Doppler Shift

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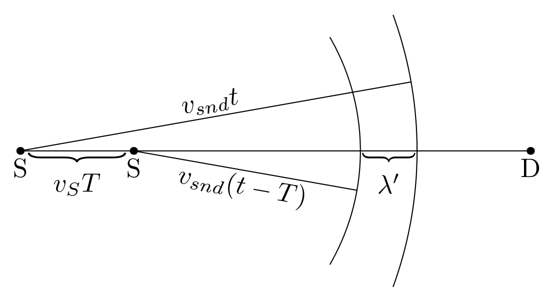
1 Introduction

This article gives a quick derivation of the doppler shift equation by exploiting invariance principles. One situation is worked out explicitly, and then the more general situations (including moving medium) are shown to follow from time-reversal and translational invariance.

2 Moving Source, Stationary Detector

This is the easiest case to work out explicitly. Consider the following diagram, where we have the source S moving with speed v_s to the right. It sound at frequency f with one wavefront at time 0 and the next one at time T (the period). The viewed distance between the fronts, λ' will change because of the Doppler shift.



If we follow the wave fronts for a time t we note that the first front moves a distance $v_{snd}t$ while the second moves a distance $v_{snd}(t-T)$ (since it was emitted time T later), and the source moves a distance $v_S t$. From the diagram we see that we have the relation

$$\begin{aligned} v_{snd}t &= \lambda' + v_{snd}(t-T) + v_S t \\ v_{snd}t &= \lambda' + v_{snd}t - v_{snd}T + v_S t \\ 0 &= \lambda' - v_{snd}T + v_S t, \end{aligned}$$

which gives

$$\lambda' = (v_{snd} - v_S)T,$$

but $T = 1/f_S$, so we have

$$\lambda' = \frac{v_{snd} - v_S}{f_S}.$$

This wavelength $\lambda' = \lambda_{front}$ is the wavelength measured by someone at rest in the medium.

Now, we want to relate this new wavelength to the frequency heard by the detector. But the speed of sound is the same for the detector, so we still have the relation $f_D \lambda' = v_{snd}$, so we have

$$f_D = f_S \frac{v_{snd}}{v_{snd} - v_S},$$

which is the Doppler formula for a moving source. Note that the sign is appropriate for a source that moves *toward* the detector.

3 Stationary Source, Moving Detector

This case need not be worked out explicitly, but follows immediately from the first case under time-reversal symmetry. In other words, imagine time running backwards. The waves start from the detector and move toward the source, that is, the detector is now the source and vice versa[1] Thus, in our formula we just swap the roles of the detector and source to get

$$f_D = f_S \frac{v_{snd}}{v_{snd} - v_S} \rightarrow f_S = f_D \frac{v_{snd}}{v_{snd} - v_D},$$

and solving the new equation for f_D gives

$$f_D = f_S \frac{v_{snd} - v_D}{v_{snd}},$$

which is the doppler formula for a detector moving *away* from the source. So, if we change the direction for the detector we get

$$f_D = f_S \frac{v_{snd} + v_D}{v_{snd}},$$

or a detector moving *toward* the source.

[1] Technically, the waves are convering to the left now, rather than diverging, which they would if D were now the source. However, the wavelength will stay the same regardless of whether the waves diverge or converge.

4 Moving Source, Moving Detector

For the general formula we just combine the two previous ones. When the source moves there is a change of frequency so that a stationary detector would measure

$$f' = f_S \frac{v_{snd}}{v_{snd} - v_S}.$$

Now, we can consider this shifted frequency to have been emitted by a stationary source (fictionally, of course), so that the actual moving detector will measure

$$f_D = f' \frac{v_{snd} + v_D}{v_{snd}}.$$

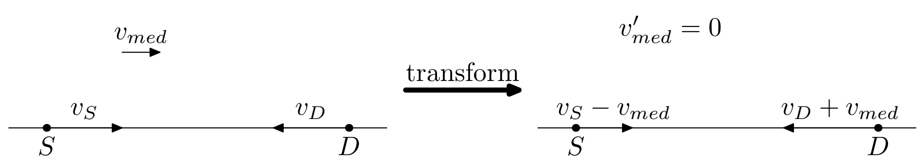
So, substitute in for f' in the second equation to get

$$\begin{aligned} f_D &= \left(f_S \frac{v_{snd}}{v_{snd} - v_S} \right) \frac{v_{snd} + v_D}{v_{snd}} \\ &= f_S \frac{v_{snd} + v_D}{v_{snd} - v_S}, \end{aligned}$$

which is the general formula, with signs appropriate for both source and detector moving toward each other.

5 Moving Medium

We now generalize this result one step further by allowing the medium in which the sound travels to move with constant speed v_{med}. This time will will exploit Galilean invariance by making a transformation to a frame of reference moving along with the medium:



In the new reference frame the velocities of the source and detector have changed, but now the medium is at rest. We are effectively adding a velocity of v_{med} (to the left) to every velocity vector present. Now we can immediately apply our previous result to get

$$\begin{aligned} f_D &= f_S \frac{v_{snd} + (v_D + v_{med})}{v_{snd} - (v_S - v_{med})} \\ &= f_S \frac{(v_{snd} + v_{med}) + v_D}{(v_{snd} + v_{med}) - v_S}, \end{aligned}$$

in other words, we just replace the speed of sound in the medium with the speed of sound measured in the original frame of reference, taking into account the motion of the medium.

Note that in problems where the wave is reflected by the detector back to the source, the speed of sound will be different in each case because it's going 'up stream' on way and 'down stream' the other. So, you'll have to do the calculations twice!