Elementary Solution of the Damped Oscillator

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1 Introduction

Differential equations are ubiquitous in physics and therefore it is imperative to have methods of solving them. There exist general theories for solving equations of the type considered here, but such theories are beyond the freshman level where one typically first encounters them. Thus, the solutions are simply stated and student gains little understanding.

The solution presented here solves the damped harmonic oscillator using elementary methods and a little sleight of hand. A little familiarity with complex analysis is needed at one point, but otherwise everything is elementary calculus.

2 Under Damped Case

Newton's Law for a spring system with linear damping reads

$$-kx - bv = ma,$$

for a block of mass m attached to a spring of constant k with damping coefficient b. Using the definitions of velocity and acceleration we can write this as the differential equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0,$$

which we will write as

$$\left\{ \frac{d^2}{dt^2} + \frac{b}{m}\frac{d}{dt} + \frac{k}{m} \right\} x = 0,$$

by factoring out the x.

Enter the sleight of hand. We can think of the expression on the left hand side as a polynomial in the 'variable' d/dt. We proceed by making the substitution $y = d/dt$ and then completing the square

$$\begin{aligned} y^2 + \frac{b}{m}y + \frac{k}{m} &= y^2 + 2\left(\frac{b}{2m}\right) + \left(\frac{b}{2m}\right)^2 - \left(\frac{b}{2m}\right)^2 + \frac{k}{m} \\ &= \left(y + \frac{b}{2m}\right)^2 + \frac{k}{m} - \left(\frac{b}{2m}\right)^2. \end{aligned}$$

So now our differential equation reads

$$\left\{\left(\frac{d}{dt}+\frac{b}{2m}\right)^2+\omega^2\right\}x=0,$$

where we have set

$$\omega^2=\frac{k}{m}-\left(\frac{b}{2m}\right)^2.$$

We are assuming here that $\omega^2>0$.

Now we we just move one term to the other side to get

$$\left(\frac{d}{dt}+\frac{b}{2m}\right)^2 x=-\omega^2 x,$$

and we take the square root of this expression to get

$$\left(\frac{d}{dt}+\frac{b}{2m}\right)x=\pm i\omega x.$$

Note that we now have two first order equations to solve (one for each sign). We seek solutions to the equations

$$\frac{dx}{dt}=\left(-\frac{b}{2m}\pm i\omega\right)x,$$

which have the obvious solutions

$$\begin{aligned}
x &= \exp\left(-\frac{b}{2m}\pm i\omega\right)t \\
&= \exp\left(-\frac{b}{2m}t\right)\exp\left(\pm i\omega t\right) \\
&= \exp\left(-\frac{b}{2m}t\right)\big(\cos(\omega t)\pm i\sin(\omega t)\big).
\end{aligned}$$

Thus our two solutions are (using Euler's formula)

$$\begin{aligned}
x_1 &= A_1\exp\left(-\frac{b}{2m}t\right)\big(\cos(\omega t)+i\sin(\omega t)\big), \\
x_2 &= A_2\exp\left(-\frac{b}{2m}t\right)\big(\cos(\omega t)-i\sin(\omega t)\big),
\end{aligned}$$

and our total solution (x_1+x_2) can be written

$$x=\exp\left(-\frac{b}{2m}t\right)\big((A_1+A_2)\cos(\omega t)+i(A_1-A_2)\sin(\omega t)\big).$$

Now, we need to choose A_1 and A_2 so that we get a real-valued solution, that is

$$\begin{aligned} A_1 + A_2 &\qquad \text{is real, and} \\ A_1 - A_2 &\qquad \text{is imaginary.} \end{aligned}$$

This condition has the effect of taking us from four unknown quantities (the real and imaginary part of each A) to just two, which is the appropriate number for a second order equation. Our solution is now

$$x = \exp\left(-\frac{b}{2m}t\right)\big(B\cos(\omega t) + C\sin(\omega t)\big),$$

which is the general form of the solution representing damped oscillations, and we have

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

3 Critically Damped Case

Suppose we have

$$\frac{k}{m} = \frac{b}{2m},$$

so that $\omega^2 = 0$. Then our equation takes the form

$$\left(\frac{d}{dt} + \frac{b}{2m}\right)^2 x = 0.$$

Then taking the square root gives the equation

$$\frac{dx}{dt} = -\frac{b}{2m}x,$$

which has the solution

$$x = A\exp\left(-\frac{b}{2m}t\right),$$

representing an exponential relaxation without any oscillations.

But we're not done yet! We must have two solutions since our original equation was of second order. We know that

$$(\frac{d}{dt} + \frac{b}{2m})x_1 = 0,$$

with x_1 the solution we know. But this is all we need to find the other solution. The full equation for the second solution is

$$(\frac{d}{dt} + \frac{b}{2m})\left\{(\frac{d}{dt} + \frac{b}{2m})x_2\right\} = 0,$$

which is satisfied if the expression in the brackets is equal to x_1! Thus we need to solve

$$(\frac{d}{dt} + \frac{b}{2m})x_2 = A \exp\left(-\frac{b}{2m}t\right).$$

The most reasonable thing to try is a product solution

$$x_2 = f \cdot A \exp\left(-\frac{b}{2m}t\right).$$

The equation simplifies enormously, leaving just

$$\frac{df}{dt} = 1,$$

which has the trivial solution

$$f = t + C,$$

and thus the full solution is

$$x = (At + B) \exp\left(-\frac{b}{2m}t\right),$$

where we defined $B = AC$.

4 Over Damped Case

This time suppose we have

$$\frac{k}{m} < \left(\frac{b}{2m}\right)^2.$$

Then we rewrite our equation as

$$\left\{\left(\frac{d}{dt} + \frac{b}{2m}\right)^2 - \omega^2\right\} x = 0,$$

where we now have set

$$\omega^2 = \left(\frac{b}{2m}\right)^2 - \frac{k}{m} > 0.$$

Then upon square rooting our equation we obtain

$$\left(\frac{d}{dt} + \frac{b}{2m}\right) x = \pm\omega x,$$

which is a real equation. The differential equation to solve is now

$$\frac{dx}{dt} = \left(-\frac{b}{2m} \pm \omega\right) x,$$

which has the solutions

$$
\begin{aligned}
x_1 &= A_1 \exp\left(-\frac{b}{2m} + \omega\right) t, \\
x_2 &= A_2 \exp\left(-\frac{b}{2m} - \omega\right) t,
\end{aligned}
$$

both representing a damped motion without oscillations. As always, determine A_1 and A_2 by the initial conditions.