests of gravity imply a lower limit on  $\omega$  of 600. In this regard, the scalar-tensor theory seems unlikely [26]. It is interesting to note, in passing, that in the  $\omega \rightarrow \infty$  limit, General Relativity is recovered [13].

Another possibility was raised by Cocconi and Salpeter [1,2] who argued that if inertial mass arises from an interaction with matter then if the surrounding mass distribution is anisotropic then inertial mass should mirror that anisotropy. At the time, the galactic center was largest known source of mass anisotropy. So they suggested a quadrupolar anisotropy in inertial mass should be detectable from studying variations in Zeeman transition lines when the applied magnetic field is parallel to or perpendicular to the galactic center.

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<sup>12</sup> Note that this applies only locally. Gravitational fields could otherwise be detected through tidal effects over sufficiently large regions, but here the equivalence principle ceases to apply.

Hughes [3] and Drever [4] performed magnetic resonance experiments on  $\text{Li}^7$  nuclei in the ground state, since this offered a greater resolution. The ground state nuclear spin is  $I = 3/2$ . In a strong applied magnetic field there are  $2I + 1 = 4$  magnetic sublevels,  $M_I$ , which should be equally spaced, and a single transition line should be seen. An anisotropic inertial mass will alter each energy level differently, resulting in three distinct transition lines. They found a line width of no greater than  $5.3 \times 10^{-15}$  eV. Considering the nuclear structure of  $\text{Li}^7$  to be a single  $P_{3/2}$  proton in a central nuclear potential, they relate the change in line width to a change in kinetic energy of the proton:

$$\Delta \left( \frac{p^2}{2m} \right) \simeq \frac{\Delta m}{m} \left( \frac{p^2}{2m} \right) \leq 5.3 \times 10^{-15} \text{ eV} \quad (1.12)$$

Since the proton kinetic energy  $p^2/2m$  is greater than  $\frac{1}{2}$  MeV, they set an upper limit on the anisotropy as

$$\frac{\Delta m}{m} \lesssim 10^{-20} \quad (1.13)$$

On the basis of this analysis, an anisotropic mass seems rather unlikely. However, one explanation for this null experiment was offered by Dicke [25], noting that if there is an anisotropic coupling to all forms of mass-energy, then the anisotropy should be unobservable, at least locally. The argument is based on Cocconi and Salpeter's original proposal that inertial mass could be a second-order tensor. In fact, they began from the velocity-momentum relationship  $p_i = m_{ij}u^j$ . Dicke expressed the mass as  $m_{ij} = mf_{ij}$ , writing it as an interaction with an universal tensor field. He then derived equations of motion that employed  $f_{ij}$  as metric tensor instead of the usual  $g_{ij}$ . Since  $f_{ij}$  can always be transformed to a locally Minkowskian form, any isotropy can be transformed away. Thus, if all mass-energy couples to this field in the same way, the anisotropy is invisible. In this way, Dicke asserts that this result actually upholds Mach's principle since the anisotropy is universal. However, if the anisotropic field cannot be expressed as a second-order tensor, or if the field couples differently to different forms of mass-energy, the argument breaks down.

Though Dicke's suggestion is quite compelling (not to mention beautiful in its simplicity), there is another possibility to consider. Mach's principle was motivated by the fact that it is the distant stars that appear to determine the inertial frames, not the close

stars; the matter further away, as a whole, has more inertia determining power than matter close by. Since the amount of matter on a shell of radius  $R$  goes as  $R^2$ , the inertial effect of matter must drop off less steeply than  $R^{-2}$  for the further matter to have more influence. Moreover, since the inertial reaction is an acceleration dependent interaction, arguing from electromagnetic radiation theory we expect the influence to fall off as  $1/R$  [6]. The galactic center is quite close compared to the known vastness of space. Not only that, though there is extensive clumping, there is an overall trend to uniformity at the greatest distances [26]. So, the matter that determines inertia most is least anisotropic. It is therefore not surprising for inertia to appear isotropic in this anisotropic universe.

However, though the matter at greater distances is more uniform, it is not entirely uniform. Recent measurements of the cosmic microwave background (CMB) have revealed a definite angular power spectrum [26], indicating a matter anisotropy in the region this radiation came from. We will look at this again later, but for the moment we anticipate that angular momentum states will only couple to certain multipoles in the distant anisotropy. Since the observed quadrupole in the CMB is quite small, it is desirable to investigate the effect of higher multipoles on nuclei of greater spin. We investigate this in the next section.

## 2. Physical Theory

### 2.1. Lagrangian Formulation

To begin we write the Lagrangian for a particle in gravitational and electromagnetic fields as

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}u^\mu u^\nu} + eA_\mu u^\mu \quad (2.1)$$

where the mass  $m$  is assumed to depend on angle (and therefore position),  $g_{\mu\nu}$  is the metric tensor,  $e$  is the absolute value of the electron charge,  $A_\mu$  is the electromagnetic 4-vector potential,  $c$  is the speed of light, and  $u^\nu = dx^\nu/d\tau$ , where  $d\tau$  is the proper time:  $c^2 d\tau^2 = -g_{\mu\nu}x^\mu x^\nu$ . It is assumed here that  $g_{\mu\nu}$  is the standard metric of general relativity, and is used to raise and lower all indices. Therefore the anisotropy will be inserted through  $m$  alone.

The conjugate momenta are

$$p_\mu = \frac{\partial \mathcal{L}}{\partial u^\mu} = \frac{mcg_{\mu\nu}u^\nu}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} + eA_\mu \quad (2.2)$$

and are anisotropic from the  $m$  dependence, but there is no anisotropy in the electromagnetic part. According to the action principle  $0 = \int L d\tau$ , the Euler-Lagrange equations  $\frac{d}{d\tau} \left( \frac{\partial L}{\partial u^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0$  become

$$\frac{d}{d\tau} \left( \frac{mcg_{\mu\nu}u^\nu}{\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} + eA_\mu \right) - \left( \frac{mc \frac{\partial g_{\alpha\beta}}{\partial x^\mu} u^\alpha u^\beta}{2\sqrt{-g_{\alpha\beta}u^\alpha u^\beta}} - \frac{\partial mc}{\partial x^\mu} \sqrt{-g_{\alpha\beta}u^\alpha u^\beta} + e \frac{\partial A_\nu}{\partial x^\mu} u^\nu \right) = 0 \quad (2.3)$$

We can simplify this expression by performing a coordinate transformation so that  $g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  and its first derivatives vanish at the atom:  $\frac{\partial g_{\lambda\nu}}{\partial x^\mu} = 0$ . We also note that  $\sqrt{-g_{\mu\nu}u^\mu u^\nu} = c$  which yields

$$\frac{d}{d\tau} (m\eta_{\mu\nu}u^\nu + eA_\mu) + \frac{\partial m}{\partial x^\mu} c^2 - e \frac{\partial A_\nu}{\partial x^\mu} u^\nu = 0. \quad (2.4)$$

Noting that

$$\frac{dA_\mu}{d\tau} = \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} = \frac{\partial A_\mu}{\partial x^\nu} u^\nu \quad (2.5)$$

and rearranging, we obtain

$$\frac{d}{d\tau} (mu_\mu) = eF_{\nu\mu}u^\nu - \frac{\partial m}{\partial x^\mu} c^2 \quad (2.6)$$

where  $F_{\nu\mu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$  is the electromagnetic field tensor. Even in the absence of applied forces (here electromagnetic) the change in mechanical momentum (and energy) is not zero in this ‘‘inertial’’ frame (it is inertial in the sense that  $g_{\mu\nu} = \eta_{\mu\nu}$ ). In accordance with Mach’s principle we attribute this force to an interaction with the surrounding mass distribution.

To estimate this force, let us assume an inertial potential built up from contributions of masses  $m_i$  at distance  $r_i$  throughout the universe:  $\Phi = \sum \frac{m_i}{r_i}$ . If a test particle changes position by  $\Delta r$  then  $r_i \rightarrow |\vec{r}_i - \Delta \vec{r}|$ . Performing a Taylor series to first order we have

$$\frac{1}{|\vec{r}_i - \Delta \vec{r}|} \approx \frac{1}{r_i} + \frac{\vec{r}_i \cdot \Delta \vec{r}}{r_i^3} \leq \frac{1}{r_i} + \frac{\Delta r}{r_i^2} \quad (2.7)$$

and

$$\Phi \rightarrow \Phi' \approx \sum \frac{m_i}{r_i} \left( 1 + \frac{\Delta r}{r_i} \right) \quad (2.8)$$

Therefore we have

$$\frac{\Delta\Phi}{\Phi} \approx \frac{\Delta r}{r_i} \quad (2.9)$$

Since the matter at the greatest distances is most significant we can replace  $r_i$  with the Hubble radius  $c/H_0$  yielding

$$\frac{\Delta\Phi}{\Phi} \approx \frac{\Delta r}{4222\text{Mpc}} \quad (2.10)$$

which for laboratory dimensions (10m) is of the order  $10^{-27}$  and completely negligible for particle dynamics. This is good, since a force of this type of appreciable strength would have been well known by now if it existed.

## 2.2. Hamiltonian and Quantum Test Theory

Previously we saw that the Machian force due to the position dependence of the mass was vanishingly small. Despite this, the direct anisotropy of the mass itself should be large enough to have observable effects on nuclear spectra. Thus we desire to write a Hamiltonian to describe its effect on nuclear states. To proceed we take the non-relativistic limit of the conjugate momenta in Eq. (2.2) which yields

$$p_i = mv_i + eA_i \quad (i = 1, 2, 3) \quad (2.11)$$

which is what one normally finds in an electromagnetic field, but here  $m$  is orientation dependent.

The Hamiltonian, after some rearrangement, is

$$\mathcal{H} = p_i u^i - \mathcal{L} = mc^2 + \frac{1}{2m} (\vec{p} - q\vec{A})^2 + V(\vec{r}) \quad (2.12)$$

where  $V(\vec{r})$  is a spherically symmetric nuclear potential that includes the Coulomb interaction. The quantum mechanical Hamiltonian  $H$  is found by replacing coordinates and momenta with the appropriate operators.

The actual Hamiltonian, including all internal atomic interactions, is more complicated, but we need only consider the following. With an isotropic mass and in the absence of an external field, the energy levels of the atom exhibit a fine structure due to internal magnetic interactions between electrons and nucleus, the spins,  $I$  of the nucleus and  $J$



of the electrons, being tightly coupled. However, in a large applied field<sup>13</sup> the nuclear and electron spins decouple, resulting in, for each value of  $m_J$ , a set of equispaced levels corresponding to the different values of  $m_I$  [27]. Therefore, we are interested in magnetic dipole transitions with  $\Delta m_J = 0$ ,  $\Delta m_I = \pm 1$ , which would yield a single spectral line since the levels are evenly spaced. Therefore, we need only calculate the effect of the anisotropic mass (taken as a perturbation) in an  $m_I$  state of the nucleus.

If the full quantum mechanical Hamiltonian is expanded out one obtains

$$H = mc^2 + \frac{P^2}{2m} + H_{hf} \quad (2.13)$$

where  $mc^2$  is the rest energy,  $\frac{P^2}{2m}$  is the kinetic energy of a proton<sup>14</sup>, and  $H_{hf}$  is the hyperfine Hamiltonian, which includes all magnetic interactions, including those with the external field. For a proton in the nucleus,  $mc^2 = 938 \text{ MeV} \gg P^2/2m \gg H_{hf}$  (in the sense of the relative sizes of the eigenvalues [28]) and is the dominant contribution to the Hamiltonian. Therefore we will consider only  $mc^2$  to be anisotropic and replace  $m$  everywhere else with  $m_0$ , the isotropic contribution to the mass<sup>15</sup>.

Now we explicitly represent the mass  $m$  as an expansion in the spherical harmonics:

$$m(\theta, \varphi) = \sum_{l,m} a_{lm} Y_m^l(\theta, \varphi) = a_{00} Y_0^0(\theta, \varphi) + \sum_{l>0,m} a_{lm} Y_m^l(\theta, \varphi) \quad (2.14)$$

where we identify the isotropic mass  $m_0 = a_{00} Y_0^0$ . Since the anisotropic contribution is assumed to be extremely small compared to the isotropic part we can treat it as a perturbation. We therefore write

$$mc^2 = m_0 c^2 \left( 1 + \sum c_{lm} Y_m^l \right) \quad (2.15)$$

where  $c_{lm} \equiv a_{lm}/m_0 \ll 1$ . From now on the sum will be understood to be over  $l > 0$  and over all  $m$ . The Hamiltonian now has the form

$$H = H_0 + W \quad (2.16)$$

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<sup>13</sup> Large here means that  $\mu_B H_0 \gg \mu_I \overline{H(0)}$  or  $H_0 \gg 10^{-3} \overline{H(0)}$ , where  $H_0$  is the applied field,  $\overline{H(0)}$  is the average field at the nucleus due to the electrons, and  $\mu_B$  and  $\mu_I$  are the magnetic moments of the electrons and nucleus respectively [27].

<sup>14</sup> This quantity is really the square of the sum of the true kinetic momentum and  $eA$ . For an applied lab field of 4700G,  $eA$  is of the order  $10^{-12} \text{ MeV}/c$  and negligible compared to the kinetic momentum.

<sup>15</sup> It should be noted that  $m$  also appears in  $H_{fs}$  as part of the magnetic moments,  $\mu_I = e\hbar I g_I / m_p$ , for example ( $m_p$  is the proton mass and  $g_I$  the nuclear g-factor).

where

$$H_0 = m_0 c^2 + \frac{P^2}{2m_0} + H_{hf} \quad (2.17)$$

and

$$W = m_0 c^2 \sum c_{lm} Y_m^l. \quad (2.18)$$

We therefore want to find, according to first order perturbation theory, the expectation value

$$\langle W \rangle = m_0 c^2 \left\langle \sum c_{lm} Y_m^l \right\rangle \quad (2.19)$$

taken in the unperturbed  $m_I$  states. A typical term in the sum is the expectation value

$$\langle I, m_I | Y_m^l | I, m_I \rangle \quad (2.20)$$

Since the spherical harmonics are tensor operators, we can use the Wigner-Eckart theorem to factor this expectation value into a part independent of the  $m$ 's and a 3-j symbol:

$$\langle I, m_I | Y_m^l | I, m_I \rangle = (-1)^{I-m_I} \langle I || Y^l || I \rangle \begin{pmatrix} I & l & I \\ -m_I & m & m_I \end{pmatrix}. \quad (2.21)$$

For  $I$  integral, the reduced matrix element can be expressed as [29]

$$\langle I || Y^l || I \rangle = (-1)^I (2I+1) \sqrt{\frac{2l+1}{4\pi}} \begin{pmatrix} I & l & I \\ 0 & 0 & 0 \end{pmatrix} \quad (2.22)$$

while for  $I$  half integral, it becomes [29]

$$\langle I || Y^l || I \rangle = (-1)^{I+\frac{1}{2}} (2I+1) \sqrt{\frac{2l+1}{4\pi}} \begin{pmatrix} I & l & I \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \frac{1}{2} [1 + (-1)^l]. \quad (2.23)$$

The 3-j symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

is different from zero only when the following are both met [29]

1.  $m_1 + m_2 + m_3 = 0$
2. the vectors  $\vec{j}_1 + \vec{j}_2 + \vec{j}_3 = 0$ .

The second condition implies that the following triangle inequalities are satisfied:

$$j_1 + j_2 \geq j_3 ; \quad j_1 + j_3 \geq j_2 ; \quad j_2 + j_3 \geq j_1 \quad (2.24)$$

and the sum  $j_1 + j_2 + j_3$  is an integer.

Applying this to the 3-j symbol in Eq. (2.21), the expectation value will be zero unless both  $m = 0$  and  $l \leq 2I$ . The 3-j symbol in the reduced matrix elements add nothing here, but the matrix element itself will be zero unless  $l$  is even, which can be seen in the final term in Eq. (2.23). Though less obvious, the 3-j in Eq. (2.22) is also zero unless  $l$  is even [29]. Thus the amount of coupling a particular nucleus can have to the matter anisotropy is extremely limited.

For example, the Hughes-Drever experiments used  $\text{Li}^7$  nuclei with a nuclear spin  $I = 3/2$ . The coupling is limited then to  $l \leq 2I = 3$ . Moreover, since  $l$  must be even, this experiment is therefore only sensitive to the quadrupole ( $l = 2$ ) anisotropy. In general, the dipole and all other odd-order multipoles are unobservable at first order in any nucleus of any spin, but all higher even-order multipoles are observable in nuclei of high enough spin; in particular, every integer increase in  $I$  allowed one further multipole to be coupled.

### 2.3. Hughes-Drever and $\text{Li}^7$

If we set  $I = 3/2$  and  $l = 2$  in Eq. (2.21), and use an algebraic form for the 3-j symbol [30] we can find the total energy shift in an  $m_I$  state to be (after some algebra)

$$\Delta E_{m_I} = \langle W \rangle = m_0 c^2 \langle c_{20} Y_0^2 \rangle = \frac{m_0 c^2 c_{20}}{8\sqrt{5}\pi} (5 - 4m_I^2). \quad (2.25)$$

Though the levels were evenly spaced in the absence of anisotropy, the presence of  $m_I^2$  splits the levels unequally. Thus the energy difference between adjacent levels depends on  $m_I$  and is given by

$$\Delta E_{m_I} - \Delta E_{m_I-1} = \frac{m_0 c^2 c_{20}}{2\sqrt{5}\pi} (1 - 2m_I). \quad (2.26)$$

The  $m_I = 1/2 \rightarrow -1/2$  transition remains unperturbed, while the  $3/2 \rightarrow 1/2$  and  $-1/2 \rightarrow -3/2$  transitions are shifted in opposite, but equal, amounts. Using this and bounding (the absolute value of) the energy difference in Eq. (2.26) by the largest shift unobserved by Hughes, we get an upper limit on  $c_{20}$  of

$$c_{20} \lesssim \frac{\sqrt{5}\pi}{m_0 c^2} (5.3 \times 10^{-15} \text{eV}) = 2.2 \times 10^{-23} \quad (2.27)$$

where we consider the total angular momentum of the nucleus to be carried by a single unpaired proton.

### 3. Anisotropy and the Cosmic Microwave Background (CMB)

Now we desire to correlate any possible inertial mass anisotropy with the CMB temperature fluctuations. With a limit obtained on  $c_{20}$  we desire to express corresponding limits for the higher  $c_{l0}$  values based on the assumption that they arise from an anisotropy of matter at large distances using the CMB data. Then experimental limits on energy shifts for future experiments on greater spin may be derived.

#### 3.1. Origin and Interpretation of the CMB

The observed cosmic microwave background is a relic of an event known as recombination when the temperature of the universe cooled sufficiently to allow the formation of neutral hydrogen [26]. Prior to this time the free electrons were highly coupled to radiation via Thompson-scattering so that the mean free path of photons was infinitesimal. However, once the electrons become captured by protons the coupling ceased and the photons were free to travel cosmic distances. These photons were thus initially emitted at a temperature  $\sim 3000\text{K}$ , but a nearly perfect blackbody spectrum of  $\sim 2.7\text{K}$  is observed. Though the radiation received was emitted over a shell in the sky of finite thickness, this is compensated for by the temperature decrease due to the cosmic expansion. Thus, the individual blackbody spectra from each layer in the shell superimpose exactly to form a single blackbody spectrum (since blackbody spectra of different temperatures are nested, the superposition of several spectra of differing temperatures would not yield a blackbody).

Now, modern cosmology is based on the assumption of large-scale homogeneity in the universe. If the universe were perfectly homogeneous, then the CMB would be likewise. However, the matter distribution around us exhibits clumping which is understood as Gaussian quantum fluctuations that were stretched into real density perturbations by the expansion of the universe. Such perturbations in the matter distribution lead to gravitational perturbations that alter the temperature of photons through redshift and time dilation. This is known as the Sachs-Wolfe effect and dominates the anisotropy at large scales<sup>16</sup>.

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<sup>16</sup> The next contribution to the anisotropy comes from Doppler shifting which does not contribute appreciably until  $l \gtrsim 30$  [26].

The temperature on the sky is naturally expanded in spherical harmonics, or more appropriately the temperature contrast  $\delta T/T_0 = (T(\theta, \varphi) - T_0)/T_0$ , where  $T_0$  is the average temperature:

$$\delta T/T_0(\theta, \varphi) = \sum t_{lm} Y_m^l(\theta, \varphi).$$

The redshift leads to a temperature fluctuation away from average at any point in the sky of  $\delta T/T_0 = \delta\Phi/c^2$ , where  $\Phi$  is the Newtonian gravitational potential. The time dilation effect is similar:  $\delta t/t = \delta\Phi/c^2$ , which expresses the difference between the actual and apparent times of emission of the radiation. Noting that temperature scales as  $1/a$ , where  $a$  is the scale factor of the universe, and that  $a$  depends on time as  $t^{2/3}$  yields  $\delta T/T_0 = -(2/3)\delta\Phi/c^2$ . Combing the two yields a net redshift of

$$\frac{\delta T}{T_0} = \frac{\delta\Phi}{3c^2} \quad (3.1)$$

Poisson's equation can be used to link the potential fluctuation to a matter fluctuation as

$$\nabla^2 \delta\Phi = 4\pi G \rho_0 \delta \quad (3.2)$$

where  $\delta \equiv \frac{\rho - \rho_0}{\rho_0}$  is the density contrast. Using the Sachs-Wolfe relation to replace  $\delta\Phi$  with  $\delta T$ , and with  $\delta T$  expanded in spherical harmonics  $\nabla^2 \rightarrow -l(l+1)/r_{LS}^2$ , where  $r_{LS}$  is the distance to the last scattering surface. If we also expand the density contrast in spherical harmonics as  $\delta(\theta, \varphi) = \sum \delta_{lm} Y_m^l(\theta, \varphi)$  and then equate the terms of the two expansions we get

$$\frac{-3c^2}{r_{LS}^2} l(l+1) t_{lm} Y_m^l = 4\pi G \rho_0 \delta_{lm} Y_m^l$$

or

$$\delta_{lm} = \frac{-3c^2 l(l+1)}{4\pi G \rho_0 r_{LS}^2} t_{lm}$$

#### 4. The Future

It is regrettable that this project remains uncompleted at this point. The framework has been laid above, but there are several steps left before completion. It has been shown that an inertial mass that depends arbitrarily on direction is *a priori* possible, the dependence presumably arising from an interaction with anisotropic matter at great distances, in the spirit of Mach's Principle. Even though the angular dependence is quite general, the possible couplings to matter in experiments of the Hughes-Drever type is severely restricted, their actual experiment probing only the quadrupolar anisotropy. Based on their experiment a limit was set on the quadrupolar term in the inertial mass anisotropy. It remains to compare expansion coefficients between the inertial mass and matter anisotropy at great distance through the WMAP data of the CMB. From this comparison, limits can be set on higher coefficients to see whether these multipoles are accessible to observation. Finally, it is desirable to propose a specific coupling scheme between local inertia and distant matter, possibly through  $1/R$  type interaction. In this manner it may be possible to predict absolute values of the local multipole coefficients, and therefore be more certain that the Hughes-Drever tests have actually ruled out such anisotropy or are yet too insensitive.

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