On the Relation between Real and Complex Jacobian Determinants.

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Abstract

When considering maps in several complex variables one may want to consider whether the maps are immersive, submersive, or locally diffeomorphic. These same questions are easily formulated in terms of functions of real variables using the Jacobian determinant. This article uses the natural correspondence between complex and real maps to extend the real result to the complex case, expressing this result entirely in terms of the complex functions (the complex Jacobian). To do this we employ a result of Sylvester on the determinant of block matrices.

1 Introduction

A complex analytic map $f : \mathbb{C}^n \to \mathbb{C}^m$ can be regarded as a real map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$. This is done by identifying each $\mathbb{C}$ with the plane $\mathbb{R}^2$. If we label the domain coordinates as $z_i = x_i + iy_i$ and range coordinates as $w_j = u_j + iv_j$ then

$$\begin{array}{rcl} w_j & = & f_j(z_i) \\ & \downarrow & \\ u_j & = & F_{2j-1}(x_i, y_i) \\ v_j & = & F_{2j}(x_i, y_i) \end{array} \tag{1}$$

The functions F_{2j-1} and F_{2j} correspond respectively to the real and imaginary parts of f_j. They can always be written in terms of the real variables x_i, y_i without the imaginary unit.

Example: Take the map $w = \bar{z}_1 z_2$. The corresponding functions F are

$$\begin{array}{l} u = F_1 = \Re(\bar{z}_1 z_2) = x_1 y_1 + x_2 y_2 \\ v = F_2 = \Im(\bar{z}_1 z_2) = x_1 y_2 - x_2 y_1, \end{array} \tag{2}$$

which, incidentally, corresponds the the dot and cross product of z_1 and z_2 considered as vectors in $\mathbb{R}^2$.

Given a real map $F : \mathbb{R}^n \to \mathbb{R}^m$ the Jacobian, J, represents the differential of F. Given coordinates on the domain and range, the Jacobian is expressed as the matrix of partial derivatives. If $y_j = F(x_i)$ then

$$J_{ji} = (DF)_{ji} = \left| \frac{\partial y_j}{\partial x_i} \right|. \tag{3}$$

If $m = n$ and the Jacobian matrix is square, and the determinant of J represents the distortion of volumes induced by the map F. If the determinant is nonzero then F is non-singular and locally a diffeomorphism (it could fail to be one-to-one). If $m > n$ and $\det J^t J \neq 0$ this map is an immersion, while if $m < n$ and $\det JJ^t \neq 0$ the map is a submersion. These conditions guarantee that J has maximal rank in each case.

The central question we will consider is for a complex map f with associated real map F, how can one compute the relevant Jacobian without using the associated map. Since the complex equations are half the dimension and most likely simpler to write, such a relationship would make the computations vastly easier.

2 Jacobians of maps of $\mathbb{C} \to \mathbb{C}$

If we have a one dimensional complex map $w = f(z)$, the real Jacobian $J_{\mathbb{R}}$ is the matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{4}$$

Such a differentiable complex map is subject to the Cauchy-Riemann conditions, which are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{5}$$

which gives the Jacobian matrix the structure

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{6}$$

This represents the fact that a complex number $a + ib$ can be represented with the real 2×2 matrix above, which preserves the algebraic structure (addition and multiplication of two complex numbers corresponds to addition and matrix-multiplication of their corresponding matrices). Since the derivative of a complex function is a complex number, it must have that structure. It is important to note that, since complex multiplication is commutative, multiplication of matrices of this form is also commutative:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{pmatrix}$$

In any case, the real Jacobian matrix is thus the matrix representation of the derivative dw/dz, which we can call the complex Jacobian, $J_{\mathbb{C}}$. Thus we have the association

$$J_{\mathbb{C}} \to J_{\mathbb{R}}, \tag{7}$$

by replacing the one complex entry with its 2×2 real matrix representation. Thus we have

$$\det J_{\mathbb{R}} = a^2 + b^2 = |dw/dz|^2 = |\det J_{\mathbb{C}}|^2. \tag{8}$$

While this relation is almost trivial in one complex dimension, we will next show that the relationship $\det J_{\mathbb{R}} = |\det J_{\mathbb{C}}|^2$ holds in general for maps between spaces of the same dimension.

3 Jacobians of maps of $\mathbb{C}^n \to \mathbb{C}^n$

For a map $f : \mathbb{C}^n \to \mathbb{C}^n$, the complex Jacobian is the matrix of partial derivatives

$$J_{\mathbb{C}} = \left| \frac{\partial w_i}{\partial z_j} \right|. \tag{9}$$

We construct the corresponding real map F by replacing each w_i and z_j with their corresponding real variables. Thus the real Jacobian is formed by replacing each complex entry in the complex Jacobian with its corresponding real 2×2 matrix representation,

$$\frac{\partial w_i}{\partial z_j} \to \left| \begin{array}{cc} \frac{\partial u_i}{\partial x_j} & \frac{\partial u_i}{\partial y_j} \\ \frac{\partial v_i}{\partial x_j} & \frac{\partial v_i}{\partial y_j} \end{array} \right|. \tag{10}$$

Now, $J_{\mathbb{R}}$ has the structure of a block matrix, each block the 2×2 representation of a complex number. As previously pointed out, each block, as a matrix, commutes under multiplication with each other block (since complex numbers commute). We want to take advantage of this structure when computing the determinant.

The first non-trivial case is when $J_{\mathbb{R}}$ is 4×4, or 2×2 counting blocks rather than entries. We then want to compute

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{11}$$

when all matrices A, B, C, D commute. It is tempting to to take the "determinant" of the block matrix, so get the 2×2 matrix $AD - BC$, and then take its determinant. This turns out to be correct (for commutative matrices)! In fact, Sylvester proved this formula in general, which we state without proof[1]:

Theorem (Sylvester). *Given a field F and a commutative sub-ring R of ${}^nF^n$ ($n \times n$ matrices over F), then*

$$\det_F M = \det_F(\det_R M). \tag{12}$$

[1] Proof may be found in Sylvester's article, which can be obtained at http://www.mth.kcl.ac.uk/~jrs/gazette/blocks.pdf

What this formula says is if M has a block structure, where the blocks commute under matrix multiplication, one can compute the determinant by first computing the "determinant of the blocks" (i.e., the usual determimant considering each block matrix as a number), and the computing the determinant of the resulting matrix. In our case the field is the real numbers $F = \mathbb{R}$ and the commutative sub-ring is the set of 2×2 matrices representing complex numbers (which commute under multiplication).

Armed with this result we can then associate the real and complex Jacobian determinants as follows. We have (writing out the determinant of the blocks)

$$\det J_{\mathbb{R}} = \det_{\mathbb{R}}(\det_{\mathbb{C}} J_{\mathbb{R}}) = \det \textstyle\sum_\sigma \operatorname{sgn}(\sigma) \prod_i A_{\sigma(i),i}, \tag{13}$$

using the usual determinant formula, where each σ is a permutation of the numbers $1, \ldots, n$, and each A_{ji} is a 2×2 matrix. Now, we will denote by A the sum of products of A_{ji}'s, which must be a matrix representation of a complex number. Taking the determinant of this matrix is equivalent to taking the norm of the associated complex number $\det A = |z|^2$. We can then associated to each A_{ji} its associated complex number, and since we must get the same complex number either way (the representation preserves the algebraic structure), we have

$$\det J_{\mathbb{R}} = \det A = |z|^2 = \left| \sum_\sigma \operatorname{sgn}(\sigma) \prod_i z_{\sigma(i),i} \right|^2 = |\det J_{\mathbb{C}}|^2, \tag{14}$$

which is remarkably simple! Finally, we note that this expression can be rewritten using the adjoint as

$$|\det J_{\mathbb{C}}|^2 = \det J_{\mathbb{C}} \overline{\det J_{\mathbb{C}}} = \det J_{\mathbb{C}} \det J_{\mathbb{C}}^\dagger = \det J_{\mathbb{C}} J_{\mathbb{C}}^\dagger = \det J_{\mathbb{C}}^\dagger J_{\mathbb{C}}. \tag{15}$$

4 Jacobians of maps of $\mathbb{C}^n \to \mathbb{C}^m$

In this section we will consider maps $f : \mathbb{C}^n \to \mathbb{C}^m$, where $n \neq m$. The real and complex Jacobian matrices are related by replacing each complex entry with its associated 2×2 real matrix representative. We will focus now on the Jacobian determinants. It is reasonable to expect that the transpose be replaced by the adjoint. This will indeed turn out to be the case.

Suppose now that $n < m$, so that the injectivity condition requires checking $\det J_{\mathbb{R}}^t J_{\mathbb{R}}$. When multiplying block matrices, we can multiply block-by-block, so that

$$B_{ji} = (A^t)_{kj} A_{ki} \tag{16}$$

is the ji-block in $J^t J$. Note that each matrix A_{ij} is transposed, and in addition the order of indices is reversed. For example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}. \tag{17}$$

Now we can write

$$\det J^t J = \det \sum_\sigma \operatorname{sgn}(\sigma) \prod_i B_{\sigma(i),i}, \tag{18}$$

similar to before. Following the reasoning in the previous section verbatim, we rewrite this as

$$|w|^2 = \left| \sum_\sigma \operatorname{sign}(\sigma) \prod_i w_{\sigma(i),i} \right|^2 = |\det B|^2, \tag{19}$$

for some matrix B, and w_{ji} is the complex number corresponding to B_{ji}.

Now, recalling the structure of the matrices A, if $z \leftrightarrow A$ then $\bar{z} \leftrightarrow A^t$. Then we have

$$w_{ji} \quad \leftrightarrow \quad B_{ji} \tag{20}$$

$$\bar{z}_{kj} z_{ki} \quad \leftrightarrow \quad (A^t)_{kj} A_{ki}, \tag{21}$$

and the determinant becomes

$$\left| \sum_\sigma \operatorname{sgn}(\sigma) \prod_i \bar{z}_{\sigma(i),k} z_{ki} \right|^2 = |\det J_{\mathbb{C}}^\dagger J_{\mathbb{C}}|^2. \tag{22}$$

The determinant of $J_{\mathbb{C}}^\dagger J_{\mathbb{C}}$ is automatically real, so the modulus is unncessary and we have

$$\det J_{\mathbb{R}}^t J_{\mathbb{R}} = (\det J_{\mathbb{C}}^\dagger J_{\mathbb{C}})^2. \tag{23}$$

When $n > m$ the argument is essentially isomorphic, and the result becomes

$$\det J_{\mathbb{R}} J_{\mathbb{R}}^t = (\det J_{\mathbb{C}} J_{\mathbb{C}}^\dagger)^2. \tag{24}$$

Finally, we note that both these expressions agree with each other and with our previous results on square Jacobians in the degenerate case $n = m$.

5 Conclusion

This article has explored the relation between the real and complex Jacobians associated to a map of several complex variables. Simple relations between these Jacobians were derived using Sylvester's theorem on the determinant of block matrices. Thus, one can easily find whether a complex map is of full rank using the complex Jacobian directly instead of having to convert these expression to the corresponding real Jacobians.