Master Analytic Representation for A_1

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Abstract

The Master Analytic Representation for the root space A_1 is constructed. This gives all of the unitary irreducible representations of the two real forms of this root space, $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. This procedure is carried out in a generalization of Schwinger's presentation for angular momentum.

Consider the lie algebra $\mathfrak{su}(2)$. In the defining representation, this algebra consists of all traceless, anti-Hermitian complex matrices. An arbitrary element thus has the form

$$\mathfrak{su}(2) \overset{\mathfrak{def}}{\rightarrow} \frac{i}{2} \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}, \tag{1}$$

where $a_i \in \mathbb{R}$. A lie algebra can be placed in a canonical form by considering the eigen-equation $\mathrm{ad}_Z X = \lambda X$, with characteristic equation $\|\mathrm{ad}_Z - \lambda I\| = 0$. For $\mathfrak{su}(2)$ this eigen-equation is

$$\lambda^2 + \phi_2(a) = 0, \tag{2}$$

where $\phi_2(a) = \langle a, a\rangle$. This equation has no solutions over $\mathbb{R}$, so we are led to consider the complexification of $\mathfrak{su}(2)$, which is $\mathfrak{su}(2)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})$.

The complexification of an algebra $\mathfrak{g}$ is the algebra $\mathfrak{g} + i\mathfrak{g}$. The lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of all traceless complex matrices. Such a matrix can be decomposed uniquely as

$$\begin{aligned} X &= \frac{X - X^\dagger}{2} + \frac{X + X^\dagger}{2} && (3)\\ &= \frac{X - X^\dagger}{2} + i\frac{X + X^\dagger}{2i} && (4)\\ &= X_1 + iX_2, && (5) \end{aligned}$$

which is a sum of two anti-Hermitian traceless matrices, that is elements of $\mathfrak{su}(2)$.

Having complexified the algebra, we may solve for the roots as $\lambda = \pm i||a||$, which gives one non-zero root and its negative. We note that the secular equation had one independent coefficient, ϕ_2, so the algebra has rank $l = 1$, thus

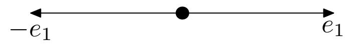

Figure 1: A_1 root space diagram.

there is one zero root, which is missing since we used $\mathfrak{def}$ rather than $\mathfrak{reg}$. The root space for $\mathfrak{sl}(2,\mathbb{C})$ is shown in Fig. 1.

The eigenspaces spanned by $H, E_{\pm 1}$ are found to be

$$\begin{aligned} H &= X_a/\sqrt{2} &(6)\\ E_1 &= X_b/\sqrt{2} &(7)\\ E_{-1} &= X_c/\sqrt{2}, &(8)\end{aligned}$$

where the X's are the generators in the defining representation of $\mathfrak{sl}(2,\mathbb{C})$,

$$\mathfrak{sl}(2,\mathbb{C}) \stackrel{\mathfrak{def}}{\rightarrow} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{9}$$

In the eigenspace decomposition, the Cartan-Killing inner product has the form

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{array}{l} H \\ E_1 \\ E_{-1} \end{array} \end{array}, \tag{10}$$

which can be diagonalized by introducing the linear combinations $E_\pm = (E_1 \pm E_{-1})/\sqrt{2}$,

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{array}{l} H \\ E_+ \\ E_- \end{array} \end{array}. \tag{11}$$

The matrices $E_\pm$ are represented by

$$E_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}. \tag{12}$$

We will now proceed to enumerate the real forms of $\mathfrak{sl}(2,\mathbb{C})$, or of the root space A_1. These are all the sub-algebras $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}_\mathbb{C} = \mathfrak{h} + i\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{g}$. Such a sub-algebra obviously satisfies the following commutation relations:

$$\begin{aligned} [\mathfrak{h},\mathfrak{h}] &\subset \mathfrak{h} &(13)\\ [\mathfrak{h},i\mathfrak{h}] &\subset i\mathfrak{h} &(14)\\ [i\mathfrak{h},i\mathfrak{h}] &\subset \mathfrak{h}. &(15)\end{aligned}$$

The compact form of $\mathfrak{g}$ is given by taking

$$\mathfrak{h} = \mathrm{span}_\mathbb{R}(iH \oplus iE_+ \oplus E_-). \tag{16}$$

Since these generators satisfy the commutation relations

$$[H, E_\pm] = E_\mp \tag{17}$$
$$[E_\pm, E_\mp] = \mp H, \tag{18}$$

it is easily checked that $\mathfrak{h}$ closes under commutation and is in fact a sub-algebra (which is always the case for the compact form). The Cartan-Killing form is now negative definite, hence this sub-algebra is compact. The basis vectors are

$$iH = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sqrt{2}X_3 \tag{19}$$
$$iE_+ = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{2}X_1 \tag{20}$$
$$E_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sqrt{2}X_2, \tag{21}$$

where the X_i are the generators of the defining representation of $\mathfrak{su}(2)$, hence the compact real form of $\mathfrak{sl}(2, \mathbb{C})$ is $\mathfrak{su}(2)$.

Having found the compact form, we can proceed to enumerate the remaining real forms by decomposing the compact form as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h}$ and $\mathfrak{p}$ obey the same commutation relations as $\mathfrak{h}$ and $i\mathfrak{h}$ in eqns 13-15. In particular $\mathfrak{h}$ must form a sub-algebra of $\mathfrak{g}$. In the present case it is simple enough to check the possibilities. One choice is $\mathfrak{h} = H$ and $\mathfrak{p} = E_+ \oplus E_-$ (meaning the span of the these elements). We obtain a real form by the analytic continuation $\mathfrak{g} = \mathfrak{h} + \mathfrak{p} \to \mathfrak{h} + i\mathfrak{p} = \mathfrak{g}'$. These new basis vectors are then

$$iH = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sqrt{2}X_3 \tag{22}$$
$$i^2E_+ = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{2}iX_1 \tag{23}$$
$$iE_- = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sqrt{2}iX_2, \tag{24}$$

which are the generators of $\mathfrak{su}(1, 1)$, as an element in the algebra is written as

$$\frac{1}{2}\begin{pmatrix} ia_3 & -a_1 + ia_2 \\ -a_1 - ia_2 & -ia_3 \end{pmatrix} \tag{25}$$

Now, suppose $E_- \in \mathfrak{h}$. Since $[E_-, H] \in E_-$ we must have $H \in \mathfrak{h}$. Now, since $[E_+, E_-] \in H$, we must have $E_- \in \mathfrak{h}$ as well, but then $\mathfrak{p} = \{0\}$. Similarly, if start with just $E_+ \in \mathfrak{h}$ we must first include H and then E_-. Thus we conclude that the only real forms of $\mathfrak{sl}(2, \mathbb{C})$ are $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$.

Now that we have the two real forms we will proceed to construct the master analytic representation for A_1, which gives all of the unitary irreducible representations (UIR) of its real forms. This construction can be seen as an extension of Schwinger's representation for angular momentum in terms of bilinear products of two mode boson creation and annihilation operators.

We will now write J_i for the $\mathfrak{su}(2)$ generators rather than X_i used previously (actually, we have $iJ_i = X_i$). The generators for $\mathfrak{su}(1,1)$ are related as in eqns 22-24 by $K_{1,2} = iJ_{1,2}$ and $K_3 = J_3$. It is convenient to introduce the linear combinations $J_\pm = J_1 \pm iJ_2$ and $K_\pm = K_1 \pm iK_2$ which satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [K_3, K_\pm] = \pm K_\pm \tag{26}$$
$$[J_+, J_-] = 2J_3 \qquad [K_+, K_-] = -2K_3 \tag{27}$$

The two mode bilinear boson algebra consists of the bilinear product of creation an annihilation operators $a_i^\dagger a_j$, $i, j \in \{1, 2\}$. The basis elements of each mode are $|n_i\rangle$, and the two mode bases are $|n_i\rangle \otimes |n_j\rangle \equiv |n_i, n_j\rangle$. The creation and annihilation operators for mode i act on basis states as

$$a_i^\dagger \left|n_i\right\rangle = \sqrt{n_i + 1}\left|n_i + 1\right\rangle, \quad a_i \left|n_i\right\rangle = \sqrt{n_i}\left|n_i - 1\right\rangle. \tag{28}$$

It is evident that if we make the identifications

$$\begin{aligned} \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) &= J_3 = K_3 && (29)\\ a_1^\dagger a_2 &= J_+ = -iK_+ && (30)\\ a_2^\dagger a_1 &= J_- = -iK_-, && (31) \end{aligned}$$

then we have algebra isomorphisms, as all commutators are identically satisfied. We will also identify basis states according to

$$\left| \begin{array}{c} j \\ m \end{array} \right\rangle = \left|n_1, n_2\right\rangle, \tag{32}$$

with $j = (n_1 + n_2)/2$ and $m = (n_1 - n_2)/2$.

In their defining representations, the lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ exponentiate onto the groups $SU(2)$ and $SU(1,1)$, both of which have a single-valuedness condition for 4π rotations around the z-axis, that is $\exp i\phi(\pm 1/2) = 1$ and $\phi = 4\pi n$. Since this representation is faithful, all representations must obey this same identification, thus an element in the algebra $i\phi J_3$, J_3 diagonal, will exponentiate onto a group operation

$$\begin{pmatrix} e^{im_1\phi} & & & \\ & e^{im_2\phi} & & \\ & & \ddots & \\ & & & e^{im_n\phi} \end{pmatrix}. \tag{33}$$

and we must have $m_j 4\pi = (2m_j)2\pi$, or $2m_j \in \mathbb{Z}$. The same conclusion follows from considering K_3.

What's more, if we consider the commutation relations for the $J_\pm$ generators we have (for eigenstates)

$$\begin{aligned}
[J_3, J_\pm] |m\rangle &= \pm J_\pm |m\rangle & (34)\\
J_3 J_\pm |m\rangle - J_\pm J_3 |m\rangle &= \pm J_\pm |m\rangle & (35)\\
J_3 J_\pm |m\rangle - m J_\pm |m\rangle &= \pm J_\pm |m\rangle & (36)\\
J_3 J_\pm |m\rangle &= (m \pm 1) J_\pm |m\rangle\,, & (37)
\end{aligned}$$

which shows that $J_\pm |m\rangle$ is an eigenstates with eigenvalue shifted by one. Thus the spectrum of m values consists of integer of half-integer values, where two consecutive eigenvalues differ by exactly one. Since the representations are assumed irreducible, there can be no 'gaps' in the spectrum. Specifically, the m spectrum in any particular representation is

$$S = \{m_0 + n,\, 2m_0, n \in \mathbb{Z},\, -\infty \le n_0 < n < n_1 \le \infty\}. \tag{38}$$

For the matrix elements with $J_\pm$ in the photon representation we have

$$\begin{aligned}
\langle n_1', n_2'| J_\pm |n_1, n_2\rangle &= \left\langle n_1', n_2' \middle| \begin{array}{c} a_1^\dagger a_2 \\ a_2^\dagger a_1 \end{array} \middle| n_1, n_2 \right\rangle & (39)\\
&= \left\langle n_1', n_2' \middle| \begin{array}{c} \sqrt{n_1+1}\sqrt{n_2} \\ \sqrt{n_2+1}\sqrt{n_1} \end{array} \middle| \begin{array}{c} n_{1+1}, n_2 \\ n_1, n_{2+1} \end{array} \right\rangle & (40)\\
\left\langle \begin{array}{c} j' \\ m' \end{array} \middle| J_\pm \middle| \begin{array}{c} j \\ m \end{array} \right\rangle &= \left\langle \begin{array}{c} j' \\ m' \end{array} \middle| \sqrt{(j \pm m + 1)(j \mp m)} \middle| \begin{array}{c} j \\ m \pm 1 \end{array} \right\rangle & (41)\\
&= \sqrt{(j \pm m + 1)(j \mp m)}\delta^{j',j}\delta_{m', m\pm 1}. & (42)
\end{aligned}$$

The corresponding expectation value for $K_\pm$ is found by inserting a factor of i.

To construct a unitary representation we want to the group operation $\exp i\theta \cdot J$ to be a unitary operator, that is

$$\begin{aligned}
(\exp i\theta \cdot J)^{-1} &= (\exp i\theta \cdot J)^\dagger & (43)\\
\exp -i\theta \cdot J &= \exp -i\theta \cdot J^\dagger & (44)\\
J &= J^\dagger, & (45)
\end{aligned}$$

that is, J must be Hermitian. J_3 is diagonal, thus manifestly Hermitian. The non-trivial requirement is for J_1 and J_2. We can write this requirement in terms of the ladder operators as

$$J_\pm^\dagger = J_1^\dagger \pm (iJ_2)^\dagger = J_1 \mp iJ_y = J_\mp. \tag{46}$$

This relates the expectation values as

$$\begin{aligned}
\langle n_1 + 1, n_2 - 1| J_+ |n_1, n_2\rangle &= \sqrt{n_2(n_1+1)} & (47)\\
\downarrow \dagger \quad & \quad \downarrow \dagger & (48)\\
\langle n_1, n_2| J_- |n_1 + 1, n_2 - 1\rangle &= \sqrt{n_2(n_1+1)}, & (49)
\end{aligned}$$

where each row is the actual expectation value, but the two rows must be equal be taking the adjoint, thus the n's must satisfy

$$\sqrt{n_2(n_1+1)} = \sqrt{n_2(n_1+1)}^*, \tag{50}$$

or the expectation must be real.

If we consider the $n_i \in \mathbb{R}$ then this is satisfied exactly when the expression under the root is non-negative, or when both factors have the same sign, thus

$$\begin{array}{ccc} n_2 & \geq & 0 \\ n_1 & \geq & -1 \end{array} \quad \text{or} \quad \begin{array}{ccc} n_2 & \leq & 0 \\ n_1 & \leq & -1 \end{array}. \tag{51}$$

The same considerations apply with the expectation values

$$\begin{aligned} \langle n_1-1, n_2+1 | \, J_- \, | n_1, n_2 \rangle &= \sqrt{n_1(n_2+1)} && (52) \\ \downarrow \dagger \quad & \quad \downarrow \dagger && (53) \\ \langle n_1, n_2 | \, J_+ \, | n_1-1, n_2+1 \rangle &= \sqrt{n_1(n_2+1)}, && (54) \end{aligned}$$

which gives the analogous inequalities

$$\begin{array}{ccc} n_1 & \geq & 0 \\ n_2 & \geq & -1 \end{array} \quad \text{or} \quad \begin{array}{ccc} n_1 & \leq & 0 \\ n_2 & \leq & -1 \end{array}. \tag{55}$$

These inequalities are best illustrated on the $n_1 n_2$-plane as in Fig. 2. The commonly shaded regions, I and III, are the allowed regions by Hermiticity.

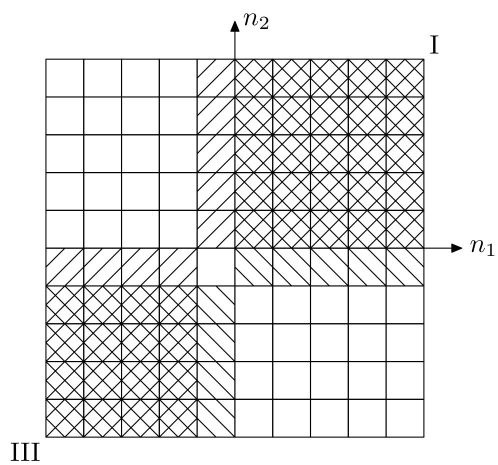

Figure 2: Hermiticity inequalities for $\mathfrak{su}(2)$.

The action of $J_\pm$ on states is to move them on diagonal lines in the plane, conserving $n_1 + n_2$. In order for Hermiticity to be enforced, a state cannot be shifted outside of the quadrants I or III. This is automatically enforced for any state lying on a boundary defined by the ground state of either $n_i = 0$ in I or $n_i = -1$ in III, since the appropriate shift operator will annihilate such a state. These are the only states annihilated by the ladder operators, thus the only allowable spectra are those that include such a boundary state as only these guarantee that the group representation remains unitary.

Since both boundaries are reachable by shifting, a spectrum must contain states with both $n_i = 0$ in I and $n_i = -1$ in III. Since both n's shift by ± 1, the allow spectrum thus requires the n_i be integers for $\mathfrak{su}(2)$. Thus, the UIR are given by two classes of discrete series

$$2j + 1 \in \mathbb{N}, \qquad m = 0, \pm 1, \ldots, \pm |j| \tag{56}$$
$$2j' + 1 \in \mathbb{Z}^-, \qquad m' = 0, \pm 1, \ldots, \pm |j' + 1|. \tag{57}$$

However, these two series are not independent. We can map a j' representation into the j representation with the same dimensionality by $j = -j' - 1$ and $m' = m$. Then the operators $J_\pm, J_3$ all have exactly the same spectra, and thus this map is an isomorphism of representations. For example,

$$\begin{aligned} \left\langle \begin{array}{c} -j'-1 \\ m' \pm 1 \end{array} \middle| J_\pm \middle| \begin{array}{c} -j'-1 \\ m' \end{array} \right\rangle &= \sqrt{((-j'-1) \mp m')((-j'-1) \pm m' + 1)} \\ &= \sqrt{(j \mp m)(j \pm m + 1)} \\ &= \left\langle \begin{array}{c} j \\ m \pm 1 \end{array} \middle| J_\pm \middle| \begin{array}{c} j \\ m \end{array} \right\rangle \end{aligned} \tag{58}$$

Thus, we conclude that all of the UIR are in fact given by the discrete series

$$D^j : 2j + 1 \in \mathbb{N}, \quad m = 0, \pm 1, \ldots, \pm |j|. \tag{59}$$

We now turn to the representations of $\mathfrak{su}(1,1)$. As the single-valuedness conditions must apply here as well, we again have a discrete, contiguous integer or half-integer spectrum for m-values. We again want the representations to be unitary, which again requires that $K_\pm^\dagger = K_\mp$. As before, we use this property in the matrix elements to deduce the relations

$$i\sqrt{n_2(n_1+1)} = (i\sqrt{n_2(n_1+1)})^* \tag{60}$$
$$i\sqrt{n_1(n_2+1)} = (i\sqrt{n_1(n_2+1)})^*, \tag{61}$$

which requires, again, that the matrix elements be real, but in this case requires that the square root be pure imaginary, so that the quantity under the square root must be a non-positive integer. This leads to the two sets of inequalities

$$\begin{array}{rcl} n_2 & \geq & 0 \\ n_1 & \leq & -1 \end{array} \quad \text{or} \quad \begin{array}{rcl} n_2 & \leq & 0 \\ n_1 & \geq & -1 \end{array} \tag{62}$$

and

$$\begin{array}{rcl} n_1 & \geq & 0 \\ n_2 & \leq & -1 \end{array} \quad \text{or} \quad \begin{array}{rcl} n_1 & \leq & 0 \\ n_2 & \geq & -1 \end{array}. \tag{63}$$

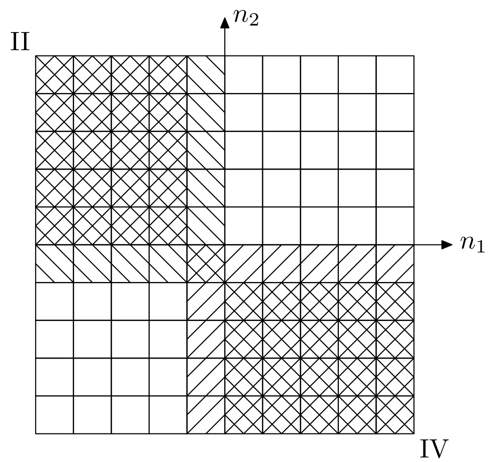

Figure 3: Hermiticity inequalities for $\mathfrak{su}(1,1)$.

These inequalities are also best represented on the n_1n_2-plane in Fig. 3 and show the quadrants II and IV, as well as the central square region.

We again require that Hermiticity be enforced as states are moved by the ladder operations, which is only satisfied in regions II and IV if an edge state is included. For example, the upper edge of IV is defined by $n_2 = -1$, thus we must include the state $|n_1, -1\rangle$, but since $2m = (n_1 - n_2) \in \mathbb{Z}$, we have $n_1 = 2m - 1 \in \mathbb{Z}$ as well, and we again have states defined on the lattice sites. Thus the total j is given by $2j = n_1 - 1$ or $2j + 1 = n_1 \in \mathbb{N}$. The first allowed value of m is given by $m = (n_1 + 1)/2 = ((2j + 1) + 1)/2 = (j + 1)$. Finally, there is no outer edge to this series, so we have a discrete series bounded below:

$$D^j_+ : 2j + 1 \in \mathbb{N}, \qquad m = |j| + 1, |j| + 2, \ldots \tag{64}$$

Nearly the same considerations applies for the left edge of IV, defined by $n_1 = 0$. This gives rise to the same series D^+_j, but with $2j = n_2 \in \mathbb{Z}^-$. This time the first m value is given by $m = -n_2/2 = -j$, which gives

$$D^j_+ : 2j \in \mathbb{Z}^- \qquad m = |j|, |j| + 1, \ldots \tag{65}$$

Finally, if we apply this procedure in II we obtain a discrete series that is bounded above. The right border is defined by $n_1 = -1$ and the bottom by $n_2 = 0$. In the first case we get $2j + 1 = n_2 \in \mathbb{N}$ with first m value $-j - 1$. In the second case we get $2j = n_1 \in \mathbb{Z}^-$ with first m value j. The two series are

$$D^j_- : 2j + 1 \in \mathbb{N} \qquad m = -|j| - 1, -|j| - 2, \ldots \tag{66}$$

$$D^j_- : 2j + 1 \in \mathbb{Z}^- \qquad m = -|j|, -|j| - 1, \ldots \tag{67}$$

Now consider the central box defined by $-1 \leq n_1, n_2, \leq 0$. States in this square are allowable by Hermiticity. However, single-valuedness requires that

$2m = n_1 - n_2 \in \mathbb{Z}$, or $n_2 = n_1 + \mathbb{Z}$, which can only be satisfied in this unit-square if $n_1 = n_2$, that is, along the diagonal. There are no further conditions since a state starting in the central region will never leave the allowed Hermiticity region. This gives the 'complementary series'

$$D^j : -1 \leq 2j+1 \leq 1, \qquad m \in \mathbb{Z}. \tag{68}$$

We can further attempt to relax the n_i to make them complex. The matrix elements of the shift operators can be written

$$\begin{aligned}
\left\langle \begin{matrix} j \\ m \pm 1 \end{matrix} \middle| K_\pm \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle &= i\sqrt{(j \mp m)(j \pm m + 1)} && (69)\\
&= i\sqrt{j^2 + j - m^2 \mp m} && (70)\\
&= i\sqrt{(j^2 + j + 1/4) - (m^2 \pm m + 1/4)} && (71)\\
&= i\sqrt{(j+1/2)^2 - (m \pm 1/2)^2} && (72)\\
&= \sqrt{(m \pm 1/2)^2 - (j+1/2)^2}, && (73)
\end{aligned}$$

which must be real. If we write $j + 1/2 = i\beta$, $\beta \in \mathbb{R}$, then

$$\left\langle \begin{matrix} j \\ m \pm 1 \end{matrix} \middle| K_\pm \middle| \begin{matrix} j \\ m \end{matrix} \right\rangle = \sqrt{(m \pm 1/2)^2 + \beta^2}, \tag{74}$$

which always satisfies the constraint. Since the m values must be integers, it's easy to invert this argument and show that this is the only allowable choice for complex j. These series must also live in the central 'square' and require $n_1 = n_2$ as before, thus $j = n_1 = n_2 = 1/2 + i\beta$ in that square. This gives rise to the 'principle series'

$$D^j : 2j+1 = i\beta, \qquad m \in \mathbb{Z}. \tag{75}$$

As in the $\mathfrak{su}(2)$ case there are equivalences between some of these representations. It is apparent from our previous calculations that the matrix elements are invariant under the simultaneous replacement $j' = j - 1$ and $m' = m$. This may be regarded as the mapping $(n_1, n_2) \to (-n_1 - 1, -n_2 - 1)$ which is an inversion about the line $n_2 = -n_1 - 1/2$. We can then write the discrete series as

$$\begin{array}{lll} \text{discrete}+: & 2j+1 \in \mathbb{N}, & m = j+1, j+2, \ldots \\ \text{discrete}-: & 2j+1 \in \mathbb{N}, & m = -j+1, -j+2, \ldots \end{array} \tag{76}$$

The complementary series reduces to

$$\text{complementary}: \quad 0 \leq 2j+1 \leq 1, \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{77}$$

but the principle series remains unchanged because it lies on the inversion axis. There is however some redundancy in the principle series as the matrix elements depend only on $(j+1/2)^2 = -\beta^2$, so that $\pm\beta$ gives rise to the same representations with the same values of m. Finally, there is an accidental degeneracy when $\beta = 0$ and this representation is equivalent to one in the complementary series.

In summary we have the following four series of representations of $\mathfrak{su}(1,1)$

$$\begin{array}{rll} \text{discrete}+: & 2j+1 \in \mathbb{N}, & m = j+1, j+2, \ldots \\ \text{discrete}-: & 2j+1 \in \mathbb{N}, & m = -j+1, -j+2, \ldots \\ \text{complementary}: & 0 \leq 2j+1 \leq 1, & m = 0, \pm 1, \pm 2, \ldots \\ \text{principle}: & 2j+1 = i\beta, \beta \geq 0, & m = 0, \pm 1, \pm 2, \ldots \end{array} \tag{78}$$

which are represented together in Fig. 4.

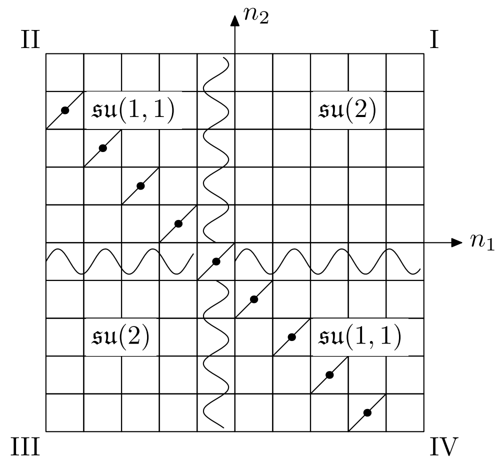

Figure 4: All UIR of the real forms of A_1. The point marked with a dot denote the principle series. The diagonal lines are the complementary series. The lattice points in the four quadrants are the discrete series. The wavy lines are the 'no-mans-land' which separates the $\mathfrak{su}(2)$ from $\mathfrak{su}(1,1)$.

This concludes the master analytic representation of A_1, which consists of all the UIR of the real forms of $\mathfrak{sl}(2,\mathbb{C})$, which are $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. These representations were constructed through the Schwinger representation of angular momentum in a lattice representing bilinear products of two photon operators. By enforcing the Hermiticity conditions on the operators (the group operations are unitary), all the UIR were constructed.