From Quantum Mechanics to Maxwell's Equations

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1 Introduction

The concept of a field is central to both classical and quantum physics, although the respective notions of field are very different. A classical field is an object that transforms under the Poincaré group (scalar, vector, tensor, ...) and is constrained to be a solution of a particular differential equation. On the other hand, a quantum field is an arbitrary linear superposition of basis states in a Hilbert space. A quantum field does not obey a constraining equation since all states are physical and accessible. The classical description apparently allows for unphysical states which the differential equations are designed to suppress. That the quantum description is fundamental suggests that field equations are not arbitrary, but are determined by the quantum nature of the field. Our task is to demonstrate this explicitly in the case of the electromagnetic field.

2 The Electromagnetic Field

The classical electromagnetic field is described by a pair of 3-vector fields, $\mathbf{E}$ and $\mathbf{B}$, (or a single anti-symmetric second rank tensor) which are functions of space and time. The field therefore has a total of $2 \cdot 3 = 6$ degrees of freedom at every point. It is more convenient to take the Fourier transform of the fields and consider them as functions of momentum rather than position; thus the electromagnetic field has six degrees of freedom per 4-momentum.

On the other hand, quantum mechanics describes the electromagnetic field as a massless spin-1 field (arbitrary superposition of photon states). Since a massless particle has no rest frame, there are only two choices for the spin - to be aligned with or against the direction of motion. These yields two independent 'helicity' states per 4-momentum. We see that the classical description has four superfluous states per 4-momentum. We will see that it is the role of the Maxwell equations to eliminate these spurious degrees of freedom. This in effect explains why the Maxwell equations have the form they do.

3 The Poincaré Group

The invariance group of Minkowski space-time (special relativity) is the Poincaré or inhomogeneous Lorentz group, $IO(1,3)$. The reason is as follows. The Minkowski metric η is indefinite with signature 1,3:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The linear transformations that preserve this metric are determined by the preservation of inner products of 4-vectors,

$$u \cdot v = u^t \eta v = (\Lambda u)^t \eta (\lambda v) = u^t \left(\Lambda^t \eta \Lambda\right) v,$$

so that we must have $\eta = \Lambda^t \eta \Lambda$. Since these matrices preserve a metric with signature 1,3 it is named $O(1,3)$ (generalized orthogonal group[1]), otherwise known as the Lorentz group.

In addition to rotations and boosts (pure Lorentz transformations), this group also includes the two discrete operations of time reversal $\mathscr{T}$ and parity $\mathscr{P}$, both of which have negative determinant. If we restrict the determinant to be $+1$ only, this is the special orthogonal group $SO(1,3)$. This restricts $\mathscr{T}$ and $\mathscr{P}$ to occur in the combination $\mathscr{T}\mathscr{P}$, since each has negative determinant. We can further suppress these operations completely and the result is called the proper orthochronous Lorentz group, $SO^+(1,3)$. This group is connected (every group element may be reached from any other by a continuous path) and will be the primary object of our investigations. The groups $O(1,3)$ and $SO(1,3)$ are not connected since one cannot continuously change from det $+1$ to -1 or continuously reverse the direction of time.

Since all these transformations are linear they preserve the origin and are called homogeneous. The laws of physics are also believed to be invariant under translations of spacetime (conservation of momentum and energy). If we allow translations of the origin we obtain affine or inhomogeneous transformations. The full symmetric group is the Lorentz group plus translation and is $IO(1,3)$, or the Poincaré group. We will be interested in $ISO^+(1,3)$, the proper orthochronous Poincaré group. In a slight abuse of notation, we will refer to $ISO^+(1,3)$ as the Poincaré group hereafter.

We will represent an element of $IO(1,3)$ as a pair (Λ, a) consisting of a homogeneous Lorentz transformation Λ and a displacement 4-vector a. A matrix is always a linear (homogeneous) transformation so one may doubt the ability to represent an inhomogeneous transformation as a matrix. However, there is a standard trick for doing this by adding on an "extra dimension". The Lorentz transformation is a 4×4 matrix and we may represent the inhomogeneous

[1] Just as the usual orthogonal group $O(3)$ leaves the standard Euclidean metric in 3-dimensions invariant.

transformation as a 5×5 matrix as

$$(\Lambda, a)v = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda v + a \\ 1 \end{pmatrix}.$$

The extra 1 is just along for the ride.

We now consider the product of two transformations: (Λ_1, a_1) followed by (Λ_2, a_2)

$$\begin{aligned} (\Lambda_2, a_2)(\Lambda_1, a_1) &= \begin{pmatrix} \Lambda_2 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & a_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Lambda_2 \Lambda_1 & \Lambda_2 a_1 + a_2 \\ 0 & 1 \end{pmatrix} \\ &= (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2), \end{aligned}$$

which is quite understandable. In particular, we have $(I, a)(\Lambda, 0) = (\Lambda, a)$, and on the other hand $(\Lambda, 0)(I, b) = (\Lambda, \Lambda b)$. Now, if we set $\Lambda b = a$ (or $a = \Lambda^{-1} b$) then we get

$$(I, a)(\Lambda, 0) = (\Lambda, a) = (\Lambda, 0)(I, \Lambda^{-1} a),$$

which will be important later on.

4 Representations

While we usually think of a group as a particular collection of matrices, this is really a *representation* of a group rather than the group itself, which is an abstract object. A *representation* of a group G is a mapping of G into a set of matrices which preserves group multiplication. In other words, if g and h are group elements and $M(g)$ is the matrix representing g, then for M to furnish a representation we must have $M(gh) = M(g)M(h)$, where we have group multiplication on the left and matrix multiplication on the right. It is important to make this distinction between groups and their representations since groups have many different representations and it is through these various representations that groups enter into physics.

This observation is of central importance for field theories. If a physical theory is invariant under some group, the fields must transform under some respresentation of that group. For exmaple, if a theory is rotationally invariant, then the new state obtained by rotating some intitial state must be expressible as some linear combination of basis states. Different fields transform under different representations, so knowing the possible representations tells us something about the possible fields. Since physical theories are invariant under the Poincaré group we will study its representations.

A few familiar examples are the transformation rules for scalars and tensors. For a scalar $\phi \to 1\phi$. That is, it transforms trivially under a group G, or transforms under the trivial representation (everything in the group goes to 1). Another example is a tensor, $T^{ab} \to \Lambda^c{}_a \Lambda^d{}_b T^{ab}$, which transforms under the direct product representation (each Λ is matrix representing a Lorentz transformation). However, there are more general representations than these, and determining all representations of a group is generally a very difficult problem.

Two constraints on this problem must be mentioned. The representations we are after are for the quantum fields. Since inner products between states correspond to certain obervables which must be invariant, we require representation matrices to preserve them:

$$\langle\psi'|\,\psi'\rangle = \langle\psi|\,U^\dagger U\,|\psi\rangle = \langle\psi|\,\psi\rangle \to U^\dagger U = 1,$$

which requires the matrices to by unitary. Second, we requires that the representations be irreducible. This requires some comment.

Irreducibility roughly means that the representation cannot be broken down into smaller simpler parts. More specifically, if the representation (matrices) acts on a vector space V then the operators leave no non-trivial subspace of V invariant. If it did leave a subspace U invariant it would leave its orthogonal complement invariant[2] as well and we could simplify the representation by looking at U and $U^\perp$ separately. As matrices the representation would look like

$$\Gamma(g) = \begin{pmatrix} \Gamma^1(g) & 0 \\ 0 & \Gamma^2(g) \end{pmatrix},$$

which is completely reducible in the matrix sense. We have two different representations stuck together in a trivial way - the subspaces U and $U^\perp$ are completely independent. Since one can always put representations together in this trivial way, it makes sense to restrict attention to those that cannot be so trivially decomposed.

The irreducible representations are the simplest ones, and since any (unitary) representation is either fully reducible or irreducible, all representations are constructable from the irreducible ones. We may think of elementary particles as corresponding to irreducible representations and composite particles to the reducible ones. In summary, our goal is the enumeration of all the unitary irreducible representations (UIR) of the Poincaré group and then determining which belongs to the electromagnetic field.

4.1 The Rotation Group

A particular representation problem is routinely solved in quantum mechanics in the theory of spin and angular momentum. There the problem was to find how quantum states behaved under rotations, that is, the group $SO(3)$. What one finds is that the representations are labeled by j, which is an integer or half-integer and corresponds to the total spin, which is quantized. More specifically, there was a vector space of dimension $2j+1$ with basis states (vectors) $\left|{j \atop m}\right\rangle$ with $|m| \leq j$.

[2] This conclusion requires a metric and is not true for arbitrary non-unitary representations.

In an angular momentum j representation, a rotation for angle $||\boldsymbol{\alpha}||$ about the axis $\boldsymbol{\alpha}$ is represented by the matrix

$$D^j(\boldsymbol{\alpha}) = \exp\left\{-\mathbf{J}^{(j)} \cdot \boldsymbol{\alpha}\right\},$$

where $\mathbf{J}^{(j)}$ is the appropriate collection of angular momentum operators. When $j = 1/2$ they are $i/2$ times the Pauli matrices. We say that the matrices D^j furnish a ($2j + 1$-dimensional) representation of $SO(3)$ and that the states $\left|{}^{j}_{m}\right\rangle$ transform under the D^j representation. We have

$$D^j \left|{}^{j}_{m}\right\rangle = \left|{}^{j}_{m'}\right\rangle D^j_{mm'},$$

which expresses the image of the state $\left|{}^{j}_{m}\right\rangle$ under the rotation as a linear combination of other basis states m' for fixed j.

Of course, not all these states are actually representations of $SO(3)$. For j an integer these states do correspond to $SO(3)$ and may be realized by the spherical harmonics. The spherical harmonics $Y^l_m(\theta, \phi)$ transform under the unitary irreducible representations of $SO(3)$. Note then that the usual representation of $SO(3)$ as 3×3 matrices is reducible. The half-integral j (along with the integral j) are representations of $SU(2)$, the universal cover of $SO(3)$.

Any object which transforms according to integral j is known as a spherical tensor of rank j. This corresponds to the usual scalar and vector in $\mathbb{R}^3$ for $j = 0, 1$. A usual tensor of rank 2 has $1+3+5 = 9$ components and is reducible. It may be regarded as the sum of two vectors (spin 1) and breaks up into a subspaces of spin 0, 1, and 2 with 1,3, and 5 components respectively. These three pieces are the trace, the antisymmetric part, and the traceless symmetric part. This decomposition is accomplished by the identity

$$T_{ij} = \frac{1}{3}T^k_k\delta_{ij} + \frac{1}{2}\left(T_{ij} - T_{ji}\right) + \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}T^k_k\delta_{ij}\right).$$

The factor of $1/3$ compensates for the fact that δ_{ij} has trace three. In the same way an arbitrary manifestly covariant rank n Euclidean tensor reduces into representations with $j = 0, 1, \ldots, n$. For example, rank 3 is obtained from coupling three spin 1's. This is accomplished by

$$1 + (1 + 1) = 1 + (0, 1, 2) = (1), (0, 1, 2), (1, 2, 3),$$

which yields 1 spin 0, 3 spin 1, 2 spin 2, and 1 spin 3 for $1+3\cdot3+5\cdot2+7 = 27$ components.

4.2 Abelian Groups

The Poincaré group contains the inhomogeneous space-time translations which form an abelian (commutative) subgroup. We will discuss the representations of such abelian groups here.

Let Γ be a representation of an abelian group G, $a \to \Gamma(a)$ is the matrix representative for a. We will first show that if the representation is irreducible then the vector space on which Γ acts is one-dimensional[3]. By definition of commutativity we have $\Gamma(a)\Gamma(b) = \Gamma(b)\Gamma(a)$. Now, consider $\Gamma(a)$. This matrix must have at least one eigenvector $|\psi_a\rangle$ with eigenvalue λ_a. But then we have

$$\begin{aligned}\Gamma(a)\Gamma(b)\,|\psi_a\rangle &= \Gamma(b)\Gamma(a)\,|\psi_a\rangle \\ &= \lambda_a\Gamma(b)\,|\psi_a\rangle\,.\end{aligned}$$

We conclude that for every b, $\Gamma(b)\,|\psi_a\rangle$ is an eigenvector of $\Gamma(a)$ with eigenvalue λ_a. Since $\Gamma(c)\Gamma(b)\,|\psi_a\rangle = \Gamma(cb)\,|\psi_a\rangle$, this subspace of eigenvectors is invariant. We know this subspace is not empty, so by irreducibility it must be the whole space and thus we can write $\Gamma(a) = \lambda_a I$.

We repeat this procedure for each $\Gamma(b)$ and corresponding eigenvector $|\psi_b\rangle$. In this way we conclude that every matrix is diagonal. But then every one-dimensional subspace is invariant, and this is a contradiction unless the whole space is one dimensional, which is what we wanted to show.

Since each representation is one dimensional it is a 1×1 matrix or simply a complex number $a \to \Gamma(a) = \exp(ik(a))$ for some complex function k. Now we must determine what k is. We must have (since Γ is a representation)

$$\begin{aligned}&a + b \to \Gamma(a+b) = e^{ik(a+b)} \\ &a + b \to \Gamma(a)\Gamma(b) = e^{ik(a)}e^{ik(b)} = e^{ik(a)+ik(b)},\end{aligned}$$

so that we see that k must be linear, $k(a+b) = k(a) + k(b)$. We can then write $k(a) = k_\mu a^\mu = k \cdot a$, and represent k as a (complex) 4-vector. Finally, for unitary representations we must have

$$1 = \left(e^{ik\cdot a}\right)\left(e^{ik\cdot a}\right)^\dagger = e^{ia\cdot(k-k^*)},$$

so that k_μ must be a real 4-vector. If we label the basis states by $|k\rangle$ we have

$$\Gamma(a)\,|k\rangle = e^{ik\cdot a}\,|k\rangle\,.$$

The transformation properties are trivial since every state is an eigenstate. We conclude that the collection of all UIR of an n parameter abelian (translation) group is in a one-to-one correspondence with the collection of all real n-vectors k, which is isomorphic to the vector space $\mathbb{R}^n$.

4.3 Lorentz Group

Next we shall determine the representations of the proper orthochronous Lorentz Group $SO^+(1,3)$. As we shall see, these representations cannot describe particle states, but rather fields. If we attempt to copy the procedure for the rotation group we are led to consider infinitesimal operations $\Lambda = I + M$, M small, which

[3]This is an easy consequence of Schur's Lemma which we decline to invoke.

must satisfy $\Lambda^t\eta\Lambda = \eta$. This equation gives constraints M to obey $M^t\eta+\eta M = 0$. If we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we have the equation

$$\begin{pmatrix} -A & C^t \\ -B^t & D^t \end{pmatrix} + \begin{pmatrix} -A & -B \\ C & D \end{pmatrix} = 0,$$

which gives the constraints $A = 0$, $B^t = C$, and $D = -D^t$. The most general form of M is then

$$M = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & \theta_3 & -\theta_2 \\ b_2 & -\theta_3 & 0 & \theta_1 \\ b_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} = \boldsymbol{\theta} \cdot J + \mathbf{b} \cdot \mathbf{K},$$

and the infinitesimal operators $\mathbf{J}$ and $\mathbf{K}$ obey the commutation relates

$$\begin{aligned} [J_i, J_j] &= -\epsilon_{ijk} J_k \\ [K_i, K_j] &= +\epsilon_{ijk} J_k \\ [J_i, K_j] &= -\epsilon_{ijk} K_k. \end{aligned}$$

The $\mathbf{J}$ operators are exactly the angular momentum operators of the rotation group and obey the same angular momentum commutation relations. The $\mathbf{K}$ operators are the generators of boosts (pure Lorentz transformations). The commutator of two non-collinear boosts gives rise to a rotation, which is the origin of the Thomas precession. Finally, the commutator of a boost and rotation is a boost.

Since all operators obey very similar commutation relations, one suspects that it would be possible to simplify them by making various linear combinations. This is in fact possible. Define the two sets of operators

$$\begin{aligned} \mathbf{J}^{(1)} &= \frac{1}{2}\left(\mathbf{J} - i\mathbf{K}\right) \\ \mathbf{J}^{(2)} &= \frac{1}{2}\left(\mathbf{J} + i\mathbf{K}\right). \end{aligned}$$

One can immediately verify that these operators mutually commute and obey angular momentum commutation relations:

$$\begin{aligned} \left[J_i^{(1)}, J_j^{(1)}\right] &= -\epsilon_{ijk} J_k^{(1)} \\ \left[J_i^{(2)}, J_j^{(2)}\right] &= -\epsilon_{ijk} J_k^{(2)} \\ \left[J_i^{(1)}, J_j^{(2)}\right] &= 0. \end{aligned}$$

Thus we have have decomposed the operators into two separate angular momentum subspaces[4]. But we already know all these representations! We have $\mathbf{J}^{(1)} \to D^j$ and $\mathbf{J}^{(2)} \to D^{j'}$ with corresponding basis states $\left|{}^{j}_{\mu}\right\rangle$ and $\left|{}^{j'}_{\mu'}\right\rangle$. Since the two are independent we have the product states and representations

$$D^{jj'} = D^j D^{j'} \text{ and } \left|{}^{jj'}_{\mu\mu'}\right\rangle = \left|{}^{j}_{\mu}\right\rangle \left|{}^{j'}_{\mu'}\right\rangle .$$

An arbitrary Lorentz transformation acting on a basis state will be expressible as certain linear combinations of basis states (with same j, j' values)

$$\Lambda \left|{}^{jj'}_{\mu\mu'}\right\rangle = \left|{}^{jj'}_{\nu\nu'}\right\rangle D^{jj'}_{\nu\nu';\mu\mu'},$$

where the linear combinations are the matrix elements

$$D^{jj'}_{\nu\nu';\mu\mu'} = \left\langle {}^{jj'}_{\nu\nu'} \middle| \Lambda \middle| {}^{jj'}_{\mu\mu'} \right\rangle,$$

which need to be determined.

The representations of the operators are given as in the case of the rotation group. We have (since $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ commute)

$$\begin{aligned}
\exp(\theta \cdot \mathbf{J} + \mathbf{b} \cdot \mathbf{K}) &= \exp\left((\theta + i\mathbf{b}) \cdot \mathbf{J}^{(1)} + (\theta - i\mathbf{b}) \cdot \mathbf{J}^{(2)}\right) \\
&= \exp\left((\theta + i\mathbf{b}) \cdot \mathbf{J}^{(1)}\right) \exp\left((\theta - i\mathbf{b}) \cdot \mathbf{J}^{(2)}\right) \\
&= D^j\left((\theta + i\mathbf{b}) \cdot \mathbf{J}^{(1)}\right) D^{j'}\left((\theta - i\mathbf{b}) \cdot \mathbf{J}^{(2)}\right).
\end{aligned}$$

The operator $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$ gives the total angular momentum under rotations $SO(3) \subset SO^+(1,3)$. This is thus the sum of two angular momentum j and j', which breaks up as a Clebsch-Gordon series and includes states of angular momentum J according to

$$\left|{}^{J}_{M}\right\rangle = \sum_{m,m'} \left\langle {}^{j\ j'}_{mm'} \middle| {}^{J}_{M} \right\rangle \left|{}^{j}_{m}\right\rangle \left|{}^{j'}_{m'}\right\rangle, \ |j - j'| \le J \le |j + j'|.$$

For example, a $j = j' = 1/2$ state breaks up into a scalar and a vector under rotations. This representation corresponds to the E&M 4-vector potential which is a pair $(\phi, \mathbf{A})$ in non-relativistic physics. Only states with $j = 0$ or $j' = 0$ are pure spin states. We will accordingly define two particular pure spin states of interest, $D^{j0} = D^j$ and $D^{0j'} = D^{j'}$. The E&M field is a spin-1 field, so that leaves the D^{10} and D^{01} representations. When taking representations the operators $\mathbf{J}^{(i)}$ both go to the appropriate angular momentum matrices which are anti-Hermitian.

Notice that these representations of the Lorentz group we have constructed are not unitary[5]. This is because the angular momentum matrices $J^{(j)}$ are

[4] As Lie algebras we have $\mathfrak{so}(1,3) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

[5] The Lorentz group is non-compact, and it is a theorem that all representations of a non-compact semi-simple group are infinite dimensional.

anti-Hermitian, but their coeffecient has an imaginary part, $i\mathbf{b}$. Thus, there are no physical states that transform under finite dimensional representations of the Lorentz group. These representations correspond to *fields* rather than to particles, since fields need not preserve an inner product. When we consider the full Poincaré group we will recover particle states.

As with the rotation group, the manifestly covariant rank n tensors break up into a sum of irreducibles. The $(1,0)$ and $(0,1)$ representations correspond to a pair of 3-vectors or a single anti-symmetric rank 2 tensor. Either representation of the E&M field is therefore both irreducible and manifestly covariant under Lorentz transformations. Analogous remarks are true for other pure spin j fields.

4.4 The Poincaré Group

In the previous sections we have enumerated the representations of the two major subgroups (translations and Lorentz transformations) that define the Poincaré group. We want to use this knowledge to construct the representations of the Poincaré group itself. We will see that unitary representations are again possible and correspond to particle states of definite mass and spin.

The most sensible thing to try is direct products[6]

$$\left|k; {}^{jj'}_{\mu\mu'}\right\rangle = |k\rangle \left|{}^{jj'}_{\mu\mu'}\right\rangle ,$$

and determine the action of the group elements on these states. We will *define* the action of (I,a) by acting trivially on the spin states,

$$(I,a)\left|k; {}^{jj'}_{\mu\mu'}\right\rangle = e^{ik\cdot a}\left|k; {}^{jj'}_{\mu\mu'}\right\rangle .$$

Having defined the action of (I,a), the action of $(\Lambda,0)$ is determined by the relation $(I,a)(\Lambda,0) = (\Lambda,0)(I,\Lambda^{-1}a)$. Let σ represent the spin labels for brevity. We have (using the Lorentz invariance of the inner product in the third line)

$$\begin{aligned}
(I,a)(\Lambda,0)\,|k,\sigma\rangle &= (\Lambda,0)(I,\Lambda^{-1}a)\,|k,\sigma\rangle \\
&= e^{ik\cdot\Lambda^{-1}a}(\Lambda,0)\,|k,\sigma\rangle \\
&= e^{i\Lambda k\cdot a}(\Lambda,0)\,|k,\sigma\rangle ,
\end{aligned}$$

which demonstrates that $(\Lambda,0)\,|k,\sigma\rangle$ is an eigenstate of (I,a) with eigenvalue $\exp(i\Lambda k\cdot a)$, i.e. it must be a linear combination of states $|\Lambda k,\sigma'\rangle$:

$$(\Lambda,0)\,|k,\sigma\rangle = M_{\sigma',\sigma}\,|\Lambda k,\sigma'\rangle ,$$

for a matrix M to be determined, which satisfies

$$M_{\sigma',\sigma} = \langle\Lambda k,\sigma'|\,\Lambda\,|k,\sigma\rangle .$$

[6] This is essentially the method of induced representations. It turns out that all the representations of the Poincaré group arise in this manor, but a proof is beyond this article.

These product states for pure spin j are manifestly covariant under the Lorentz subgroup, however they are reducible in the Poincaré group! To see why, note that states with momentum Λk are present in the representation whenever states with momentum k are present (just apply the transformation Λ). On the other hand, if a certain value k' is not of the form Λk, then k and k' have no matrix elements between them and this makes the representation reducible.

Naturally, this brings us to consider time-like, space-like, and null vectors separately. Since each Λk is in a representation if k is, it suffices to consider a reference 4-momentum, k_0. Let $\Lambda_0(k)$ be a standard pure boost from k_0 to k. Then we have (for an arbitrary Lorentz transformation Λ)

$$
\begin{aligned}
\Lambda \left|k, \sigma\right\rangle &= \Lambda\Lambda_0(k) \left|k_0, \sigma\right\rangle \\
&= \underbrace{\left[\Lambda_0(\Lambda k)\Lambda_0^{-1}(\Lambda k)\right]}_{1} \Lambda\Lambda_0(k) \left|k_0, \sigma\right\rangle \\
&= \Lambda_0(\Lambda k) \left[\Lambda_0^{-1}(\Lambda k)\Lambda\Lambda_0(k)\right] \left|k_0, \sigma\right\rangle .
\end{aligned}
$$

Note what the bracketed operator, H, on the last line does. We have

$$
k_0 \overset{\Lambda_0(k)}{\to} k \overset{\Lambda}{\to} \Lambda k \overset{\Lambda_0^{-1}(\Lambda k)}{\to} k_0,
$$

so the net effect is to take k_0 to itself, i.e. H is an element of the invariance group of k_0, the so-called "little group". This operation will generally operate non-trivially on the σ (though Λ_0 does not), so we have $H \left|k_0, \sigma\right\rangle = D_{\sigma',\sigma} \left|k_0, \sigma'\right\rangle$, and

$$
\Lambda \left|k, \sigma\right\rangle = D_{\sigma',\sigma}(\Lambda, k_0)\Lambda_0(\Lambda k) \left|k_0, \sigma'\right\rangle ,
$$

where we have to determine the matrix elements $D_{\sigma',\sigma}$. The problem of constructing the representations will be simplified once we identify the little group.

4.4.1 Massive Particles

First, consider the time-like case, $-k \cdot k = m^2 > 0$. A good choice of standard 4-momentum is the rest frame, $k_0 = (\pm 1, 0, 0, 0)$, where we have two choices related by time reversal. Now, any boost introduces time-dilation and changes the time component by $k_t \to \gamma k_t$ for any k. Thus only rotations are allowed ($\gamma = 1$). However, it is obvious that all rotations leave invariant the unit time-like vector since it has no spatial part. Hence the little group of k_0 is $SO(3)$, the rotation group in 3-space.

Finding the representations for massive particles is now reduced to finding the representations of $SO(3)$, which is equivalent to angular momentum states for various spin. We conclude that massive particles are the states $\left|k, {}^{j}_{m}\right\rangle$ of 4-momentum k and spin j, subject to the constraint $-k \cdot k = m^2 > 0$. Since the representations of $SO(3)$ are unitary, we see that we have obtained unitary irreducible particle states. We do not pursue massive states any further here.

4.4.2 Massless Particles

The case of primary interest is when k is null, $k \cdot k = 0$. There is no rest frame, so the next best thing is to pick a definite propagation direction, say z, so that $k_0 = (\pm 1, 0, 0, 1)$. As before there are two choices related by time reversal. The little group in less obvious in this case.

It is at least obvious that any rotation about the z-axis leaves k_0 invariant since such a transformation preserves both the z- and t-axis. These operations are the most important in what follows. However, it is not obvious is whether there are any other transformations. To determine the full little group, we pass to infinitesimal operations. We have

$$\Lambda k_0 = (I + M)k_0 = k_0 \to M k_0 = 0,$$

so that

$$M = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & \theta_3 & -\theta_2 \\ b_2 & -\theta_3 & 0 & \theta_1 \\ b_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \begin{pmatrix} \pm 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_3 \\ -\theta_2 \pm b_1 \\ \theta_1 \pm b_2 \\ \pm b_3 \end{pmatrix} = 0,$$

which gives the relations among the parameters. We notice that the two sets of matrices

$$M = \begin{pmatrix} 0 & \pm\theta_2 & \mp\theta_1 & 0 \\ \pm\theta_2 & 0 & \theta_3 & -\theta_2 \\ \mp\theta_1 & -\theta_3 & 0 & \theta_1 \\ 0 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ 0 & 0 & 0 \end{pmatrix},$$

have the same commutation relations. The latter set is easily identifiable as infinitesimal operations of $ISO(2)$, the group of rotations and translation in the Euclidean plane. The rotation subgroup here correspondence to the z-axis rotations we already determined should be in the little group. The little group can also be determined directly by examining the group operations, but this is much more algebraicly demanding.

Now, finding the representations of $ISO(2)$ is quite similar to finding those of $ISO^+(1,3)$. We have a 2-dimensional Abelian subgroup of translations that gives rise to basis states $|\kappa\rangle$. There are the two cases $\kappa \cdot \kappa > 0$ and $\kappa \cdot \kappa = 0$ (that is, $\kappa = 0$). If κ is a state in an UIR, so is $\kappa' = R(\theta)\kappa$ since $\kappa' \cdot \kappa' = \kappa \cdot \kappa$, where $R(\theta)$ is a rotation by angle θ, i.e. an element of $SO(2)$.

The first case gives rise to an infinite number of internal degrees of freedom, referred to as either "continuous" or "infinite" spin. While there are theories that incorporate these representations, there are no known particles with such internal continuous degrees of freedom, so we will ignore them[7].

This leaves $\kappa = 0$. The little group in this case is the 2-dimensional rotation group $SO(2)$ (rotations preserve the origin). This an Abelian group so the UIR are one-dimensional and of the form

$$\Gamma(\phi)\,|\xi\rangle = e^{i\phi\xi}\,|\xi\rangle\,,$$

[7]It is an important and deep question to ask why such representation do not seem to exist.

where ϕ is the rotation angle and $\xi \in \mathbb{R}$ labels the state. Since ϕ and $\phi + 2\pi$ are the same rotation, their representations must be the same, which requires that ξ be an integer. However, just as $SO(3)$ leads to half-spin representations (representations of $SU(2)$), we expect half-spin representations of $SO(2)$ as well (representations of $U(1)$). Thus we allow ξ to be an integer or half-integer[8]. In this case ϕ and $\phi + 4\pi$ are the same rotation. Of course, these representations are unitary.

Thus we have for a little-group operation H

$$H\left|\xi\right\rangle = e^{i\phi\xi}\left|\xi\right\rangle,$$

and our little group representations are simply

$$D_{\xi',\xi} = e^{i\phi\xi}\delta_{\xi'\xi}.$$

This index has a natural interpretation as helicity - the spin along (or against) direction of propagation. This is because 1) it is a spin index and 2) it is left invariant by rotations about the direction of motion (about the z-axis). We conclude that the UIR of massless particles are the states $|k, \xi\rangle$, where k is a null 4-momentum and ξ is a integer of half-integer helicity index. Any physical state is an arbitrary linear superposition of these basis states. Next we shall see why the covariant states are not all allowable - they include non-helicity states, and only helicity states correspond to particle states.

5 Maxwell's Equations

We are now nearly ready to demonstrate the origin of Maxwell's equations. We will fix k_0 to be the forward time vector. Note that we have described the UIR of massless particles in terms of a helicity index. We want to relate these states to the manifestly covariant (classical) representations of the Poincaré group. Consider the action of a little group operation in the D^{j0} representation, using the relations $b_1 = \theta_2$ and $b_2 = -\theta_1$:

$$\begin{aligned}
\exp\left[(\theta - ib)\mathbf{J}\right] &= \exp\left[(\theta_3 J_3 + (\theta_1 - i\theta_2)J_1 + (\theta_2 + i\theta_1)J_2\right] \\
&= \exp\left[(\theta_3 J_3 + (\theta_1 - i\theta_2)(J_1 + iJ_2)\right] \\
&= \exp\left[(\theta_3 J_3 + (\theta_1 - i\theta_2)(J_+)\right] \\
&= \begin{pmatrix} e^{ij\theta_3} & * & \cdots & * \\ 0 & e^{i(j-1)\theta_3} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-ij\theta_3} \end{pmatrix}
\end{aligned}$$

In the spin basis, the operator J_3 is diagonal with eigenvalues the angular momentum m-values and exponentiates onto the diagonal while the shift up

[8]This makes sense since we are essentially finding the representations of the complexification $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{so}^+(1,3)_\mathbb{C}$, just as $\mathfrak{su}(2) \simeq \mathfrak{so}(3)_\mathbb{C}$.

operator J_+ is upper triangular and exponentiates into the upper triangular part. The details of the $*$ stuff is unimportant. While most of the states transform in a complex manor, for the $\left|\begin{smallmatrix} j0 \\ j0 \end{smallmatrix}\right\rangle$ state we have simply

$$D^{j0}\left[(\theta - ib)\mathbf{J}\right]\left|\begin{smallmatrix} j0 \\ j0 \end{smallmatrix}\right\rangle = e^{ij\theta_3}\left|\begin{smallmatrix} j0 \\ j0 \end{smallmatrix}\right\rangle,$$

since it sees only the 1-1 matrix element. This state is therefore an eigenstate and has the same transformation properties as the helicity state $|\xi\rangle$ for $j = \xi > 0$ if we identify $\theta_3 = \phi$.

An analogous calculation in the D^{0j} representation gives

$$\begin{aligned}\exp\left[(\theta + ib)\mathbf{J}\right] &= \exp\left[(\theta_3 J_3 + (\theta_1 + i\theta_2)(J_-)\right] \\ &= \begin{pmatrix} e^{ij'\theta_3} & 0 & \cdots & 0 \\ * & e^{i(j'-1)\theta_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & e^{-ij'\theta_3} \end{pmatrix}.\end{aligned}$$

This time the operator J_- appears and exponentiates into the lower triangular part. In this case the only simply transforming state is $\left|\begin{smallmatrix} 0 & j' \\ 0 & -j' \end{smallmatrix}\right\rangle$ for which we get

$$D^{0j}\left[(\theta + ib)\mathbf{J}\right]\left|\begin{smallmatrix} 0 & j' \\ 0 & -j' \end{smallmatrix}\right\rangle = e^{-ij\theta_3}\left|\begin{smallmatrix} 0 & j' \\ 0 & -j' \end{smallmatrix}\right\rangle,$$

which transforms the same as the helicity state $|\xi\rangle$ for $-j' = \xi < 0$.

In summary we have the physical states

$$|k,\xi\rangle = \left|k, \begin{smallmatrix} j0 \\ j0 \end{smallmatrix}\right\rangle \text{ for } \xi > 0, \text{ and } |k,\xi\rangle = \left|k, \begin{smallmatrix} 0 & j' \\ 0 & -j' \end{smallmatrix}\right\rangle \text{ for } \xi < 0,$$

Any other non-helicity spin state in the covariant representation is therefore unphysical and must be suppressed. For the D^{j0} representations this requires eliminating all j, m states with $m \neq j$. Another way of saying this is that any physical state $|\psi\rangle$ should have no amplitudes in unallowed basis states, that is

$$\left\langle k, \begin{smallmatrix} j & 0 \\ m & 0 \end{smallmatrix}\right| \psi\rangle = 0, \text{ and } \left\langle k, \begin{smallmatrix} 0 & j' \\ 0 & -m' \end{smallmatrix}\right| \psi\Big\rangle = 0,$$

when $m \neq j$ and $m' \neq j'$ for some physical state $|\psi\rangle$.

We seek a way to eliminate these states at any 4-momentum. We can obtain the helicity through the operator $\mathbf{J}\cdot\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector of the spatial part of k. This operators gives the projection of angular momentum onto the direction of motion. If this is not j we want the amplitude to vanish, so a simple way to enforce this condition is

$$\left\{\mathbf{J}\cdot\hat{\mathbf{k}} - jI\right\}\left\langle k, \begin{smallmatrix} j & 0 \\ m & 0 \end{smallmatrix}\right| \psi\rangle = 0,$$

where I is the unit matrix. Notice that the operator is identically zero for a helicity state, so a physical state can have an arbitrary amplitude there. On the other hand, a non helicity amplitude is forced to vanish. Note that since $k \cdot k = 0$ we have $||\mathbf{k}||^2 = k_1^2 + k_2^2 + k_3^2 = k_t^2$, so we may rewrite our condition as

$$\{\mathbf{J} \cdot \mathbf{k} - jIk_t\} \left\langle k, {}^{j\,0}_{m\,0} \right| \psi \rangle = 0,$$

where we multiplied through by $||\mathbf{k}||$. The analogous condition for the other helicity differs only in the sign of the eigenvalue j,

$$\{\mathbf{J} \cdot \mathbf{k} + jIk_t\} \left\langle k, {}^{0\,j'}_{0\,m'} \middle| \psi \right\rangle = 0.$$

Finally, we obtain our equations by taking the the Fourier transform to return to spacetime coordinates,

$$\mathscr{F} \{\mathbf{J} \cdot \mathbf{k} - jIk_t\} \mathscr{F} \left\langle k, {}^{j\,0}_{m\,0} \right| \psi \rangle = 0.$$

Define the (complex) fields in spacetime by

$$\psi^j_m(x) = \mathscr{F} \left\langle k, {}^{j\,0}_{m\,0} \right| \psi \rangle = \langle x | k \rangle \left\langle k, {}^{j\,0}_{m\,0} \right| \psi \rangle .$$

The Fourier transform of the operator is

$$\mathscr{F} \{\mathbf{J} \cdot \mathbf{k} - jIk_t\} = -i \left\{ \mathbf{J} \cdot \nabla - j \frac{\partial}{\partial t} I \right\} .$$

For a spin-1 field we set $j = 1$ and the angular momentum matrices are (in the 1,0,-1 basis)

$$J_1 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; J_3 = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

Using these spin matrices the constraint equation is

$$\begin{pmatrix} \partial_3 - \partial_t & \partial_{-1} & 0 \\ \partial_{+1} & -\partial_t & \partial_{-1} \\ 0 & \partial_{+1} & -\partial_3 - \partial_t \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = 0,$$

where $\partial_{\pm 1} = \mp(\partial_x \pm i\partial_y)/\sqrt{2}$. After making a similarity transformation to a Cartesian basis the angular momentum matrices become

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

and the constraint becomes

$$\begin{pmatrix} -i\partial_t & \partial_z & -\partial_y \\ -\partial_z & -i\partial_t & \partial_x \\ \partial_y & -\partial_x & -i\partial_t \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} = 0,$$

where $\psi_0 = \psi_z$ and $\psi_{\pm 1} = \mp(\psi_x \pm i\psi_y)/\sqrt{2}$. Upon contemplation of the matrix product one sees that this may be written as

$$\nabla \times \psi = -i\frac{\partial}{\partial t}\psi,$$

where we write $\psi = (\psi_x, \psi_y, \psi_z)$. The negative helicity solution is analogous

$$\nabla \times \bar{\phi} = +i\frac{\partial}{\partial t}\bar{\phi},$$

where ϕ^j_m are the complex fields corresponding to negative helicity states. By considing time reversal one can show that $\psi^* = \phi$.

If we separate out the real and complex parts by setting $\psi = \mathbf{B} + i\mathbf{E}$ we obtain (after multiplying through by i)

$$\begin{aligned}\nabla \times (\mathbf{E} - i\mathbf{B}) &= -i\frac{\partial}{\partial t}(\mathbf{E} - i\mathbf{B}) \\ \nabla \times (\mathbf{E} + i\mathbf{B}) &= +i\frac{\partial}{\partial t}(\mathbf{E} + i\mathbf{B}),\end{aligned}$$

which are the equations of left- and right- circularly polarized light, respectively. Taking the real and imaginary parts of each equation gives the same set of two equations

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= +\frac{\partial \mathbf{E}}{\partial t},\end{aligned}$$

which are the sourceless Faraday and Ampère-Maxwell laws respectively.

What about the other two Maxwell equations? Recall that in the little-vector frame the only non-vanishing component of the field was $\psi_1 = (-\psi_x + i\psi_y)/\sqrt{2}$. Since the little vector was $k^0 = (1, 0, 0, 1)$, we see that ψ_1 is orthogonal to the spatial vector $\mathbf{k} = (0, 0, 1)$, $\mathbf{k} \cdot \psi_1 = 0$. We also have $k \cdot \psi = 0$ as 4-vectors, which is manifestly invariant. Taking the Fourier transform we obtain

$$k \cdot \psi = 0 \rightarrow \nabla \cdot \psi(x) = \nabla \cdot (\mathbf{B} + i\mathbf{E}) = 0.$$

Upon taking real and imaginary parts we have the sourceless Gauß laws. These two equations are rather auxiliary and arise from taking spatial vector representations of spin states rather than remaining in the spin representations. Indeed, half-spin fields do not have such representations and thus lack these auxiliary equations.

Notice that since we have null vectors, $k \cdot k = 0$, so any physical state obeys $\{k \cdot k\} \langle k, \sigma | \psi\rangle = 0$, which upon Fourier Transform yields the Klein-Gordon equation, $\Box^2\psi = 0$, otherwise known as the spin-1 wave equation.

It may seem arbitrary that we wrote $\psi = \mathbf{B} + i\mathbf{E}$ and in some sense it is. In the absence of sources the phase transformation $\mathbf{B} + i\mathbf{E} \rightarrow e^{i\varphi}(\mathbf{B} + i\mathbf{E})$

leaves the Maxwell equations invariant. Only in the presence of sources is this transformation eliminated (unless the sources are similarly transformed). One can decide between $\mathbf{B} + i\mathbf{E}$ and the complex conjugate $\mathbf{B} - i\mathbf{E}$ by the signs in the resulting equations.

6 Conclusion

In the previous section we succeeded in our goal of deriving the Maxwell Equations from quantum mechanics through group theoretical means. These equations are necessary to remove the superfluous states that exist in a classical covariant description of a field. The quantum fields obey no constraints are simply given as arbitrary superpositions of basis states. These states are simply given by construction the UIR of the invariance group of physics, the Poincaré group.

A Some Other Spins

Though our main purpose was to derive the Maxwell equations, the apparatus is in place to derive equations for any massless representation of the Ponicaré group. This is done by writing the constraint equation

$$-i\left\{\mathbf{J}\cdot\nabla - j\frac{\partial}{\partial t}I\right\}\psi_m^j = 0,$$

for the appropriate spin representation j. We list here some common representations and the fields they represent, but do not derive the corresponding equations.

Table 1: Some UIR of the Lorentz group.

(j, j')	field type (example)
$(0, 0)$	scalar (inflaton)
$(1/2, 0)$	left-handed Weyl spinor (neutrinos)
$(0, 1/2)$	right-handed Weyl spinor (anti-neutrinos)
$(1/2, 0) \oplus (0, 1/2)$	Dirac spinor (?)
$(1/2, 1/2)$	four-vector (E&M vector potential)
$(1, 0) \oplus (0, 1)$	tensor (E&M field)
$(1, 1)$	spin 2 field (metric tensor)
$(2, 0) \oplus (0, 2)$	tensor (linearized gravitational field)

The total spin is $j + j'$. A direct sum state $(j, 0) \oplus (0, j)$ has indefinite parity. I am unaware of any massless Dirac spinor field in nature. A massive $(1/2, 0) \oplus (0, 1/2)$ Dirac state would correspond to the leptons. The identification of neutrinos with massless Weyl spinors is according to electroweak theory in the standard model and is now known to be false. Neutrinos are actually massive Dirac spinors.

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