Paradoxical Twins

Beyond an Introduction

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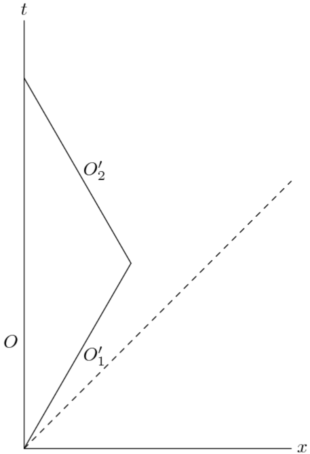
The Standard Twin Paradox



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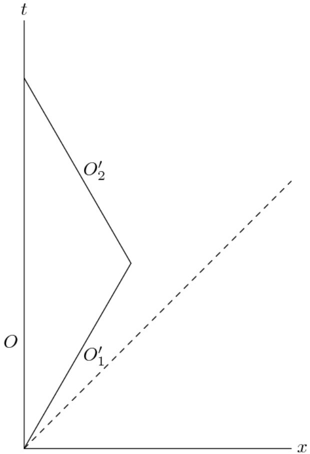

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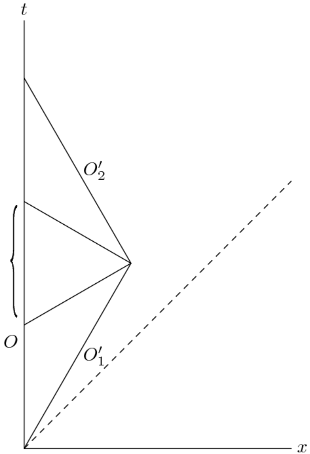

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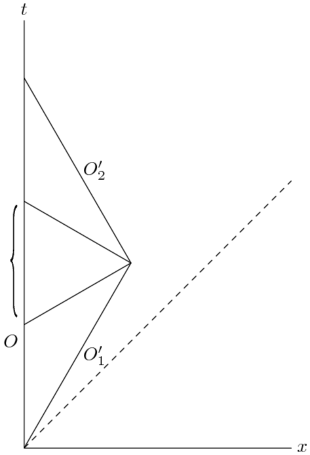

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Actually, O' does see O-clock run slowly, but there is also a sudden aging at the turn-around point. Using finite acceleration, we can account for this faster aging as a gravitational effect.

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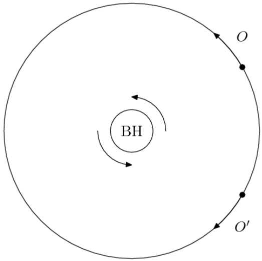

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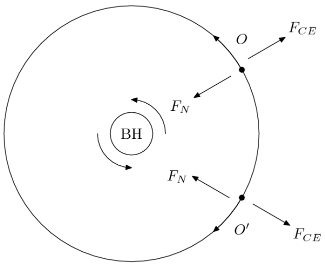

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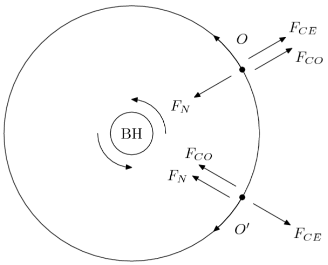

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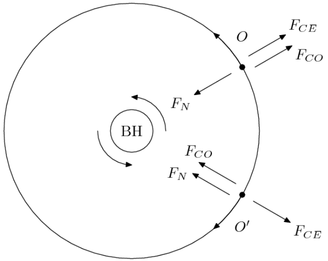

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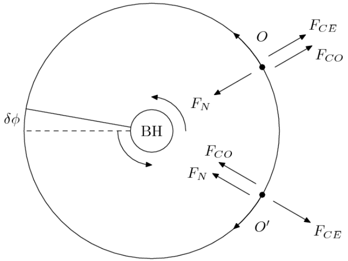

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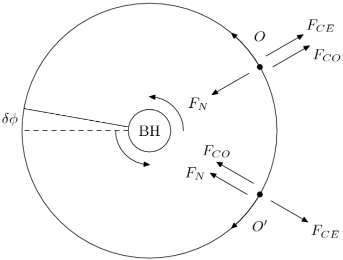

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- Since O' moves faster, she ages less!

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- It would be interesting to compute perceived aging directly in each twins' frame.

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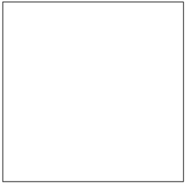

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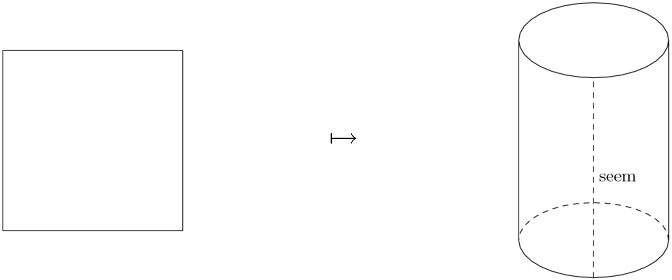
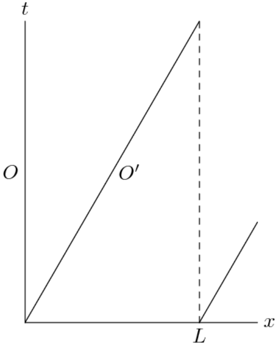


Periodic Boundary Conditions



Identify points that differ by L.

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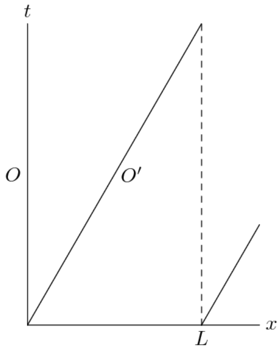
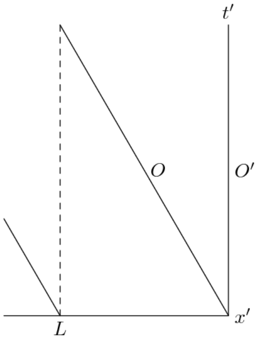


$\longrightarrow$



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Contradiction?

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- The näive Minkowski diagram for O' is incorrect since it assumed the condition $(t', x') \sim (t', x' + nL)$.

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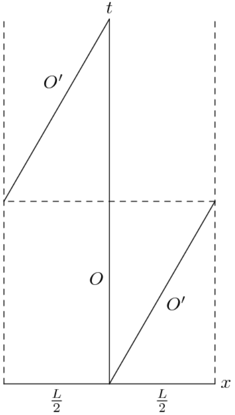

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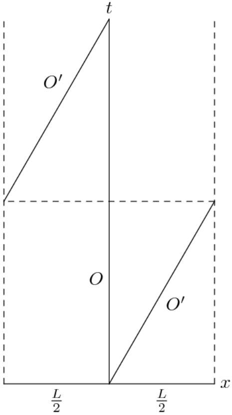

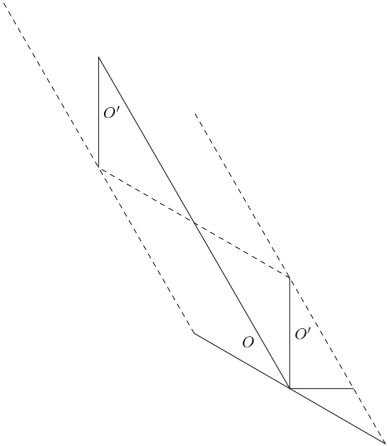

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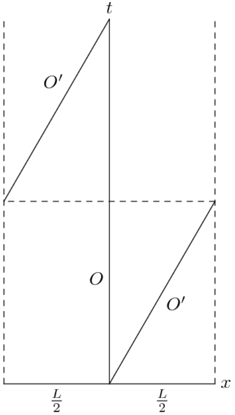

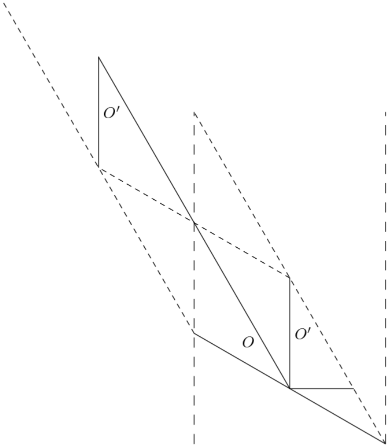

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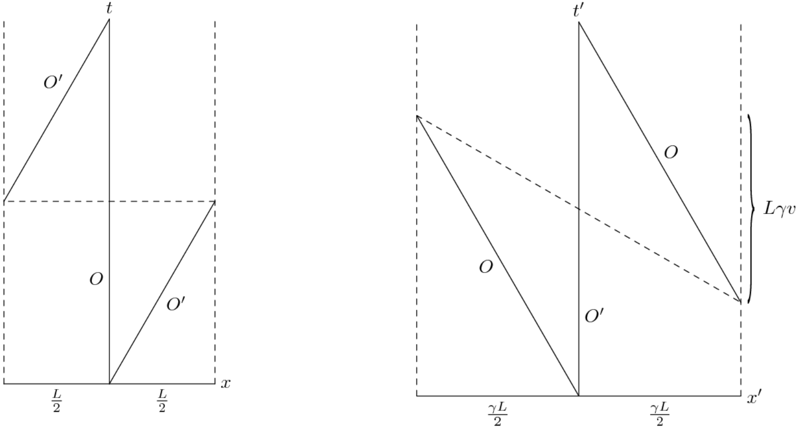

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They agree!

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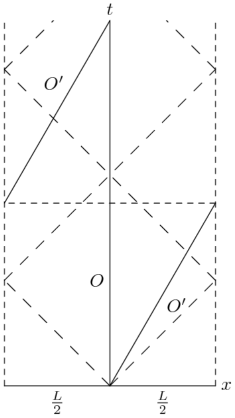
Thus the universe is shorter in one direction than the other.

(Anisotropic) Light Propagation



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- However, it is still homogeneous.
- We can make this more precise in group theory:

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The action of M on a point is then

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \left(\begin{array}{c|c} R_\theta & T \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} R_\theta \begin{pmatrix} x \\ y \end{pmatrix} + T \\ 1 \end{pmatrix}$$

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$$\begin{aligned}
\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} &\sim T_{L,0} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\
&= \left(\begin{array}{c|c} I_2 & \begin{matrix} L \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x + L \\ y \\ 1 \end{pmatrix}
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that is, g and T commute.

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$$\begin{aligned} gT &= Tg \\ \left(\begin{array}{c|c} R & \begin{matrix} v_x \\ v_y \end{matrix} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} I_2 & \begin{matrix} L \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right) &= \left(\begin{array}{c|c} I_2 & \begin{matrix} L \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} R & \begin{matrix} v_x \\ v_y \end{matrix} \\ \hline 0 & 1 \end{array}\right) \end{aligned}$$

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$$\left(\begin{array}{c|c} R & R\begin{pmatrix} L \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} R & \begin{pmatrix} L \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \hline 0 & 1 \end{array}\right)$$

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\left(\begin{array}{c|c} R & \begin{matrix} v_x \\ v_y \end{matrix} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} I_2 & \begin{matrix} L \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right) &= \left(\begin{array}{c|c} I_2 & \begin{matrix} L \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} R & \begin{matrix} v_x \\ v_y \end{matrix} \\ \hline 0 & 1 \end{array}\right) \\
\left(\begin{array}{c|c} R & R\begin{pmatrix} L \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \hline 0 & 1 \end{array}\right) &= \left(\begin{array}{c|c} R & \begin{pmatrix} L \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \hline 0 & 1 \end{array}\right)
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Identifying two that differ by T gives $S^1 \times \mathbb{R} \simeq SO(2) \times T(1)$.

The SR Cylinder

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And the symmetry group is translations again.

Cylinder Frames

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The frame we constructed for O' kept the metric diagonal, but the coordinates didn't respect the symmetry.

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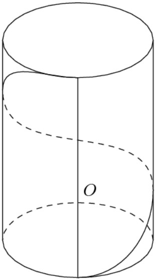

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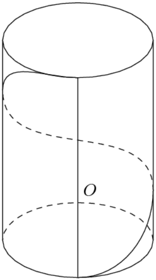
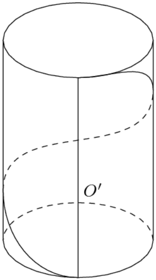


$\xrightarrow{\text{twist}}$



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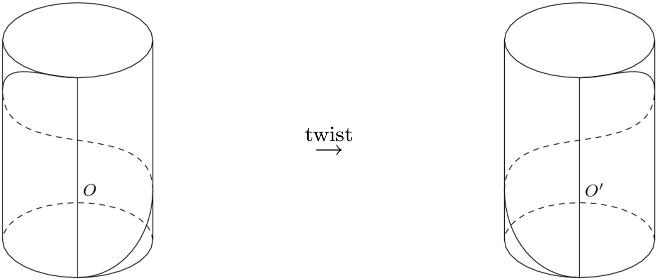

$$f : (t, [x]) \to (t, [x - vt])$$

where $[x]$ means the image of x in the cylinder.

Cylinder Frames

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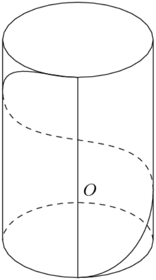
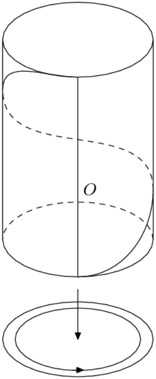
Which is not diagonal.

Topological Invariant

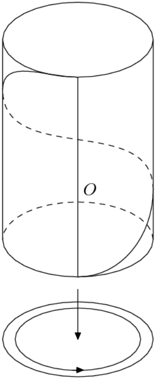


Topological Invariant



- The number of times the path wraps around the cylinder is the *winding number*.

Topological Invariant



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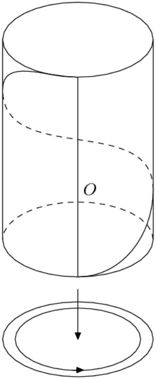

- The number of times the path wraps around the cylinder is the *winding number*.
- This makes sense as long as we project points down in a manner *orthogonal* to the space surfaces.
- This invariant describes twin aging.

Summary

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Summary

- The twin paradox is possible without acceleration.
- Differential aging is always accompanied by another asymmetry.
- The SR cylinder actually has more in common with GR.
- It possessed a preferred frame due to symmetry.

Fin

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