# On the Flux Rule

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#### Abstract

The Flux Rule for calculating the EMF due to a changing magnetic flux is critically examined. First, the rule is derived from Maxwell's equations in a way that unifies the two contributions to the flux change. Then it is shown that so-called "failures of the flux rule" are not problems with the actual rule, but rather in trying to improperly deduce a stronger local result from the weaker global result that the rule actually provides.

#### 1 Faraday's Law and the Flux Rule

The integral from of the Maxwell equations of classical electrodynamics are

$$\int_{\partial U} E \cdot dA = \int_{U} 4\pi\rho \qquad \int_{\partial U} B \cdot dA = 0 \tag{1-1}$$

$$\oint_{\partial S} E \cdot dl = -\int_{S} \frac{\partial B}{\partial t} \cdot dA \qquad \oint_{\partial S} B \cdot dl = 4\pi J + \int_{S} \frac{\partial E}{\partial t} \cdot dA, \quad (1-2)$$

where S is an arbitrary, possibly moving surface with boundary  $\partial S$ , and U is an arbitrary, possibly moving volume with boundary  $\partial U$ . By well-known methods, these equations may be transformed into their local, differential versions, which are

$$\nabla \times E = 4\pi\rho \qquad \nabla \cdot B = 0 \tag{1-3}$$

$$abla \times E = -\frac{\partial B}{\partial t} \qquad \nabla \times B = 4\pi j + \frac{\partial E}{\partial t}.$$
(1-4)

We are particularly interested in Faraday's law, expressing the relationship between an electric field and changing magnetic field. It is often claimed that this law is equivalent to the flux rule, but this is not strictly correct. Consider the time derivative of a surface integral of some arbitrary vector field V over a *moving* surface S. This may be written<sup>1</sup>

$$\frac{d}{dt} \int_{S(t)} V \cdot dA = \int_{S(t)} \left( \frac{\partial V}{\partial t} + (\nabla \cdot V)v - \nabla \times (v \times V) \right) dA, \tag{1-5}$$

<sup>&</sup>lt;sup>1</sup>See the Appendix

where v = v(x, t) is the vector field describing the motion of the surface with time. It should be noted that one cannot simply integrate the explicit change in time of the vector field; one must include contributions due to the changing region. This is a classical result in vector analysis which is not often remembered.

In the present case we are interested in the time derivative of the magnetic flux integral, which is (since  $\nabla \cdot B = 0$ )

$$-\frac{d}{dt}\int_{S(t)} B \cdot dA = -\int_{S(t)} \left(\frac{\partial B}{\partial t} - \nabla \times (v \times B)\right) dA \qquad (1-6)$$

$$= \int_{\partial S(t)} (E + v \times B) \cdot dl, \qquad (1-7)$$

where we have used Faraday's law and Green's theorem, respectively, on the two terms in the integral. If we designate this quantity by EMF we have

$$-\frac{d\Phi_B}{dt} = EMF = \int_{\partial S(t)} f \cdot dl, \qquad (1-8)$$

that is, the EMF is simply the force per charge integrated around the loop. We may safely add any electrostatic field to the E term present, as this contribution will integrate to zero over the loop. Note that this expression is valid for *any* path bounding a region, whether it corresponds to a physical path for current or not.

This expression is not just the integral of E around the loop, which many books erroneously call the EMF and is what one obtains when the variation of the integration surface is ignored. It is this error that leads some to claim that there are two unrelated source of EMF, one coming from changing magnetic fields and one from the Lorentz force law. But the fundamental approach above shows that these are unified by taking the proper derivative of the integral. Moreover, calling the  $v \times B$  term the Lorentz force is not strictly correct in general, since v here refers to the velocity of the moving region which may or may not correspond to the velocity of actual charges.

#### 2 Applications of the Flux Rule

The relation between changing magnetic flux, EMF and the integral of force per charge over a boundary is fundamental, being derived strictly from Maxwell's equations and the rules of vector calculus. Note that we have not yet mentioned anything about voltage of current readings, which is typically what this rule is used for.

When the loop of integration is a physical wire, if we multiply by q we get the total force on charges throughout the loop, from which we deduce that there is a current if and only if this total force on the charges is not zero. What one typically does is create some conducting apparatus that is hooked up to a current or volt meter. The magnetic flux through the 'circuit' is manipulated in some fashion to create and EMF or current which is then read out by the

meter. The surface is taken to be the region bounded by the wires that form the circuit and then the EMF is taken to be the voltage difference seen in the wire (see Fig. 1).



Figure 1: Typical use of the flux rule in 1-D circuit. The moving circuit give a change of flux of Bvh and drives a current I as indicated.

The reason this works is that we may identify, unambiguously, a privileged integration path with the physical path of electrons in the circuit. More specifically, if we calculate a non-zero EMF around a closed wire, we deduce that there must be local current flow by Kirchhoff's law. The charge can only move along the one dimensional wire and must therefore move with the *same constant current everywhere*, which gives the measured current. In one dimension there is no other possibility. Note that what we have done is to move from a global, integral result, to a local one by squeezing on the one dimensionality of the setup.

The source of all so-called flux rule paradoxes or failures is precisely when one attempts to answer whether a current flows in a circuit which includes a bulk conductor of some sort. The real problem here is attempting to localize the contributions to the integral so as to deduce a local current flow from the global integral value. We cannot do this when there is more that one dimension for current to move in.



Figure 2: Faraday's homopolar generator with rotating conducting disk.

The principle example is Faraday's homopolar generator (Fig. 2). Here a conducting disk of radius r rotates with uniform angular velocity  $\omega$  in the plane perpendicular to a constant magnetic field B. Consider two possible integration lines *abda* and *acbda*, where b is fixed in space and c rotates with the disk. In the first case there is no change in flux and no EMF. In the second case there is a change of flux, given by  $\omega r^2 B/2$  since the path of integration moves w.r.t the field. In typical discussions it is claimed that we have a paradox or contradiction

since these predict different voltage or current readings.

No paradox arises because we actually are in no position to predict, whatsoever, whether there is a current in the wires. A priori, neither of the two numbers computed above have anything to say about whether there is a current present or not. The two dimensional nature of the rotating disk means that more complicated local behaviors are possible. Specifically, there could be a current along *abda* and yet zero EMF along this countour since in the disk the current need not flow directly from *a* to *b*.

The flux rule only fails when we force it to say more than it is capable of saying. Remember further that the integral relationships are valid for any path, whether physical or not. Thus we get these same answers for the given integration paths regardless of whether the platform is rotating or not, indeed whether there even is a platform or not!

What we need to know is the actual path taken by electrons through the platform and that requires a more detailed analysis. When we have one dimensional wires this part is automatic since the current is constrained to be in the wires. It turns out that the current follow line<sup>2</sup> ac. Knowing this we may now localize the integral along the path acbda and conclude that there is an EMF in the external wire given by  $\omega r^2 B/2$ , which is indeed what is measured.

### 3 Conclusion

In summary, the flux rule, relating EMF to changing magnetic flux in a circuit is derived from Maxwell's equations and thus is always valid. The application of the rule to one dimensional circuits allows us infer the existence of measured voltage or current by appealing to Kirchhoff's law. However, when we have multi-dimensional conductors present Kirchhoff's law is no longer applicable and we are simply not justified in saying whether a current flows or not without first figuring out the actual path of current through the conductor. Once we realize this we may conclude that the flux rule is never violated.

## A Appendix

For the sake of completeness, we wish to derive equation 1-5. Like all complicated vector calculus identities, it is best to actually forget the vector calculus and use the machinery of differential forms, which makes everything simpler, more elegant and general. However, to keep the presentation "elementary" and "accessible", we will use the classical language. We want to find the time derivative of the integral

$$I(t) = \int_{S(t)} V(t) \cdot dA.$$
 (A-1)

 $<sup>^{2}</sup>$ Without an applied field the electrons are constrained to rotate with the disk, thus having the same angular velocity. Thus with the applied field the charges will additionally move radially outward by the Lorentz force, at least in the adiabatic limit.

We have (with the definition of derivative)

$$I'(t) = \lim_{h \to 0} \frac{I(t+h) - I(t)}{h}$$
 (A-2)

$$= \lim_{h \to 0} \frac{\int_{S(t+h)} V(t+h) \cdot dA - \int_{S(t)} V(t) \cdot dA}{h}.$$
 (A-3)

Our strategy will be to systematically pull out the various ways the first integral depends on h, write the dependence to first order in h, and then take the (then trivial) limit on each term.

First let us consider the variation of V with time. V(t+h) may be expanded for small h as

$$V(t+h) = V(t) + h \frac{\partial V}{\partial t}(t), \qquad (A-4)$$

and the integral of this latter term becomes

$$\lim_{h \to 0} \int_{S(t+h)} \frac{h}{h} \frac{\partial V}{\partial t}(t) \cdot dA = \int_{S(t)} \frac{\partial V}{\partial t} \cdot dA, \tag{A-5}$$

which is the contribution due to the change in V alone.

What is left is the effect of the change of domain. We will now make explicit the dependence of V on position and ignore the time dependence. Changing the domain of integration changes where the vector field V is evaluated spatially, thus we expect the effect of the moving domain to include the spatial variation of V. If a point x is in the original domain of integration and that point moves with speed v(x), then the new domain will be integrated at x + vh (for h small) so that

$$V(x+hv) = V(x) + hv \cdot \nabla V(x). \tag{A-6}$$

The second term integrates to

$$\lim_{h \to 0} \int_{S(t+h)} \frac{h}{h} v \cdot \nabla V(x) \cdot dA = \int_{S(t)} v \cdot \nabla V \cdot dA, \tag{A-7}$$

and what remains is

$$\lim_{h \to 0} \int_{S(t+h)} V \cdot dA - \int_{S(t)} V \cdot dA, \tag{A-8}$$

and the only part of the expression we have not examined is dA.

This is most difficult part. dA can changes in two ways. The first is due to shrinking or expanding of area and the second is due to a change in the direction of the normal. We will treat the area change first. We suppose that the surface can be described with coordinates  $u^1, u^2$  so that the area element looks like  $du^1 du^2$ . We can express these quantities in terms of the three space coordinates as

$$du^{i} = \frac{\partial u^{i}}{\partial x^{j}} dx^{j} \qquad dx^{j} = \frac{\partial x^{j}}{\partial u^{i}} du^{i}$$
(A-9)

Under the motion a point x and nearby point x + dx will move to

$$x \rightarrow x + hv(x)$$
 (A-10)

$$x + dx \quad \to \quad x + dx + hv(x + dx) \tag{A-11}$$

$$= x + hv(x) + dx + hdx \cdot \nabla v(x), \qquad (A-12)$$

and subtracting the two gives the change in the dx as

$$dx \to dx + hdx \cdot \nabla v(x). \tag{A-13}$$

The corresponding changes in du are given by

$$du^{i} \to \frac{\partial u^{i}}{\partial x^{j}} \left( dx^{j} + h \frac{\partial v^{j}}{\partial x^{k}} dx^{k} \right)$$
(A-14)

To first order in h the change in  $du^1 du^2$  is then

$$h\left(\frac{\partial u^1}{\partial x^j}du^1 + \frac{\partial u^2}{\partial x^j}du^2\right)\frac{\partial v^j}{\partial x^k}dx^k.$$
 (A-15)

If we then express the  $dx^k$  in terms of the  $du^r$ , then not all terms contribute as  $du^1 du^1$  has no meaning in a surface integral, for example. By canceling these terms we are left with

$$h\left(\frac{\partial u^{1}}{\partial x^{j}}\frac{\partial v^{j}}{\partial u^{1}} + \frac{\partial u^{2}}{\partial x^{j}}\frac{\partial v^{i}}{\partial u^{2}}\right)du^{1}du^{2} = h\frac{\partial u^{k}}{\partial x^{j}}\frac{\partial v^{j}}{\partial u^{k}}du^{1}du^{2}$$
(A-16)

$$= h \frac{\partial v^{j}}{\partial x^{j}} du^{1} du^{2}$$
 (A-17)

$$= h(\nabla \cdot v)du^1du^2.$$
 (A-18)

Finally, the change in normal. Recall that the normal is so called because it is normal or perpendicular to the tangent to the surface at the given point. We will calculate the new normal by evaluating the change in tangent plane. Points are moved, to lowest order by  $x \to x + hv$ . When points are moved by a map, tangent vectors at the point are moved by the derivative of the map, which is  $I + h\nabla v$ , where I is the identity matrix. If we write  $u^1, u^2$  for the tangent vectors spanning the tangent space we have

$$n = u^1 \times u^2 \rightarrow u^1(1 + h\nabla v) \times u^2(1 + h\nabla v)$$
 (A-19)

$$= n + h(u^1 \times (u^2 \cdot \nabla)v - u^2 \times (u^1 \cdot \nabla)v).$$
 (A-20)

Since the triple  $u^1, u^2, n$  forms a right-handed orthonormal basis, we can expand all vectors in terms of them. We will expand  $(u^i \cdot \nabla)v$  as

$$(u^{1} \cdot \nabla)v = a_{1}u^{1} + a_{2}u^{2} + a_{n}n$$
 (A-21)

$$(u^2 \cdot \nabla)v = b_1 u^1 + b_2 u^2 + b_n n, \tag{A-22}$$

where

$$a_1 = u^1 \cdot ((u^1 \cdot \nabla)v) = (u^1 \cdot \nabla)(u^1 \cdot v),$$
 (A-23)

and so on. Using the orthonormality and right-handedness the new normal simplifies to

$$n(1 + h(a_2 + b_1)) - h(b_n u^1 + a_n u^2).$$
(A-24)

The first term changes the normal in the direction of the normal and can be ignored (more specifically, if we renormalize the new n to be unit length, this term disappears). Thus we are left with

$$n - h\left(u^1(u^1 \cdot \nabla)(v \cdot n) + u^2(u^2 \cdot \nabla)(v \cdot n)\right).$$
(A-25)

This expression enters the integral as  $V \cdot n$  which will become

$$V \cdot n - h\left((V^1 \cdot \nabla)(v \cdot n) + (V^2 \cdot \nabla)(v \cdot n)\right), \qquad (A-26)$$

using  $V \cdot u^i = V^i$ . We almost have  $(V \cdot \nabla)(v \cdot n)$  except for  $(V_n \cdot \nabla)(v \cdot n)$ , but this latter part is the change in normal in its own direction, which is zero to order h. Finally, combining both contributions to the change in dA we have

$$(1 + h(\nabla \cdot v)(V \cdot n) - h(V \cdot \nabla)(v \cdot n))dA.$$
(A-27)

The term corresponding to 1dA will have no h dependence so in the limit  $h \rightarrow 0$  the integral will vanish. The term on the right will combine with all our other contributions to give

$$h\left((v \cdot \nabla)V - (V \cdot \nabla)v + (\nabla \cdot v)V\right) \cdot dA,\tag{A-28}$$

and all this is left is some simplification. We will use an identity to remove the first two 'convective derivatives':

$$(B \cdot \nabla)A - (A \cdot \nabla)B = \nabla \times (A \times B) - A(\nabla \cdot B) + B(\nabla \cdot A), \qquad (A-29)$$

with A = V and B = v, which gives

$$h\left(\nabla \times (V \times v) + (\nabla \cdot V)v\right) \cdot dA,\tag{A-30}$$

and the integral becomes

$$\lim_{h \to 0} \int_{S(t+h)} \frac{h}{h} \left( \nabla \times (V \times v) + (\nabla \cdot V)v \right) \cdot dA$$
 (A-31)

$$= \int_{S(t+h)} \left( \nabla \times (V \times v) + (\nabla \cdot V)v \right) \cdot dA, \tag{A-32}$$

so, finally, putting back in the term with the time partial derivative we get

$$\frac{d}{dt} \int_{S(t)} V \cdot dA = \int_{S(t)} \left( \frac{\partial V}{\partial t} + (\nabla \cdot V)v - \nabla \times (v \times V) \right) dA, \qquad (A-33)$$

as promised.