Equivariant Differential Embeddings

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Takens[1] has shown that a dynamical system may be reconstructed from scalar data taken along some trajectory of the system. A reconstruction is considered successful if it produces a system diffeomorphic to the original. However, if the original dynamical system is symmetric, it is natural to search for reconstructions that preserve this symmetry. These generally do not exist. We demonstrate that a differential reconstruction of *any* nonlinear dynamical system preserves at most a two-fold symmetry.

I. INTRODUCTION

Symmetry is an important property enjoyed by many equations describing physical phenomena. Common examples include the Lorenz[2], Burke and Shaw[3,4], Kremliovsky[5], and Thomas[6] dynamical systems. Each system models a measurable physical dynamics, but a typical experiment records only a single variable. Time delay and differential embedding techniques can be used to attempt the reconstruction of the entire original phase space. We are interested in determining what constraints the symmetry of a nonlinear dynamical system imposes on this reconstruction process. Specifically, the questions this paper addresses are the following: for a differential embedding constructed from a single observation function, 1) is the reconstructed dynamics equivariant; 2) if yes, under which group is it equivariant; and 3) under which representation of that group?

In short, equivariance provides an extremely tight constraint on the embedding problem. Specifically, we shall show that only two possibilities exist when attempting to reconstruct an equivariant dynamics, either 1) the reconstruction has no symmetry; or 2) the reconstruction is equivariant under the parity representation of $\mathbb{Z}_2$, the cyclic group of order two. In other words, regardless of the symmetry of the original system, the construction possesses at most a two-fold symmetry. It most cases this precludes the possibility of an actual embedding since the loss of symmetry prevents the reconstruction from being one-to-one. That is not to say that embeddings do not exist; they just cannot preserve symmetry.

The organization of this paper is as follows. Sec. II provides background material and motivation. Sec. III reviews the relevant theory of group representations. Sec. IV reviews the structure theory for equivariant dynamical systems, while Sec. V introduces a structure theory for differential mappings (dynamical system reconstructions). The structure of equivariant reconstructions is worked out in Secs. VI and VII. Implications of this theory for the embedding problem are given in Sec. VIII. Finally, Sec. IX states our conclusions.

II. BACKGROUND

A dynamical system is a set of first order ordinary differential equations or, equivalently, a smooth vector field on a manifold. The vector field generates a flow $\varphi_t(x)$ which is the unique solution to the differential equations. We are interested in autonomous dynamical systems on Euclidean space $\mathbb{R}^n$, which have the form $\dot{x}^i = v^i(x)$. We regard the vector field v as a map $v : \mathbb{R}^n \to \mathbb{R}^n$ associating to each point $x \in \mathbb{R}^n$ the vector $v(x)$.

A group G may act on $\mathbb{R}^n$ as a set of linear transformations. Such an action is through a *representation* Γ of G. A dynamical system $\dot{x} = v(x)$ is said to be symmetric or *equivariant* under G if there exists a *faithful* representation Γ of G acting on $\mathbb{R}^n$ such that the following diagram commutes for every $g \in G$

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ v\ } & \mathbb{R}^n \\ {\scriptstyle \Gamma(g)}\downarrow & & \downarrow{\scriptstyle \Gamma(g)} \\ \mathbb{R}^n & \xrightarrow{\ v\ } & \mathbb{R}^n. \end{array} \tag{1}$$

This relation states the the vector field "looks the same" when viewed from a point x as is does from any symmetry related point $\Gamma(g)(x)$. The representation is required to be faithful to eliminate trivial equivariance, which is simply invariance.

The Lorenz and Kremliovsky dynamical systems are both equivariant under $\mathbb{Z}_2$, the cyclic group of order two. The Lorenz system is given by the equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= Rx - y - xz \\ \dot{z} &= -bz + xy, \end{aligned} \tag{2}$$

which are equivariant under the transformation $R_z(\pi) : (x, y, z) \mapsto (-x, -y, z)$, equivalent to a π rotation about the z-axis. We say that the Lorenz system is rotationally equivariant. The Kremliovsky system is given by the equations

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= bx + z(x^2 - c), \end{aligned} \tag{3}$$

which are equivariant under the transformation $P : (x, y, z) \mapsto (-x, -y, -z)$, which is a spatial inversion.

We say that the Kremliovsky system is parity equivariant. The representations $R_z(\pi)$ and P are inequivalent in $\mathbb{R}^3$. The two systems therefore possess distinct symmetries even though they are both equivariant under faithful representations of the same group, $\mathbb{Z}_2$.

An observation function for a dynamical system is a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ that measures some observable of the system. The values of an observation function are recorded along some trajectory of the system; what one records is the composition $f \circ \varphi_t(x_0)$ for some initial condition x_0 at various times t, typically evenly spaced.

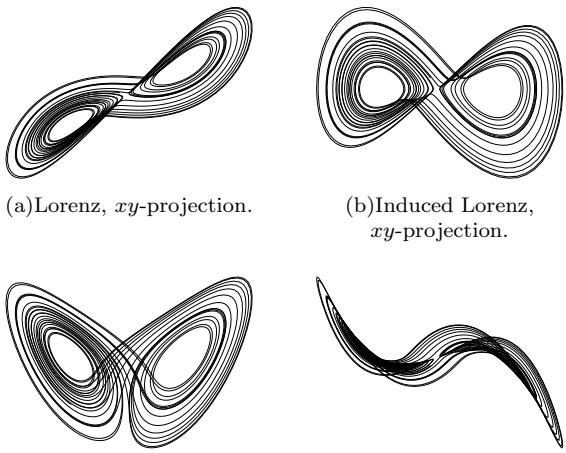
Given an observation function, a "differential mapping" of the dynamical system into $\mathbb{R}^m$ may be defined by the formula

$$x \mapsto \left(f(x), \left.\frac{d}{dt}\right|_0 f(\varphi_t(x)), \ldots, \left.\frac{d^{m-1}}{dt^{m-1}}\right|_0 f(\varphi_t(x)) \right), \quad (4)$$

where the notation indicates that derivatives are to be evaluated at $t = 0$. A theorem of Takens[1] states that for a generic dynamical system (of dimension n) and generic function f, this mapping is an embedding when $m = 2n+1$. This mapping is called a differential or Takens embedding. While smaller values of m may provide embeddings, Takens' theorem does not guarantee this.

When an observation function is discretely sampled at an interval Δt that is sufficiently small, linear combinations of k adjacent terms in the time series are good approximations to the signal and its first $k-1$ derivatives. Thus differential embeddings can be approximated by discretely sampled experimental data. In the sequel we investigate the equivariant properties of dynamical systems under differential mappings only.

As an example to motivate the present analysis, consider the Lorenz system, Eq. (2) (for details, see[7]). The coordinate function x and all of its derivatives transform under the parity representation P of $\mathbb{Z}_2$. A differential mapping of the Lorenz system using x as the observation function results in the induced Lorenz system, which is equivariant under P. The symmetries of the two attractors in $\mathbb{R}^3$ are compared in Fig. 1. An important consequence of this difference of symmetry is that this differential mapping does not provide an embedding of the entire Lorenz system into $\mathbb{R}^3$ (though it does in higher dimensions). We return to this point in Sec. VIII.



(a)Lorenz, xy-projection. (b)Induced Lorenz, xy-projection.

(c)Lorenz, xz-projection. (d)Induced Lorenz, xz-projection.

FIG. 1. Projections of the Lorenz and induced Lorenz attractors. The first row shows that both attractors possess $(x, y) \to -(x, y)$ symmetry. The bottom row shows that Lorenz has no z symmetry while the induced system has $(x, z) \to -(x, z)$ symmetry.

III. GROUP REPRESENTATIONS AND SCHUR'S LEMMAS

The structure of equivariant dynamical systems and their differential embeddings depends on the structure of the underlying equivariance group, G. We will assume that G is a finite group. Let Γ be a representation of G acting on the linear space V. Then Γ is said to be reducible if there exists a proper subspace $U \subset V$ that is invariant under Γ, that is $\Gamma(g)(u) \in U$ for every $u \in U$. If V has no proper invariant subspaces then Γ is said to be irreducible.

A representation Γ is said to be fully reducible if whenever U is a proper invariant subspace, there exists a complementary subspace which is also invariant. This means that in the proper basis the matrices $\Gamma(g)$ are simultaneously block diagonal. It is a fundamental fact that representations of finite groups are always fully reducible[8]. In this case every representation is a direct sum of irreducibles.

When speaking of irreducibility it is important to specify the field. A representation that is irreducible over $\mathbb{R}$ may be reducible over $\mathbb{C}$. Examples are provided by the representations of the cyclic groups $\mathbb{Z}_p$ for $p > 2$ as planar rotations through angle $2\pi/p$ (this is discussed further in Sec. IV). As we are concerned with real representations on real vector spaces ($\mathbb{R}^n$), irreducibility will be understood over $\mathbb{R}$ unless otherwise noted.

Two more fundamental results that are instrumental to the following analysis are Schur's lemmas, which describe the structure of homomorphisms between irreducible representations. Though applicable in more general settings, in the context of group representations they take the following form[8].

Schur's First Lemma: Suppose that Γ is an irreducible representation of a group G acting on a vector space V. If there exists a linear map $M : V \to V$ that commutes with Γ for every $g \in G$,

$$\begin{array}{ccc} V & \xrightarrow{M} & V \\ {\scriptstyle \Gamma(g)}\downarrow & & \downarrow{\scriptstyle \Gamma(g)} \\ V & \xrightarrow{M} & V, \end{array} \quad (5)$$

then M is a multiple of the identity, $M = \lambda I$.

Schur's Second Lemma: Suppose that Γ^1 is an irreducible representation of a group G acting on a vector space V^1 and that Γ^2 is an irreducible representation of G acting on V^2. If there exists a linear map $M : V^1 \to V^2$ that commutes with Γ^i for every $g \in G$,

$$\begin{array}{ccc} V^1 & \xrightarrow{M} & V^2 \\ {\scriptstyle \Gamma^1(g)}\downarrow & & \downarrow{\scriptstyle \Gamma^2(g)} \\ V^1 & \xrightarrow{M} & V^2, \end{array} \tag{6}$$

then either M is zero or an isomorphism. In the latter case the two representations Γ^1 and Γ^2 are equivalent.

IV. THE STRUCTURE OF EQUIVARIANT DYNAMICAL SYSTEMS

This section reviews the structure theory of equivariant dynamical systems[9], since this is not widely known. Let the representation Γ^D define an action of the group G on $\mathbb{R}^n$. Then Γ^D acts on the coordinate functions x^i of $\mathbb{R}^n$. Denote by $\mathbb{R}[x]$ the set of all polynomials in variables $x^1, \ldots, x^n$. This set is a ring under the operations of polynomial addition and multiplication. The action of Γ^D on the monomials x^i induces an action on all of $\mathbb{R}[x]$ in a natural way. This representation is denoted by Γ^R.

Let $p \in \mathbb{R}[x]$ be a polynomial. If p is invariant under Γ, $p(\Gamma x) = p(x)$, then p is said to be an invariant polynomial. Otherwise p is said to be covariant. Since Γ^R is fully reducible, each polynomial p can be decomposed into components, each belonging to an invariant subspace transforming under a particular irreducible representation. The invariant polynomials all belong to the same subspace, which transforms under the trivial representation $\Gamma(g) = I_n$. The sets of invariant and covariant polynomials each possess a basis set of polynomials from which all others may be constructed through the ring operations[10]. They are called an integrity basis and a ring basis, respectively.

An arbitrary function f on $\mathbb{R}^n$ may be decomposed with respect to the action Γ^D of G on $\mathbb{R}^n$ into a sum of an invariant and a covariant function. The invariant part may be written as $h_0(p)$, where h_0 is a (not necessarily polynomial) function of the integrity basis polynomials, p. The covariant part may be further decomposed as $\sum_r h_r(p)q^r$, where $r \geq 1$, the q^r are polynomials in the ring basis, and the h_r are functions of the invariant polynomials. If we define $q^0 \equiv 1$ as a ring basis polynomial representing the invariant irreducible subspace, an arbitrary function f may be written as $f = h_r(p)q^r$, where $r \geq 0$ and summation is implicit over the repeated index.

Now consider a dynamical system $\dot{x}^i = v^i$ equivariant under the representation Γ^D of G. Each component of the vector field may be expanded in the ring basis as $v^i = h^i{}_r(p)q^r$. The behavior of the dynamical system under the group operation g is determined by

$$\begin{aligned} gv^i &= gh^i{}_r(p)q^r \\ gv^i &= h^i{}_r(p)gq^r \\ \Gamma^D(g^{-1})^i{}_j v^j &= h^i{}_r(p)\Gamma^R(g^{-1})^r{}_s q^s \\ \Gamma^D(g^{-1})^i{}_j h^j{}_s q^s &= h^i{}_r(p)\Gamma^R(g^{-1})^r{}_s q^s, \end{aligned} \tag{7}$$

where in the second line we used invariance of the $h^i{}_r$, in the third the definitions of the representations Γ^D and Γ^R, and in the last the expansion of v^j in the ring basis.

The last line must hold for each basis element q^s in the ring basis separately. The resulting equation may be expressed as the commutative diagram

$$\begin{array}{ccc} \mathbb{R}[x] & \xrightarrow{h} & \mathbb{R}^n \\ {\scriptstyle \Gamma^R(g)}\downarrow & & \downarrow{\scriptstyle \Gamma^D(g)} \\ \mathbb{R}[x] & \xrightarrow{h} & \mathbb{R}^n, \end{array} \tag{8}$$

demonstrating that h intertwines the two representations Γ^D and Γ^R. We may regard $\mathbb{R}^n$ as a subspace of $\mathbb{R}[x]$ spanned by the monomials x^i. Since both Γ^D and Γ^R are fully reducible, Schur's first lemma may be applied to the restriction of h to the irreducible subspaces. The conclusion is that h is multiplication by a constant (that is, an invariant polynomial) between equivalent representations and zero otherwise. This severely restricts the structure of the functions $h^i{}_j$ that define an equivariant dynamical system.

For example, consider the representation $\Gamma^D = R_z(\pi)$ of $\mathbb{Z}_2$, the equivariance group of the Lorenz system. The invariant polynomials z, x^2, xy, and y^2 span an integrity basis. The ring basis polynomials x and y each transform under the non-trivial one dimensional representation $\mathbb{Z}_2 \to \{1, -1\}$. The most general form of a three dimensional dynamical system equivariant under $\Gamma^D = R_z(\pi)$ is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & h^1{}_2 & h^1{}_3 \\ 0 & h^2{}_2 & h^2{}_3 \\ h^3{}_1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \tag{9}$$

where each $h^i{}_j$ is a arbitrary function of the invariant polynomials. The Lorenz system is defined by the choices $h^1{}_3 = -h^1{}_2 = \sigma$, $h^2{}_2 = R - z$, $h^2{}_3 = -1$, and $h^3{}_1 = -bz + xy$.

V. THE STRUCTURE OF DIFFERENTIAL MAPPINGS

This section describes two properties of differential mappings that restrict the structure of equivariant embeddings of dynamical systems. These are 1) the canonical form of the image dynamical equations; and 2) the preservation of transformation properties under differentiation.

The differential mapping F in Eq. (4) is constructed from the consecutive derivatives of a single observation

function f. When the image dynamical system is well defined (for example, when the mapping is an embedding) the new vector field V at $F(x)$ is given by

$$\begin{aligned} V^i &= \frac{\partial F^i}{\partial x^j} v^j \\ &= \left.\frac{d}{dt}\right|_0 F^i(\varphi_t(x)) \end{aligned} \tag{10}$$

It is immediate from the definition that $V^1 = F^2$. For V^2 we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 F^2(\varphi_t(x)) &= \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 f(\varphi_s(\varphi_t(x))) \\ &= \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 f(\varphi_{s+t}(x)) \\ &= \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds'}\right|_t f(\varphi_{s'}(x)) \\ &= \left.\frac{d^2}{dt^2}\right|_0 f(\varphi_t(x)) \\ &= F^3(x), \end{aligned} \tag{11}$$

where $s' = s + t$. By induction we have the general rule that $V^i = F^{i+1}$ for $i < m$.

Therefore the image dynamical system always has the canonical form

$$\begin{aligned} \dot{F}^1 &= F^2 \\ \dot{F}^2 &= F^3 \\ &\vdots \\ \dot{F}^{m-1} &= F^m \\ \dot{F}^m &= h(F^1, \ldots, F^m), \end{aligned} \tag{12}$$

for some function h. We can express this canonical form by $\dot{F}^i = M^i{}_j F^j + \delta^i{}_m h(F)$, where above the last row M is an upper shift matrix (unit super-diagonal) and the bottom row is zero,

$$M = \left(\begin{array}{cccc} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \hline & & 0 & \end{array}\right). \tag{13}$$

Next we consider how the derivatives of the observation function f transform under a group operation g. By definition of derivative (recalling that φ_t is the flow generated by v)

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 f(\varphi_t(gx)) &= \lim_{t=0} \frac{f(gx + tv_{gx}) - f(gx)}{t} \\ &= \lim_{t=0} \frac{f(g(x + tv_x)) - f(gx)}{t} \\ &= \left.\frac{d}{dt}\right|_0 f(g(\varphi_t(x))), \end{aligned} \tag{14}$$

where in the second line we used the assumption of equivariance. It follows that if f is invariant under g then so is its time derivative since $f \circ g = f$. Suppose $f = q^i$ is a ring basis polynomial. In this case

$$\left.\frac{d}{dt}\right|_0 q^i(g\varphi_t(x)) = \Gamma^i{}_j(g) \left.\frac{d}{dt}\right|_0 q^j(\varphi_t(x)), \tag{15}$$

which just says the derivative of q^i transforms under the same representation as q^i. In the general case of a linear combination of covariant polynomials multiplied by arbitrary invariant polynomials, the derivative of f transforms the same as f, that is, it is composed of the same representations. This follows at once from the linearity of the derivative, the chain rule, and the special cases already considered.

Consider again the Lorenz system with observation function x, which transforms under the parity representation of $\mathbb{Z}_2$. The differential mapping $F(x, y, z) = (X, Y, Z)$ of the Lorenz system into $\mathbb{R}^3$ constructed using x is given by

$$\begin{aligned} X &= x \\ Y &= \sigma(y - x) \\ Z &= \sigma(R + \sigma - z)x - \sigma(1 + \sigma)y, \end{aligned} \tag{16}$$

and it is apparent that the coordinate functions (X, Y, Z) transform under the P representation of $\mathbb{Z}_2$. The canonical equations of motion are satisfied with h given by[11,12]

$$\begin{aligned} &b\sigma(R-1)X - b(1+\sigma)Y - (1+b+\sigma)Z \\ &- X^2 Y - \sigma X^3 + \frac{Y}{X}\left(Z + (1+\sigma)Y\right). \end{aligned} \tag{17}$$

The canonical equations are also equivariant under P.

VI. THE STRUCTURE OF EQUIVARIANT REPRESENTATIONS

This section applies the structure built up in the past two sections to constrain the symmetry of equivariant dynamical systems under differential mappings. First, we demonstrate that equivariance requires that an observation function be composed of polynomials transforming under a single representation. Next, we demonstrate that this representation is necessarily abelian, in fact cyclic. Finally, we show that this representation is one dimensional. We conclude that if the image of an equivariant dynamical system is itself equivariant, the equivariance group representation is necessarily one dimensional.

Suppose that $f = F^1$ is an observation function and that $F = (F^1, \ldots, F^m)$ is the corresponding differential mapping. Since the original dynamical system is equivariant, the image system will be equivariant under G if the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^m \\ {\scriptstyle \Gamma^D(g)}\downarrow & & \downarrow{\scriptstyle \Gamma^{D'}(g)} \\ \mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^m. \end{array} \tag{18}$$

Recall that the definition of equivariance requires that $\Gamma^{D'}$ be faithful. As we shall see, Eq. (18) is often satisfied by an unfaithful representation $\Gamma^{D'}$. In this case $\Gamma^{D'}$ provides a faithful representation of some group G' homomorphic to G. Specifically, if $\rho : G \to \Gamma^{D'}$ is the homomorphism defining the representation, then $G' \cong G/\ker\phi$. We say that the image system is equivariant under G' rather than G.

For instance, the Lorenz system is equivariant under $\mathbb{Z}_2$ acting as π rotations around the z-axis. The coordinate function z is invariant under this action and a differential mapping constructed using this function results in a dynamical system without symmetry. It is equivariant under the identity representation $\rho : \mathbb{Z}_2 \to I_3$. We will return to this example in Sec. VII.

As in Sec. IV we expand each component F^i in the ring basis of $\mathbb{R}^n$ as $F^i = h^i{}_j(p)q^j$. By essentially the same reasoning that led to Eq. (8) we obtain the diagram

$$\begin{array}{ccc} \mathbb{R}[x] & \xrightarrow{h} & \mathbb{R}^m \\ {\scriptstyle \Gamma^R(g)}\downarrow & & \downarrow{\scriptstyle \Gamma^{D'}(g)} \\ \mathbb{R}[x] & \xrightarrow{h} & \mathbb{R}^m, \end{array} \tag{19}$$

showing that h intertwines Γ^R and $\Gamma^{D'}$, that is $h\Gamma^R = \Gamma^{D'}h$.

Using full reducibility, decompose Γ^R and $\Gamma^{D'}$ into a direct sum of irreducible representations, $\Gamma^R = \mathrm{diag}\left(\Gamma^{(l_1)}, \ldots, \Gamma^{(l_s)}\right)$, and $\Gamma^{D'} = \mathrm{diag}\left(\Gamma^{(k_1)}, \ldots, \Gamma^{(k_r)}\right)$. Similarly decompose $\mathbb{R}[x]$ and $\mathbb{R}^m$ into the corresponding invariant subspaces on which the irreducible representations act, $\mathbb{R}[x] = U_1 \oplus \cdots \oplus U_s$ and $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_r$. Let the indices of $h^i{}_j$ refer now to invariant subspaces rather than matrix elements so that $h^i{}_j : U_j \to V_i$ is a linear map for each i, j. Schur's second lemma requires that each $h^i{}_j$ be an isomorphism when non-zero, in particular U_j and V_i have the same dimension. We obtain the commutative diagram

$$\begin{array}{ccc} U_j & \xrightarrow{h^i{}_j} & V_i \\ {\scriptstyle \Gamma^{(l_j)}}\downarrow & & \downarrow{\scriptstyle \Gamma^{(k_i)}} \\ U_j & \xrightarrow{h^i{}_j} & V_i, \end{array} \tag{20}$$

for each pair of indices (i, j).

Using the decompositions given by the previous paragraph, Eq. (19) can be written in the block form

$$\begin{pmatrix} h^1{}_1\Gamma^{(l_1)} & h^1{}_2\Gamma^{(l_2)} & \cdots \\ h^2{}_1\Gamma^{(l_1)} & h^2{}_2\Gamma^{(l_2)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \Gamma^{(k_1)}h^1{}_1 & \Gamma^{(k_1)}h^1{}_2 & \cdots \\ \Gamma^{(k_2)}h^2{}_1 & \Gamma^{(k_2)}h^2{}_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

The components of F are built from covariant polynomials. Suppose that $f = F^1$ contains a polynomial q^r transforming under some representation, which we assume to be $\Gamma^{(l_1)}$ without loss of generality. Then some $h^i{}_1$ is non-zero and therefore an isomorphism. Assume without loss of generality that $i = 1$. We then have $h^1{}_1\Gamma^{(l_1)} = \Gamma^{(k_1)}h^1{}_1$, which shows that $\Gamma^{(l_1)}$ and $\Gamma^{(k_1)}$ are isomorphic and therefore the same representation.

Now, by the results of Sec. IV every component of F must contain a covariant polynomial transforming under the same representation $\Gamma^{(l_1)}$. This in turn requires that $h^i{}_1$ is non-zero (and therefore an isomorphism) for every value of i. The first column of Eq. (21) then yields the equation $h^i{}_1\Gamma^{(l_1)} = \Gamma^{(k_i)}h^i{}_1$ for every i, which shows that *every* irreducible representation $\Gamma^{(k_i)}$ appearing in $\Gamma^{D'}$ is the same and equal to the representation $\Gamma^{(l_1)}$. In the same way, comparing the remaining columns shows that every representation of Γ^R is equal to $\Gamma^{(l_1)}$ as well. A very strong result follows: each component of F must be composed of polynomials transforming under a *single* irreducible representation.

It turns out that this representation cannot be arbitrary; it is necessarily abelian as we now show. Recall the canonical form $\dot{F}^i = M^i{}_j F^j + \delta^i{}_m h(F)$ of the image differential equations, where M is given by Eq. (13). Equivariance under Γ yields the equation

$$\left(\Gamma^i{}_j M^j{}_k - M^i{}_j \Gamma^j{}_k\right) F^k = \delta^i{}_m h(\Gamma F) - \Gamma^i{}_m h(F). \tag{22}$$

The LHS is manifestly linear in F and the RHS must be linear in F as well. When $i \neq m$ the delta vanishes and we must have $\Gamma^i{}_m h(F)$ be linear in F. Since h is always non-linear in cases of interest (we are studying nonlinear dynamical systems) we see that $\Gamma^i{}_m = 0$ and therefore $\Gamma^i{}_j M^j{}_k = M^i{}_j \Gamma^j{}_k$ when $i \neq m$. By writing $M^i{}_j = \delta^{i+1}{}_j$, it follows immediately that $\Gamma^i{}_j = \Gamma^{i+1}{}_{j+1}$, which says that Γ is Toeplitz in the basis spanned by the F^i.

That every matrix in Γ is simultaneously Toeplitz implies that Γ is an abelian representation. The components of an $n \times n$ Toeplitz matrix A are completely determined by the values along the anti-diagonal, which can be considered as a vector of length $2n - 1$. In index notation we may write $A_{ij} = a_{i-j+n}$, in terms of the vector a. Similarly let $B_{ij} = b_{i-j+n}$. If A and B belong to Γ then both products AB and BA belong to Γ and must be Toeplitz.

Now the components AB and BA are given in terms of the vectors a and b by

$$\begin{aligned} (AB)_{ij} &= \sum_{k=1}^{n} a_{n+i-k} b_{n-j+k} \\ (BA)_{ij} &= \sum_{l=1}^{n} b_{n+i-l} a_{n-j+l}. \end{aligned} \tag{23}$$

In the expression for BA, the sum over l may be rewritten as a sum over k by setting $l = n + 1 - k$. A term from the this sum is now given by $a_{2n+1-k-j} b_{k+i-1}$. The anti-diagonal of a matrix is specified by the index condition $i + j = n + 1$. This relation can be used to swap i and j in the terms giving BA, yielding $a_{2n+1-k-j} b_{k+i-1} \to a_{n+i-k} b_{n-j+k}$, which is exactly the

form of the terms giving AB. Thus the two matrices have identical anti-diagonals. But since the anti-diagonal determines the entire matrix, the two matrices are identical. We conclude that A and B commute.

Thus the representation Γ is necessarily abelian for *any* equivariance group G. In particular, if a dynamical system is equivariant under a non-abelian group G, the largest equivariance group of any image system constructed by a differential mapping is the abelianization $\tilde{G} = G/G^{(1)}$, which is the quotient of the group by its commutator subgroup $G^{(1)} = [G, G]$. This is because if $G' = G/N$ is any abelian quotient of G then $G^{(1)} \leq N$. In other words, $\tilde{G}$ is the largest abelian homomorphic image of G. It follows that a differential mapping for a non-abelian G cannot provide an embedding equivariant under G since group elements representing non-trivial commutators are mapped to the identity.

For example, the alternating group A_4 (the group of all even permutations on four objects) is a non-abelian group of order twelve. The commutator subgroup is isomorphic to the vierergruppe V_4 and the abelianization is $\tilde{A}_4 \cong A_4/V_4 \cong \mathbb{Z}_3$, the cyclic group of order three[13]. Therefore a differential mapping of a dynamical system equivariant under A_4 will have at most a three-fold symmetry. For $n \geq 5$ A_n is non-abelian and simple[13]. Since A_n is non-abelian, $A_n^{(1)}$ is not trivial. Since A_n is simple $A_n^{(1)}$ must then be equal to all of A_n, and the abelianization $\tilde{A}_n \cong A_n/A_n$ is trivial. A differential mapping of a dynamical system equivariant under A_n for $n \geq 5$ never has symmetry. Remarkably, we will see that differential mappings for A_4 and $A_3 \cong \mathbb{Z}_3$ equivariant dynamical systems never have symmetry either.

Finally, we show that Γ must be one dimensional. To this end we momentarily extend to the complex plane. Schur's first lemma implies that every irreducible representation of an abelian group is one dimensional over $\mathbb{C}$. There are thus two possibilities for Γ. Either the representation is one dimensional over $\mathbb{R}$ and therefore irreducible over $\mathbb{C}$, or two dimensional over $\mathbb{R}$ and expressible as the direct sum of a one dimensional complex representation and its complex conjugate, $\Gamma = \Gamma^{(i)} \oplus \bar{\Gamma}^{(i)}$.

We now suppose that Γ is two dimensional. In the decomposition $\Gamma = \Gamma^{(j)} \oplus \bar{\Gamma}^{(j)}$, the complex irreducible representation $\Gamma^{(j)}$ is one dimensional and unitary and therefore a complex number of modulus one, which can be written $\Gamma^{(j)}(g) = \exp i\phi(j, g)$. It follows that Γ is similar to a real 2×2 rotation matrix

$$\Gamma = \begin{pmatrix} \exp i\phi & 0 \\ 0 & \exp -i\phi \end{pmatrix} \simeq \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \qquad (24)$$

Note that every 2×2 rotation matrix is manifestly Toeplitz. We may think of Γ as providing a homomorphism of G onto a finite subgroup of $SO(2)$. Such a subgroup is not only abelian, it is necessarily cyclic.

All of the irreducible representations of cyclic groups are known[8]. If we let g denote the generator of the cyclic group of order p then there are exactly p inequivalent irreducible representations of $\mathbb{Z}_p$ over $\mathbb{C}$. They are given by

$$\Gamma^{(q)}(g^m) = \epsilon^{mq}, \qquad (25)$$

where ϵ is a primitive p-th root of unity and $0 \leq q < p$. The representation $q = 0$ is always the identity. Setting $z = x + iy$, the invariant basis polynomials for $\Gamma^{(0)}$ are $\bar{z}z$, z^p, and $\bar{z}^p$. The covariant polynomials for $\Gamma^{(j)}$, $j > 1$, are z^j and $\bar{z}^{p-j}$. Since real representations are formed by the direct sum of a complex representation and its complex conjugate, q and $p-q$, the real basis polynomials are the real and imaginary parts of the corresponding complex polynomials.

In the defining representation on $\mathbb{R}^2$, the x and y coordinates transform under the $\Gamma = \Gamma^{(1)} \oplus \bar{\Gamma}^{(1)}$ representation. The only other polynomials that transform under this representation are the real and imaginary parts of $\bar{z}^{p-1}$. If a dynamical system is equivariant under Γ then in a two dimensional subspace on which Γ acts the equations of motion have the complex form $\dot{z} = \xi z + \zeta \bar{z}^{p-1}$, with ξ and ζ functions of invariant polynomials. In terms of the real variables we have

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \zeta_1 & \zeta_2 \\ -\zeta_2 & \zeta_1 \end{pmatrix}\begin{pmatrix} \Re(\bar{z}^{p-1}) \\ \Im(\bar{z}^{p-1}) \end{pmatrix}. \qquad (26)$$

Notice that the real an imaginary parts of $\bar{z}^{p-1}$ are non-linear in x and y when $p > 2$.

Now if the image of a dynamical system under a differential mapping is equivariant under Γ, then as was shown in Sec. VI the image phase space $\mathbb{R}^m$ must decompose as $\mathbb{R}^m = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$ with the same representation Γ of $\mathbb{Z}_p$ acting on each factor $\mathbb{R}^2$. In each subspace the equations of motion must have the form of Eq. (26). This is a second canonical form for the equations of motion (Eq. (12) being the first).

Denote by Y the coordinates defining this decomposition so that (Y^{2k-1}, Y^{2k}) spans the k-th subspace. These coordinates are related to the canonical coordinates F by some invertible linear transformation, $Y^i = P^i{}_j F^j$. We wish to show that the two canonical forms of the equations are consistent only when h is linear.

The differential equations in the Y coordinates are given by

$$\begin{aligned} \dot{Y}^i &= P^i{}_j \dot{F}^j \\ &= P^i{}_j M^j{}_k F^k + P^i{}_m h(F) \\ &= P^i{}_j M^j{}_k (P^{-1})^k_l Y^l + P^i{}_m h(P^{-1}Y) \\ &= N^i{}_j Y^j + C^i \tilde{h}(Y), \end{aligned} \qquad (27)$$

where $N^i{}_j$ and C^i are constants and $\tilde{h} = h \circ P^{-1}$ is a non-linear function of Y. For simplicity in the following we will drop the tilde and write h for $\tilde{h}$.

The function h may be uniquely written as $h = h_r(p)q^r$ in terms of invariant and covariant polynomials. If we

identify $(Y^{2i-1}, Y^{2i}) = (x, y)$ for any i, then the most general form of h consistent with Eq. (26) is

$$h = h_1 x + h_2 y + h_3 \Re(\bar{z}^{p-1}) + h_4 \Im(\bar{z}^{p-1}), \tag{28}$$

where the h_i are functions of invariant polynomials. Using this decomposition of h, Eq. (27) becomes in the $(Y^{2i-1}, Y^{2i}) = (x, y)$ subspace

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} N_{11} + C_1 h_1 & N_{12} + C_1 h_2 \\ N_{21} + C_2 h_1 & N_{22} + C_2 h_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &+ \begin{pmatrix} C_1 h_3 & C_1 h_4 \\ C_2 h_3 & C_2 h_4 \end{pmatrix}\begin{pmatrix} \Re(\bar{z}^{p-1}) \\ \Im(\bar{z}^{p-1}) \end{pmatrix}.\end{aligned} \tag{29}$$

Comparing this to Eq. (26) leads to the equations $\zeta_1 = C_1 h_3 = C_2 h_4$ and $\zeta_2 = C_1 h_4 = -C_2 h_3$. These equations require that $C_1^2 = -C_2^2$, or $C_1 = C_2 = 0$, which in turn implies that $\zeta_1 = \zeta_2 = 0$. We conclude that that this equation is satisfied only if h is linear. But if h is linear then the image dynamical system is linear and uninteresting. We therefore conclude that for non-linear systems the representation Γ must be one dimensional.

For completeness, we note that in the linear case two dimensional equivariant embeddings do exist. Consider the simple two dimensional dynamical system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x,\end{aligned} \tag{30}$$

which is equivariant under $SO(2)$ and therefore every $\mathbb{Z}_p$ acting as rotations through angle $2\pi/p$. For $p > 2$ the complex representation $\Gamma^1 = \{1, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}\}$ is faithful. The complex basis polynomial is $z = x + iy$, and the monomials x and y form a basis for the two dimensional real representation. Suppose that x is chosen as the observation function. Then since $\dot{x} = y$ the differential mapping is $F = (x, y)$ which is just the identity. The image system is in this case identical to the original system and manifestly equivariant under the same representation of the same symmetry group.

As an application of the results of this section, consider the Thomas system[6], which is defined by the differential equations

$$\begin{aligned}\dot{x} &= -bx + ay - y^3 \\ \dot{y} &= -by + az - z^3 \\ \dot{z} &= -bz + ax - x^3.\end{aligned} \tag{31}$$

These equations have a six-fold symmetry. They are equivariant under the parity representation P of $\mathbb{Z}_2$ with generator $g_2 = -I_3$ as well as the $C_3 = R_{\mathbf{u}}(2\pi/3)$ representation of $\mathbb{Z}_3$, where $\mathbf{u} = (1, 1, 1)$. The generator of C_3 is the cyclic permutation matrix

$$g_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{32}$$

Since the Thomas system is equivariant under both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ it is equivariant under their direct product $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_3$ with generator $g_6 \equiv g_2 g_3 = g_3 g_2$. This generator can also be described by a $2\pi/6$ rotation about $\mathbf{u}$ followed by a reflection in the plane perpendicular to $\mathbf{u}$. The generators of the two subgroups are recovered as $C_3 = g_6^2$ and $P = g_6^3$.

A more convenient representation of the system is given by transforming to new variables defined by the linear transformation[9]

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{33}$$

which makes Z the new rotation axis so that projection onto the XY-plane exhibits the six-fold symmetry. Basis polynomials for both subgroups can be constructed and have degree at most three. Each basis polynomial has definite transformation properties under the two generators C_3 and P. The transformation properties of these polynomials and the equivariance properties of the images constructed from them are summarized in Table I.

TABLE I. Transformation properties for basis polynomials of degree at most two for the symmetries of the Thomas system, P and C_3. Cov and Inv denote covariance and invariance respectively. The final column gives the symmetry of the image system using the corresponding basis polynomial as observation function. An I denotes the identity representation or invariance.

Polynomial	P	C_3	Image
X, Y	Cov	Cov	P
Z	Cov	Inv	P
$X^2 + Y^2$	Inv	Inv	I
$X^2 - Y^2, 2XY$	Inv	Cov	I

All four combinations of invariance and covariance between the two subgroups exist. The coordinate functions X and Y are covariant polynomials of both symmetries and are therefore covariant polynomials of the complete symmetry group $\mathbb{Z}_6$. However, in accordance with the results of this section, no differential mapping constructed from any of these functions can possess more than the $\mathbb{Z}_2$ symmetry. A direct calculation shows that differential mappings constructed from X or Y have parity symmetry, and visual inspection shows no apparent rotational symmetry.

VII. THE STRUCTURE OF ONE DIMENSIONAL REPRESENTATIONS

The previous section demonstrated that the only nontrivial equivariance group representations for differential mappings are one dimensional. In this case every basis polynomial must be an eigenvector with eigenvalue

$\lambda = \pm 1$. Since all components of the mapping F transform under the same representation, each component is a simultaneous eigenvector with the same eigenvalue. If $\lambda = 1$ then the image is equivariant under the trivial representation $\Gamma(g) = I_m$ for every g. The image system is no longer equivariant under G, but rather invariant. We say that F has modded out the symmetry of the dynamical system. In this case, the nicest possible behavior for F is providing a $|G| \to 1$ local diffeomorphism[9]. We noted in Sec. VI that constructing a differential mapping of the Lorenz system using the z coordinate results in an image without symmetry. This mapping is in fact a $2 \to 1$ local diffeomorphism[9].

On the other hand if $\lambda = -1$ then the image coordinates transform under a representation satisfying $\Gamma(g) = \pm I_n$ and $\Gamma(g^2) = I_n$ for every g. In this case Γ furnishes the parity representation of $G \cong \mathbb{Z}_2$ in $\mathbb{R}^m$. This representation defines a group homomorphism $G \to \mathbb{Z}_2$.

The necessary and sufficient condition for the existence of such a homomorphism is the existence of a normal subgroup $N \triangleleft G$ with $|N| = |G|/2$, since by Lagrange's Theorem we have $|G/N||N| = |G|$ and $\mathbb{Z}_2$ is the unique group of order two. We see immediately that when the order of G is odd that no such homomorphism can exist. In particular, if a dynamical system is equivariant under $\mathbb{Z}_p$, p odd, its image under any differential mapping cannot be equivariant.

When $|G|$ is even such a homomorphism may or may not exist, depending on the group. For example the alternating group A_4 has order twelve but has no subgroup of order six[13], so possesses no homomorphism onto $\mathbb{Z}_2$. One could also note that the abelianization is $\tilde{A}_4 \cong \mathbb{Z}_3$, which possesses no homomorphism onto $\mathbb{Z}_2$. Since there is no homomorphism of A_4 onto $\mathbb{Z}_2$, the image of an A_4 equivariant dynamical system under any differential mapping cannot have symmetry.

Notice that A_4 is non-abelian. Abelian groups of even order always possess a normal subgroup of half the group order, which we now show. By the fundamental theorem of finite abelian groups we can write G as a direct product of cyclic groups. Since the order of a direct product is the product of the orders, at least one summand $\mathbb{Z}_r$ must have even order. If the generator of this subgroup is h, then h^2 generates a cyclic subgroup of order $r/2$. But every subgroup of an abelian group is normal, which establishes the claim.

Consider again the Lorenz system, equivariant under the representation $\Gamma = R_z(\pi)$ of $\mathbb{Z}_2$. The basis set of invariant polynomials is given by z, x^2, xy, and y^2, while the basis set of covariant polynomials, which transform under P, is given by x and y. Constructing a differential mapping using an invariant polynomial results in an image without symmetry. For instance, using z results in a $2 \to 1$ local diffeomorphism onto the proto-Lorenz system[9]. On the other hand, using a covariant function such as x results in a parity equivariant image, the induced Lorenz system. In no case is it possible to construct an image transforming under the same representation as the original Lorenz system, $R_z(\pi)$. This agrees with previous results[7], obtained using different techniques. Similar remarks would hold for any $R_z(\pi)$ equivariant dynamical system, such as the Burke and Shaw system.

It is worth stressing this last observation. If one constructs a differential mapping of *any* equivariant dynamical system and the image system is equivariant, it is necessarily *parity* equivariant, regardless of the original symmetry. This is congruent with the results of the Thomas system in Sec. VI. In particular this means that a differential *embedding* of a system equivariant under a group of order greater than two cannot be equivariant under a faithful representation of the symmetry group. In general, symmetries are not preserved by differential embeddings constructed from a single observation function.

VIII. IMPLICATIONS FOR EMBEDDINGS

An important consequence of the foregoing analysis is that in almost all cases equivariant differential mappings are not embeddings. This is immediate if the symmetry of the original system has order $|G| > 2$. Specifically, the action of G partitions the original phase space into $|G|$ symmetry related domains. Since the image system has only two symmetry related domains, the original domains are mapped onto the image domains in a $|G|/2 \to 1$ fashion. If the image system is invariant, these domains are mapped in a $|G| \to 1$ fashion.

Even when $|G| = 2$ one may fail to obtain an embedding when the original representation of $\mathbb{Z}_2$ is not the parity representation. Every representation of $\mathbb{Z}_2$ acting in $\mathbb{R}^n$ is given in the appropriate basis by $\Gamma = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$. Representations are distinguished by their signature, that is, the number of $+$ signs in this matrix. Since the coordinate directions corresponding to the $+$ signs are left invariant (and those corresponding to the $-$ signs covariant), representations are distinguished by the dimension of their invariant subspace. The parity representation leaves only the origin (zero dimensional subspace) invariant.

A differential mapping must map the symmetry invariant set (not to be confused with the dynamical invariant set) of the original system onto that of the image system. When the original invariant set has nonzero dimension, this identification obviously precludes an embedding. However, in many cases this invariant set may be considered disjoint from the flow. In the case of the Lorenz system, the z-axis is the stable manifold of the central fixed point and is generally ignored (excised) when discussing embeddings.

Even with this understanding trouble still arises. Denote by x and y the invariant and covariant coordinates, respectively, so that $\Gamma(x, y) = (x, -y)$. Let F be the differential mapping between spaces of the same dimension. If J denotes the Jacobian at (x, y), then at $(x, -y)$ we

have

$$\left.\frac{\partial F(x,y)}{\partial(x,y)}\right|_{(x,-y)} = \left.\frac{\partial F(x,-y)}{\partial(x,-y)}\right|_{(x,y)} = -\left.\frac{\partial F(x,y)}{\partial(x,-y)}\right|_{(x,y)}, \tag{34}$$

so that the Jacobian determinants are related by

$$|J(x,-y)| = (-1)^{\#x}|J(x,y)|, \tag{35}$$

where $\#x$ is the number of invariant coordinates. We see that if $\#x$ is odd, the Jacobian determinants at (x,y) and at $\Gamma(x,y)$ have opposite sign, and so the Jacobian must become degenerate somewhere along any curve connecting these two points. This presents an obstruction to obtaining an embedding into a space of the same dimension as the original system. We note, however, that this condition on the Jacobian is not an obstruction to finding an embedding in higher dimensions.

For example, the Lorenz system has the z-axis as a one-dimensional invariant subspace. Therefore no equivariant differential mapping of Lorenz into $\mathbb{R}^3$ can be an embedding. This is true for any $R_z(\pi)$ equivariant dynamical system. However, for the Lorenz system, a differential mapping constructed from the x coordinate does provide an embedding into $\mathbb{R}^4$ and higher dimensions. This is worked out explicitly in[7].

The general theory presented in this paper provides the following implications for the four dynamical systems listed in the introduction: an equivariant embedding of the Kremliovsky system Eq. (3) is possible that preserves the parity symmetry; an equivariant embedding of the Lorenz system Eq. (2) or the Burke and Shaw system is possible, but the symmetry necessarily changes from rotation to parity; an equivariant embedding of the Thomas system Eq. (31) is not possible.

Finally, we note that while differential mappings typically destroy symmetry, it is sometimes possible to recover the lost symmetry. If one has an invariant (non-equivariant) image it is possible to construct a lift of the image system to a covering system with any prescribed symmetry. If the original symmetry group and representation are known, then a lift to a system equivariant under this symmetry is possible. This two part process of generating an invariant image and lifting to an equivariant system yields an embedding of the original system which preserves symmetry. For details of this construction, see[9,14,15].

IX. CONCLUSIONS

This paper has considered the embedding problem for equivariant dynamical systems. Equivariant dynamical systems possess a rather rigid structure that constrains this problem. We have shown that for any dynamical system equivariant under any representation of any discrete equivariance group, there are only two possibilities when attempting to construct equivariant images under differential mappings: either 1) the image is invariant; or 2) the image is equivariant under the parity representation of $\mathbb{Z}_2$. An immediate corollary is that the only symmetry that can be preserved under a differential mapping is parity symmetry.

It follows that in almost all cases differential mappings are not embeddings. This is always the case if the original symmetry has order $|G| > 2$, since symmetry related domains in the original system are mapped onto symmetry domains in the image in a $|G| \to 2$ or $|G| \to 1$ fashion. Even if $|G| = 2$, an equivariant differential mapping of an n-dimensional system into $\mathbb{R}^n$ will fail to be an embedding if the dimension of the symmetry invariant subspace is odd. Embeddings in the same dimension are only possible when the symmetry invariant subspace has even dimension, such as when the original system is already parity equivariant. The symmetry of an equivariant dynamical system typically cannot be preserved under differential embedding.

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