Lie Groups, Physics, and Geometry Solutions: Chapter 2

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Problem 2.1 *Construct the analytic mapping $\phi(x, y)$ for the parametrization of $SL(2;\mathbb{R})$ given in Fig 2.2.*

With an eye to 2.3 we see that we can write such a matrix as

$$M = SR,$$

where S is real, symmetric, and unimodular, and R is a rotation matrix. We want to represent the symmetric matrix as a point on the hyperboloid $z^2 - x^2 - y^2 = 1$, which we can do as follows. For S we have the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

and the unimodular condition is $ac - b^2 = 1$. We can identify this condition with the hyperboloid condition by $ac - b^2 = z^2 - x^2 - y^2$, which we can satisfy by setting $b = y$, and thus $ac = z^2 - x^2 = (z - x)(z + x)$, so that we have $a = z - x$ and $c = z + x$. But, by choosing the upper sheet up of the hyperboloid, $z = +\sqrt{1 + x^2 + y^2}$ is unique. Thus a general matrix looks like

$$\begin{bmatrix} \sqrt{1+x^2+y^2} - x & y \\ y & \sqrt{1+x^2+y^2} + x \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Now, we ϕ such that

$$\begin{array}{ccccc} g(x', y', \theta') & \circ & g(x, y, \theta) & = & g(\phi(x, y, \theta; x', y', \theta')) \\ S & \circ & S' & = & S''. \end{array}$$

...I suppose this is doable with the Caley-Hamilton theorem as in problem 2.

Problem 2.2 *Construct the inversion mapping for the parametrization of $SL(2;\mathbb{R})$ given in Fig 2.2. show that*

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = - \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}.$$

We want to find S' and O' such that $(SO)^{-1} = S'O'$. We know from the next problem that S' can be found as

$$S' = \sqrt{(SO)^{-1}((SO)^{-1})^t},$$

which is generally non-trivial to do. However, we can streamline this computation using the Caley-Hamilton theorem, that every square matrix satisfies its characteristic equation. If M is a 2×2 matrix, then

$$M^2 - (\mathrm{tr\,M})\mathrm{M} + (\det \mathrm{M})\mathrm{I}_2 = 0,$$

since the characteristic equation of a 2×2 matrix can always be written in terms of the trace and determinant. If $M = A^2$, then

$$\begin{aligned} (\mathrm{tr\,M})\mathrm{M} &= A + \sqrt{\det A} I_2 \\ (\mathrm{tr\,M})^2 &= \mathrm{tr\,A} + 2\sqrt{\det \mathrm{A}}. \end{aligned}$$

In the present case our matrices are unimodular and we have easily that

$$M = \sqrt{A} = \frac{A + I_2}{\sqrt{2 + \mathrm{tr\,A}}},$$

which is remarkably simple.

In our case we have

$$S = \begin{bmatrix} x - z & y \\ y & z + x \end{bmatrix} \quad R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

with simple inverses (since they're unimodular)

$$S^{-1} = \begin{bmatrix} x - z & -y \\ -y & z + x \end{bmatrix} \quad R^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then the matrix we wish to compute the square root of is

$$A = O^{-1}S^{-1}(O^{-1}S^{-1})^t =$$

$$\begin{bmatrix} 4z(y\cos\theta\sin\theta - x\cos^2\theta) + 2z(z+x) - 1 & -2z(2x\cos\theta\sin\theta + 2y\cos^2\theta - y) \\ -2z(2x\cos\theta\sin\theta + 2y\cos^2\theta - y) & -4z(y\cos\theta\sin\theta - x\cos^2\theta) + 2z(z-x) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2z(y\sin 2\theta - x\cos 2\theta) + 2z^2 - 1 & -2z(x\sin 2\theta + y\cos 2\theta) \\ -2z(x\sin 2\theta + y\cos 2\theta) & -2z(y\sin 2\theta - x\cos 2\theta) + 2z^2 - 1 \end{bmatrix}.$$

Where we have omitted the necessary algebra (which Maple helped with a lot). The trace is apparently $4z^2 - 2$, so that $\sqrt{2 + \mathrm{tr}} = 2z$, and our square root is

$$\begin{bmatrix} z - (x\cos 2\theta - y\sin 2\theta) & -(x\sin 2\theta + y\cos 2\theta) \\ -(x\sin 2\theta + y\cos 2\theta) & z + (x\cos 2\theta - y\sin 2\theta) \end{bmatrix},$$

and at this point a visual inspection is sufficient to show the transformation is as advertised.

Problem 2.3 *Convince yourself that every matrix M in the group $SL(n;\mathbb{R})$ can be written as the product of an $n \times n$ real symmetric unimodular matrix S and an orthogonal matrix $O \in SO(n)$: $M = SO$. Devise an algorithm for constructing these matrices. Show that $S = \sqrt{MM^t}$ and $O = S^{-1}M$. How do you compute the square root of a matrix? Show that O is compact while S and M are not compact.*

From M we can form symmetric matrix $A = MM^t$. It is indeed symmetric since

$$A^t = (MM^t)^t = (M^t)^t M^t = MM^t = A.$$

Moreover, A is positive definite. For $z \in \mathbb{C}$ we have

$$\begin{aligned} \bar{z}Az &= \bar{z}(MM^t)z \\ &= (\bar{z}M)(M^t z) \\ &= (\bar{z}M)(\bar{z}M)^\dagger \\ &= |\bar{z}M|^2 \\ &\geq 0. \end{aligned}$$

Now, since A is symmetric it is diagonalizable, that is, there exists a matrix P such that $D = P^{-1}AP$ is diagonal. Moreover, since A is positive definite, all the eigenvalues are positive, thus we can take form the square root of D by taking the square root of the diagonal entries, which will all be real (and taken to be positive). Thus we can take define the square root of A by

$$\sqrt{A} \equiv P\sqrt{D}P^{-1},$$

indeed we have

$$\begin{aligned} \sqrt{A}\sqrt{A} &= P\sqrt{D}P^{-1}P\sqrt{D}P^{-1} \\ &= P\sqrt{D}\sqrt{D}P^{-1} \\ &= PDP^{-1} \\ &= A. \end{aligned}$$

From the definition of A we can then write

$$\begin{aligned} M &= A(M^t)^{-1} \\ &= \left(\sqrt{A}\right)\left(\sqrt{A}(M^t)^{-1}\right) \\ &\equiv SO, \end{aligned}$$

where $S = \sqrt{A}$ is unimodular and symmetric, and we just need to verify that $O = \sqrt{A}(M^t)^{-1} = S^{-1}M$ is special orthogonal. On the one hand we have

$$\begin{aligned} O^t &= (S^{-1}M)^t \\ &= M^t S^{-1}, \end{aligned}$$

while on the other we have

$$\begin{aligned} O^{-1} &= (S(M^t)^{-1})^{-1} \\ &= M^t S^{-1}, \end{aligned}$$

and O is indeed orthogonal. Moreover, it's determinant must be positive since the determinants of both M and S are, and O is special.

We argue that $SO(n)$ is compact in the following way. This group is the group of rotation in $\mathbb{R}^n$, or invariance group of the sphere S^{n-1}. Any group element can be generated by a sequence of rotation about a number of axes, each of which parametrized by a circle. Thus, $SO(n)$ can be thought of as the continuous image of a finite product of circles, which are compact, and thus is compact itself.

Consider the subgroup of $SL(n;\mathbb{R})$ of the form

$$\left[\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right],$$

where $M \in SL(2,\mathbb{R})$ and I is the $(n-2)\times(n-2)$ identity matrix. Then since $SL(2;\mathbb{R})$ is unbounded, so to is $SL(n;\mathbb{R})$. In particular, it fails to be compact. Finally, since $SO(n)$ is compact, the set off all S must be non-compact.

We note that this proof doesn't apply to $n = 1$. Indeed such a matrix is 1×1 and thus equal to $+1$, and obviously compact (and boring).

Problem 2.4 *Construct the most general linear transformation $(x, y, z) \to (x', y', z')$ that leaves invariant the quadratic form $z^2 - x^2 - y^2 = 1$. Show that this linear transformation can be expressed in the form*

$$\left[\begin{array}{c} x' \\ y' \\ z' \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_1}} & a \\ & & b \\ \hline a & b & M_2 \end{array}\right]$$

where the real symmetric matrices M_1 and M_2 satisfy

$$\begin{aligned} M_1^2 &= I_2 + \left[\begin{array}{c} a \\ b \end{array}\right]\left[\begin{array}{cc} a & b \end{array}\right] = \left[\begin{array}{cc} 1+a^2 & ab \\ ba & 1+b^2 \end{array}\right] \text{ and} \\ M_2^2 &= I_1 + \left[\begin{array}{cc} a & b \end{array}\right]\left[\begin{array}{c} a \\ b \end{array}\right] = [1+a^2+b^2] \end{aligned}$$

As in the previous problem we can construct the decomposition

$$M = SO,$$

S real symmetric and O orthogonal. However, 0 may not have det 1, but in this case it is of the form $-O'$, O' having det 1, and we may put the negative with M, thus we assume $O \in SO(3)$. However, O is not arbitrary because it must preserve an indefinite quadratic form and hence metric[1] (when $M = 0$). That is

$$\begin{array}{ccc} x_3^2 - x_1^2 - x_2^2 & = & x^t \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} x \\ \downarrow & & \downarrow \\ {x'}_3^2 - {x'}_1^2 - {x'}_2^2 & = & x'^t \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} x' \\ \| & & \| \\ x_3^2 - x_1^2 - x_2^2 & = & (\Lambda x)^t \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} (\Lambda x) \\ & & \\ x_3^2 - x_1^2 - x_2^2 & = & (x^t)\Lambda^t \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} \Lambda(x), \end{array}$$

and hence

$$\begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda^t \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} \Lambda.$$

Writing O in the form

$$O = \left[\begin{array}{c|c} O_2 & B \\ \hline C & D \end{array}\right],$$

where O_2 is 2×2, we must have (since $O^t = O^{-1}$)

$$\begin{array}{ccc} \left[\begin{array}{c|c} O_2 & B \\ \hline C & D \end{array}\right] \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} & = & \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} O_2 & B \\ \hline C & D \end{array}\right] \\ \left[\begin{array}{c|c} -O_2 & B \\ \hline -C & D \end{array}\right] & = & \left[\begin{array}{c|c} -O_2 & -B \\ \hline C & D \end{array}\right], \end{array}$$

and hence B and C are 0, $D = D^{-1} = 1$, and $O_2 \in SO(2)$. So we have our desired decomposition. It remains to show we can further decompose M as shown.

To do this we force M to satisfy the same metric preserving condition with $O = 0$ (the presence of non-zero O actually gives the same result since it drops from the calculation, but excluding it from the beginning makes for a tidier calculation). We will begin by writing M as indicated in block form. We get

$$\begin{aligned} \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$-I_2$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right] &= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_1}} & a \\ & & b \\ \hline a & b & M_2 \end{array}\right] \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$-I_2$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_1}} & a \\ & & b \\ \hline a & b & M_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_1}} & a \\ & & b \\ \hline a & b & M_2 \end{array}\right] \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$-M_1$}} & -a \\ & & -b \\ \hline a & b & M_2 \end{array}\right] \\ &= \left[\begin{array}{c|c} -M_1^2 + \begin{bmatrix} a \\ b \end{bmatrix} [a\ b] & (NI_2 - M^t) \begin{bmatrix} a \\ b \end{bmatrix} \\ \hline [a\ b](NI_2 - M) & M_2^2 - a^2 - b^2 \end{array}\right], \end{aligned}$$

[1] We note that this is an extra condition on O. In the $O(3, \mathbb{R})$ case O automatically preserves the standard metric, and hence $O(3, \mathbb{R})$ is the group leaving the standard metric invariant.

from which we read off the desired conditions from the main diagonal. With those conditions satisfied, the off diagonal condition (there is only one since the result is symmetric) is automatically satisfied.

Problem 2.5 *Construct the group of linear transformations ($SO(1,1)$) that leaves invariant the quantity $(ct)^2 - x^2$. Compare this with the group of linear transformations ($SO(2)$) that leaves invariant the radius of the circle $x^2 + y^2$.*

We know that rotation matrices ($SO(2)$) leave invariant the standard metric, that is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda,$$

and we wand our new matrices $\bar{\Lambda}$ to satisfy the same equation with the indefinite metric. Now, we can write the new metric by a clever multiplication:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix},$$

Now, if we substitute this expression for I_2 on both sides, and then isolate the new metric on the left by multiplying both sides by the inverse ($i \to -i$) of the complex matrix, we find that our new matrix $\bar{\Lambda}$ is

$$\begin{aligned} \bar{\Lambda} &= \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \Lambda \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cosh(b) & \sinh(b) \\ \sinh(b) & \cosh(b) \end{bmatrix}, \end{aligned}$$

where we have set $\theta = ib$.

Problem 2.6 *Construct the group of linear transformations that leaves invariant the quantity $(ct)^2 - x^2 - y^2 - z^2$. This is the Lorentz group $O(3,1)$. Four disconnected manifolds parametrize this group. These contain the four different group operations*

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the signs $\pm$ are incoherent.

We begin as in 4, decomposing the matrix into the form SO, S symmetric and unimodular, $O \in SO(4)$. Similar reasoning applies to the previous case and we can write O in the form

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(3) \end{bmatrix}.$$

Since this group is connected and conserves parity, it includes diag$(1,1,1,1)$ and not diag$(1,-1,1,1)$, thus we have to analyze S.

With more inspiration from 4 we write S in the form

$$\left[\begin{array}{c|c} a & B \\ \hline B^t & M \end{array}\right],$$

where M is 3×3 and B is a row vector. After doing the multiplications the condition is

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -I_3 \end{array}\right] = \left[\begin{array}{c|c} a^2 - B\cdot B & B(aI_3 - M) \\ \hline [B(aI_3-M)]^t & B\otimes B - M^2 \end{array}\right],$$

written in terms of the inner and outer product. We have immediately that

$$a^2 = 1 + B\cdot B > 0,$$

so that, while a can take different signs, it cannot change signs continuously, so we have our first two disconnected pieces.

The next condition is

$$M^2 = I_3 + B\otimes B,$$

or in terms of determinant,

$$\begin{aligned} \det(M)^2 &= \det(I_3 + B\otimes B) \\ &= 1 + B\cdot B \\ &> 0, \end{aligned}$$

so, while $\det(M)$ can have either sign, it cannot change sign continuously, and here is our other disconnected piece[2].

Problem 2.7 *The group of 2×2 complex matrices with det $+1$ is named $SL(2,\mathbb{C})$. Matrices in this group have the structure $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, where α, β, γ, and δ are complex numbers. Define the matrix X by*

$$X = H(x,y,z,ct) = \begin{bmatrix} ct+z & x-iy \\ x+iy & ct-z \end{bmatrix} = ctI_2 + \sigma\cdot\mathbf{x},$$

where $\mathbf{x} = (x,y,z)$ and $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ are the Pauli spin matrices.

1. *Show that X is hermitian.*
2. *Show that the most general 2×2 hermitian matrix can be written in the form used to construct X.*
3. *If $g\in SL(2,\mathbb{C})$, show that $g^\dagger X g = X' = H(x',y',z',ct')$.*
4. *How are the new space-time coordinates related to the original coordinates?*

[2] We have used special case of the identity $\det(I + AB^t) = \det(1 + B^tA)$

5. *Find the subgroup of $SL(2,\mathbb{C})$ that leaves $t = t'$.*

6. *For any $g \in SL(2,\mathbb{C})$ write $g = kh$, where $h \in SU(2)$, $h^\dagger = h^{-1}$, has has the form $h = \mathrm{EXP}(\frac{i}{2}\sigma\cdot\theta)$ and $k \in SL(2;\mathbb{C})/SU(2)$, $k^\dagger = k$, k has the form $k = \mathrm{EXP}(\frac{1}{2}\sigma \cdot \mathbf{b})$. The three-vector $\mathbf{b}$ is called a boost vector. The three-vectors θ and $\mathbf{b}$ are real. Construct $k^\dagger H(x,y,z,ct)k = H(x',y',z',ct')$.*

7. *Show that the usual Lorentz transformation law results.*

8. *Applying $k(b')$ after applying $k(b)$ results in (a) $k(b{+}b')$, (b) two successive Lorentz transformations. show that the velocity addition law for co-linear boosts results.*

9. *If $\mathbf{b}$ and $\mathbf{b}'$ are not co-linear, $k(\mathbf{b}'k(\mathbf{b}) = k(bfb'')h(\theta)$. Compute $\mathbf{b}''$, θ. The angle θ is related to the Thomas precession.*

1. Since I and σ_i are hermitian, so is X.

2. We can write an arbitrary hermitian matrix as

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & x+iy \\ x-iy & 0 \end{bmatrix},$$

   since the hermitian condition demands the diagonal entries be self-conjugate (real), and the off-diagonal be complex conjugates. However, the pair (a,b) and $(ct+z, ct-z)$ are linearly related, hence this is the most general form.

3. Since $(g^\dagger X g)^\dagger = g^\dagger X g$, it is expressible in the form indicated.

4. Linearly related by coefficients bi-linear in the the matrix elements and their complex conjugates.

5. This is equivalent to having $g^\dagger(ctI_2 + \sigma \cdot x)g = ctI_2 + \sigma \cdot x'$, and thus $g^\dagger g = I_2$, and g is unitary, $g \in SU(2)$.

6. We need to compute

$$\exp\left(\frac{1}{2}\sigma \cdot b\right)\left(ctI_2 + \sigma \cdot x\right)\exp\left(\frac{1}{2}\sigma \cdot b\right),$$

   so we will start with exponentiation the matrix,

$$\exp\left(\frac{1}{2}\sigma \cdot b\right) = \sum \frac{(\sigma \cdot b)^k}{k!2^k}.$$

   One way to do this is expand the power series, collect terms, and hope to recognize the expansions of known functions. As I'm allergic to power series, let's find a better way. By the Caley-Hamilton theorem, any matrix satisfies its characteristic equation, which for a 2×2 matrix is second degree. Thus any equation involving M^n can by reduced until it only involves the identity and M. Thus

$$\exp(M) = f_0 I_2 + f_1 M,$$

where the f_i are unknown functions. Next, a diagonal matrix is easy to exponentiate, so let's try to diagonalize M. Then we have

$$\begin{aligned} \exp(M) &= f_0 I_2 + f_1 M \\ \exp(SDS^{-1}) &= f_0 I_2 + f_1 M \\ S\exp(D)S^{-1} &= f_0 I_2 + f_1 M \\ \exp(D) &= f_0 I_2 + f_1 D, \end{aligned}$$

where $D = S^{-1}MS$ is diagonal. So we can solve for the f_i just by finding the eigenvalues of M (we don't even need the eigenvectors!). The trace vanishes and the determinant is $-|b/2|^2$, thus the eigenvalues are $\pm|b/2|$. Thus we need to solve

$$\begin{pmatrix} e^{|b/2|} & 0 \\ 0 & e^{-|b/2|} \end{pmatrix} = f_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f_1 \begin{pmatrix} |b/2| & 0 \\ 0 & -|b/2| \end{pmatrix},$$

which is the linear system

$$\begin{aligned} e^{|b/2|} &= f_0 + |b/2| f_1 \\ e^{-|b/2|} &= f_0 - |b/2| f_1, \end{aligned}$$

which has solution

$$\begin{aligned} f_0 &= (e^{|b/2|} + e^{-|b/2|})/2 = \cosh|b/2| \\ f_1 &= (e^{|b/2|} - e^{-|b/2|})/2|b/2| = \sinh|b/2|/|b/2|, \end{aligned}$$

thus we have

$$\exp\left(\frac{1}{2}\sigma\cdot b\right) = I_2\cosh|b/2| + (\sigma\cdot\hat{b})\sinh|b/2|,$$

where the hat denotes the unit vector $\hat{b} = b/|b|$.

Going back to compute the action of this operator on either side of H, we will first check its action on ctI_2:

$$\begin{aligned} &= ct\left(I_2\cosh|b/2| + (\sigma\cdot\hat{b})\sinh|b/2|\right)^2 \\ &= ctI_2\left(\cosh^2|b/2| + \sinh^2|b/2|\right) + 2ct(\sigma\cdot\hat{b})\sinh|b/2|\cosh|b/2| \\ &= ctI_2\cosh|b| + ct(\sigma\cdot\hat{b})\sinh|b|, \end{aligned}$$

since $(\sigma\cdot\hat{b})^2 = 1$.

The action on $\sigma \cdot x$ is more complicated and we'll take it term-by-term. First, the $\cosh^2$ term will simply give

$$\cosh^2 |b/2|(\sigma \cdot x).$$

Next, the two cross-terms will give

$$\begin{aligned}
& \sinh |b/2| \cosh |b/2| \Big((\sigma \cdot x)(\sigma \cdot \hat{b}) + (\sigma \cdot \hat{b})(\sigma \cdot x) \Big) \\
= \; & \sinh |b/2| \cosh |b/2| x^i \hat{b}^j \left(\sigma_i \sigma_j + \sigma_j \sigma_i\right) \\
= \; & 2 \sinh |b/2| \cosh |b/2| x^i \hat{b}^j \delta_{ij} I_2 \\
= \; & \sinh |b| (x \cdot \hat{b}) I_2,
\end{aligned}$$

using the anti-commutation relations of the Pauli matrices. Finally, for the $\sinh^2$ term we get

$$\begin{aligned}
& \sinh^2 |b/2|(\sigma \cdot \hat{b})(\sigma \cdot x)(\sigma \cdot \hat{b}) \\
= \; & \sinh^2 |b/2|(\hat{b}^i x^j \hat{b}^k)(\sigma_i \sigma_j \sigma_k)
\end{aligned}$$

and to unravel the product of matrices will use twice the relation $\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$:

$$\begin{aligned}
\sigma_i \sigma_j \sigma_k &= \sigma_i(\delta_{jk} + i\epsilon_{jks}\sigma_s) \\
&= \delta_{jk}\sigma_i + i\epsilon_{jks}(\sigma_i \sigma_s) \\
&= \delta_{jk}\sigma_i + i\epsilon_{jks}(\delta_{is} + i\epsilon_{isr}\sigma_r) \\
&= \delta_{jk}\sigma_i + i\epsilon_{ijk} - \epsilon_{jks}\epsilon_{sri}\sigma_r \\
&= \delta_{jk}\sigma_i + i\epsilon_{ijk} - (\delta_{jr}\delta_{ki} - \delta_{ji}\delta_{kr})\sigma_r \\
&= \delta_{jk}\sigma_i + \delta_{ji}\sigma_k - \delta_{ki}\sigma_j + i\epsilon_{ijk}.
\end{aligned}$$

Now, inserting this expression we get

$$\begin{aligned}
& \sinh^2 |b/2|(\hat{b}^i x^j \hat{b}^k) \left(\delta_{jk}\sigma_i + \delta_{ji}\sigma_k - \delta_{ki}\sigma_j + i\epsilon_{ijk}\right) \\
= \; & \sinh^2 |b/2| \Big((x \cdot \hat{b})(\sigma \cdot \hat{b}) + (x \cdot \hat{b})(\sigma \cdot \hat{b}) - (\hat{b} \cdot \hat{b})(\sigma \cdot x) + i\hat{b} \cdot (\hat{b} \times x) \Big) \\
= \; & \sinh^2 |b/2|(2(\sigma \cdot \hat{b})(x \cdot \hat{b}) - \sigma \cdot x).
\end{aligned}$$

Thus, altogether this term becomes

$$\begin{aligned}
& \cosh^2 |b/2|(\sigma \cdot x) + \sinh^2 |b/2| \Big(-(\sigma \cdot x) + 2(\sigma \cdot \hat{b})(x \cdot \hat{b}) \Big) + \sinh |b|(x \cdot \hat{b}) I_2 \\
= \; & \sigma \cdot x(\cosh^2 |b/2| - \sinh^2 |b/2|) + 2\sinh^2 |b/2|(\sigma \cdot \hat{b})(x \cdot \hat{b}) + \sinh |b|(x \cdot \hat{b}) I_2 \\
= \; & \sigma \cdot x + (\cosh |b| - 1)(\sigma \cdot \hat{b})(x \cdot \hat{b}) + \sinh |b|(x \cdot \hat{b}) I_2,
\end{aligned}$$

and the transformation is

$$I_2(ct \cosh |b| + (x \cdot \hat{b}) \sinh |b|) + \sigma \cdot \Big(x + \hat{b}\big((x \cdot \hat{b})(\cosh |b| - 1) + ct \sinh |b|\big) \Big).$$

7. We can thus write the transformed ct, x as

$$
\begin{aligned}
\begin{bmatrix} ct \\ x \end{bmatrix} &\rightarrow \begin{bmatrix} ct\cosh|b| + (x\cdot\hat{b})\sinh|b| \\ x + \hat{b}\Big((x\cdot\hat{b})(\cosh|b|-1) + ct\sinh|b|\Big) \end{bmatrix} \\
&= \begin{bmatrix} \cosh|b| & \sinh|b|\hat{b}^t \\ \sinh|b|\hat{b} & I_2 + (\cosh|b|-1)\hat{b}^t\otimes\hat{b} \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}.
\end{aligned}
$$

To make this expression the same as a Lorentz transformation we must have $\cosh|b| = \gamma$ and $\sinh|b| = \gamma\beta$ by comparing the tt and tx matrix elements. But, once these identifications are made, the xx matrix elements are identical since $\hat{b} = \hat{\beta}$.

8. If b and b' are collinear then $\sigma\cdot b$ and $\sigma\cdot b'$ are proportional and thus commute. Thus

$$
\exp(\frac{1}{2}\sigma\cdot b')\cdot\exp(\frac{1}{2}\sigma\cdot b) = \exp(\frac{1}{2}\sigma\cdot(b'+b)) = \exp(\frac{1}{2}\sigma\cdot b'').
$$

With our identifications from the previous part of the problem we get

$$
\begin{aligned}
\tanh(b'') &= \tanh(b'+b) \\
&= \frac{\tanh b' + \tanh b}{1 + \tanh b \tanh b'},
\end{aligned}
$$

from which the addition law is evident.

9. Is this possible in general?

Problem 2.8 *The circumference of the unit circle is mapped into itself under the transformation $\theta \rightarrow \theta' = \theta + k + f(\theta)$, where k is a real number, $0 \leq k \leq 2\pi$, and $f(\theta)$ is periodic, $f(\theta + 2\pi) = f(\theta)$. The mapping must be $1:1$, so an additional condition is imposed on $f(\theta) : df/d\theta > -1$ everywhere. Does this set of transformations form a group? What are the properties of this group?*

We check the composition of two such transformations (which is automatically a diffeomorphism since each transformation is). We have

$$
\begin{aligned}
g_2\circ g_1(\theta) &= g_2(\theta + k_1 + f_1(\theta)) \\
&= \theta + k_1 + f_1(\theta) + k_2 + f_2(\theta + k_1 + f_1(\theta)) \\
&= \theta + (k_1 + k_2) + f_2(\theta + k_1 + f_1(\theta)),
\end{aligned}
$$

and we need only check periodicity of f_2, indeed

$$
\begin{aligned}
f_2(\theta + 2\pi + k_1 + f_1(\theta + 2\pi)) &= f_2(\theta + 2\pi + k_1 + f_1(\theta)) \\
&= f_2(\theta + k_1 + f_1(\theta)),
\end{aligned}
$$

and we're done.

Problem 2.9 *Rational fractional transformations* (a,b,c,d) *map points on the real line (real projective line* RP^1*) to the real line as follows:*

$$x \to x' = (a,b,c,d)x = \frac{ax+b}{cx+d}.$$

The transformations (a,b,c,d) *and* $(\lambda a, \lambda b, \lambda c, \lambda d) = \lambda(a,b,c,d)$, $\lambda \neq 0$ *generate identical mappings.*

1. *Compose two successive rational fractional transformations*

$$(A,B,C,D) = (a', b'c', d')(a,b,c,d),$$

   and show that the composition is a rational fractional transformation. Compute the values of A, B, C, D.

2. *Show that the transformations* $(\lambda, 0, 0, \lambda)$ *map* x *to itself.*

3. *construct the inverse transformation* $x' \to x$, *and show that it is* $\lambda(d, -b, c, a)$ *provided* $\lambda \neq 0$. *Such transformations exist if* $D = ad - bc \neq 0$.

4. *Show that the transformation degeneracy* $x' = (a,b,c,d)x = \lambda(a,b,c,d)x$ *can be lifted by requiring that he four parameters* a, b, c, d *describing these transformations satisfy the constraint* $D = ad - bc = 1$.

5. *It is useful to introduce homogeneous coordinates* (y, z) *and define the real projective coordinate* x *as the ratio of these homogeneous coordinates:* $x = y/z$. *If the homogeneous coordinates transform linearly under* $SL(2, \mathbb{R})$ *then the real projective coordinates* x *transform under rational fractional transformations:*

$$\left[\begin{array}{c} y' \\ z' \end{array}\right] = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{c} y \\ z \end{array}\right] \to x' = \frac{y'}{z'} = \frac{a(y/z)+b}{c(y/z)+d} = \frac{ax+b}{cx+d}.$$

6. *Show that a rational fractional transformation can be constructed that maps three distinct points* x_1, x_2, x_3 *on the real line to the three standard positions* $(0, 1, \infty)$, *and that this mapping is*

$$x \to x' = \frac{(x-x_1)(x_2-x_3)}{(x-x_3)(x_2-x_2)}.$$

   What matrix in $SL(2;\mathbb{R})$ *describes this mapping?*

7. *Use this construction to show that there is a unique mapping of any triple of distinct points* (x_1, x_2, x_3) *to any other triple of distinct points* (x'_1, x'_2, x'_3).

1. The composition is

$$\begin{aligned}\frac{a'\left(\frac{ax+b}{cx+d}\right)+b'}{c'\left(\frac{ax+b}{cx+d}\right)+d'} &= \frac{a'(ax+b)+b'(cx+d)}{c'(ax+b)+d'(cx+d)} \\ &= \frac{(aa'+b'c)x+(a'b+b'd)}{(ac'+cd')x+(bc'+dd')}.\end{aligned}$$

2. We have

$$\frac{ax+b}{cx+d} = \frac{\lambda x+0}{0+\lambda} = x.$$

3. Straightforward algebra gives solves for x it terms of x' as

$$x = \frac{b-dx'}{cx-a},$$

Which is unique up to an overall scale of λ. The books answer is given by $\lambda = -1$ from our expression. We note that we can write a transformation as

$$\frac{ax+b}{cx+d} = \frac{b\left(\frac{ad}{b}x+d\right)}{d(cx+d)},$$

so that if $ad = bc$, the transformation is independent of x and obviously not invertible.

4. By adjusting λ we have $ad-bc \to \lambda^2(ad-bc)$, and the constraint fixes $\lambda = \pm 1$. So we apparently need a further constraint, say making $a \geq 0$.

5. Is there a question here?

6. We can write our conditions as

$$0 = \frac{ax_1+b}{cx_1+d},\ 1 = \frac{ax_2+b}{cx_2+d},\ \infty = \frac{ax_3+b}{cx_3+d},$$

or as

$$ax_1+b=0,\ ax_2+b=cx_2+d,\ cx_3+d=0,$$

which gives us a linear system with solution

$$a = \frac{x_3-x_2}{x_1-x_2}c,\ b = -x_1\frac{x_3-x_2}{x_1-x_2}c,\ d=-x_3c,\ c\in\mathbb{R}.$$

If we choose $c = x_1 - x_2$, we get

$$\frac{(x_3-x_2)x-x_1(x_3-x_2)}{(x_3-x_2)x+x_3(x_3-x_2)} = \frac{(x_3-x_2)(x-x_1)}{(x_3-x_2)(x+x_3)},$$

which agrees with up to $\lambda = -1$. The appropriate matrix is then

$$\begin{bmatrix} \lambda x_3 - \lambda x_2 & \lambda x_1(\lambda x_2 - \lambda x_3) \\ \lambda x_1 - \lambda x_2 & \lambda x_3(\lambda x_2 - \lambda x_1)\end{bmatrix},$$

where λ satisfies

$$\lambda^3 = \frac{1}{(x_3 - x_2)(x_2 - x_1)(x_2 - x_1)}.$$

7. To do this, we first map a triple to the standard positions, and then an appropriate inverse transformation to map from the standard positions to the new triple.

Problem 2.10 *The real projective space RP^n is the space of all straight lines through the origin in $\mathbb{R}^{n+1}$. The group $SL(n+1;\mathbb{R})$ maps $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ to $x' \in \mathbb{R}^{n+1}$, with $x' \neq 0$ iff $x \neq 0$. A straight line through the origin contains $x \neq 0$ and $y \neq 0$ iff $y = \lambda x$ for some $\lambda \in \mathbb{R}$. The scale factor can always be chosen so that y is in the unit sphere S^n in $\mathbb{R}^{n+1}$. In fact, two values of λ can be chosen $\lambda = \pm 1/\left(\sum_{i=1}^{n+1} x_i^2\right)^{1/2}$. In $\mathbb{R}^3$ the straight line containing (x, y, z) can be represented by homogeneous coordinates $(X, Y) = (x/z, y/z)$ if $z \neq 0$. Straight lines through the origin of $\mathbb{R}^3$ are mapped to straight lines in $\mathbb{R}^3$ by $x \to x' = Mx$, $M \in SL(3;\mathbb{R})$. Show that the homogeneous coordinates representing the two lines containing x and x' are related by the linear fractional transformation*

$$\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X' \\ Y' \end{bmatrix} = \frac{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}}{\begin{bmatrix} m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + m_{33}}.$$

Generalize for linear fractional transformations $RP^n \to RP^n$.

Because of the different rôles of x, y and z, we will write $M \in SL(R, \mathbb{R})$ as block diagonal form, so that the transformation becomes

$$\begin{aligned}
\left[\begin{array}{c} x' \\ y' \\ \hline z' \end{array}\right] &= \left[\begin{array}{c|c} A & B \\ \hline C^t & D \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline z \end{array}\right] \\
&= \left[\begin{array}{c} A \begin{bmatrix} x \\ y \end{bmatrix} + Bz \\ \hline C^t \begin{bmatrix} x \\ y \end{bmatrix} + Dz \end{array}\right],
\end{aligned}$$

and we can simply find the fractional transformation by dividing through by z:

$$\begin{aligned}
\begin{bmatrix} x'/z' \\ y'/z' \end{bmatrix} &= \frac{A \begin{bmatrix} x \\ y \end{bmatrix} + Bz}{C^t \begin{bmatrix} x \\ y \end{bmatrix} + Dz} \\
&= \frac{A \begin{bmatrix} x/z \\ y/z \end{bmatrix} + B}{C^t \begin{bmatrix} x/z \\ y/z \end{bmatrix} + D}.
\end{aligned}$$

The generalization is automatic in this block-diagonal form, by replacing $(x, y) \to (x_1, \ldots, x_n)$ and $z \to x_{n+1}$.

Problem 2.11 *The hyperbolic two-space* $SL(2;\mathbb{R})/SO(2) \simeq \left[\begin{array}{cc} z+x & y \\ y & z-x \end{array} \right]$ *consists of the algebraic submanifold in the Minkowski* $2+1$ *dimensional space-time with metric* $(+1, -1, -1)$

$$z^2 - (x^2 + y^2) = 1.$$

This submanifold inherits the metric

$$ds^2 = dz^2 - (dx^2 + dy^2).$$

1. *Show that*

$$\begin{aligned} -ds^2 &= dx^2 + dy^2 - \left(d\sqrt{1 + x^2 + y^2}\right)^2 \\ &= \frac{1}{1 + x^2 + y^2} (dx\ dy) \left[\begin{array}{cc} 1 + y^2 & -xy \\ -yx & 1 + x^2 \end{array} \right] \left(\begin{array}{c} dx \\ dy \end{array} \right). \end{aligned}$$

2. *Introduce polar coordinates* $x = r\cos(\phi),\ y = r\sin(\phi)$*, and show*

$$-ds^2 = \frac{dr^2}{1 + r^2} + (rd\phi)^2.$$

3. *Show that the volume element on this surface is*

$$vol = \frac{rdr\, d\phi}{\sqrt{q + r^2}}.$$

4. *Repeat this calculation for* $SO(3)/SO(2)$*. This space is a sphere* $S^2 \subset \mathbb{R}^3$*: the algebraic manifold in* $\mathbb{R}^3$ *that satisfies* $z^2 + (x^2 + y^2) = 1$ *and inherits the metric* $ds^2 = dz^2 + (dx^2 + dy^2)$ *from this Euclidean space. Show that the metric and measure on* S^2 *are obtained from the results above for* H^2 *by the substitutions* $1 + r^2 \to 1 - r^2$*. Show that the disk* $0 \le r \le 1$*,* $0 \le \phi \le 2\pi$ *maps onto the upper hemisphere of the sphere, with* $r = 0$ *mapping to the north pole and* $r = 1$ *mapping to the equator. Show that the geodesic length from the north pole to the equator along the longitude* $\phi = 0$ *is* $s = \int_0^1 dr/\sqrt{1 - r^2} = \pi/2$ *and the volume of the hemisphere surface is* $V = \int_{r=0}^{r=1} \int_{\phi=0}^{\phi=2\pi} vol = \int_0^1 rdr/\sqrt{1 - r^2} \int_0^{2\pi} d\phi = 2\pi$.

1. The first equality is simply from the condition $z^2 - (x^2 + y^2) = 1$. To write the second we take the derivative of the constraint:

$$2zdz = 2xdx + 2ydy$$

so that

$$
\begin{aligned}
-ds^2 &= dx^2 + dy^2 - dz^2 \\
&= dx^2 + dy^2 - (xdx + ydy)^2/z^2 \\
&= z^{-2}\left((1 + x^2 + y^2)(dx^2 + dy^2) - (x^2dx + y^2dy + xy\,dx\,dy)\right) \\
&= z^{-2}\left((1 + y^2)dx^2 - 2xy\,dx\,dy + (1 + x^2)dy^2\right) \\
&= \frac{1}{1 + x^2 + y^2}(dx\ dy)\begin{bmatrix} 1 + y^2 & -xy \\ -xy & 1 + x^2 \end{bmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}.
\end{aligned}
$$

2. We note that we can relate the differentials by

$$
\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix}\begin{bmatrix} dr \\ d\phi \end{bmatrix},
$$

so that the new matrix is

$$
\begin{aligned}
& \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix}\begin{bmatrix} 1 + y^2 & -xy \\ -xy & 1 + x^2 \end{bmatrix}\begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} \\
= & \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix}\begin{bmatrix} 1 + r^2\sin^2\phi & -r^2\sin\phi\cos\phi \\ -r^2\sin\phi\cos\phi & 1 + r^2\cos^2\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 \\ 0 & r^2(1 + r^2) \end{bmatrix},
\end{aligned}
$$

and so

$$
\begin{aligned}
-ds^2 &= \frac{1}{1 + r^2}(dr\ d\phi)\begin{bmatrix} 1 & 0 \\ 0 & r^2(1 + r^2) \end{bmatrix}\begin{pmatrix} dr \\ d\phi \end{pmatrix} \\
&= \frac{dr^2}{1 + r^2} + (rd\phi)^2.
\end{aligned}
$$

3. In general the volume form is given by

$$
\mathrm{vol} = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n,
$$

where $|g|$ is the determinant of the metric. The determinant of the polar metric is simply $r^2/(1 + r^2)$, so that

$$
\mathrm{vol} = \frac{rdr \wedge d\phi}{\sqrt{1 + r^2}}.
$$

4. Similarly to before we have

$$
\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= dx^2 + dy^2 + (xdx + ydy)^2/z^2 \\
&= z^{-2}\left((1 - x^2 - y^2)(dx^2 + dy^2) + (x^2dx + y^2dy + xy\,dx\,dy)\right) \\
&= z^{-2}\left((1 + y^2)dx^2 + 2xy\,dx\,dy + (1 + x^2)dy^2\right) \\
&= \frac{1}{1 - x^2 - y^2}(dx\ dy)\begin{bmatrix} 1 + y^2 & xy \\ xy & 1 + x^2 \end{bmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}.
\end{aligned}
$$

We note that we can relate the differentials by

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} dr \\ d\phi \end{bmatrix},$$

so that the new matrix is

$$\begin{aligned} & \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{bmatrix} \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} \\ = & \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} 1-r^2\sin^2\phi & r^2\sin\phi\cos\phi \\ r^2\sin\phi\cos\phi & 1-r^2\cos^2\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 \\ 0 & r^2(r^2-1) \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} ds^2 &= \frac{1}{1-r^2}(dr\ d\phi) \begin{bmatrix} 1 & 0 \\ 0 & r^2(r^2-1) \end{bmatrix} \begin{pmatrix} dr \\ d\phi \end{pmatrix} \\ &= \frac{dr^2}{1-r^2} + (rd\phi)^2. \end{aligned}$$

Similarly for the volume element we have

$$\mathrm{vol} = \frac{rdr \wedge d\phi}{\sqrt{1-r^2}}.$$

The appropriate map is

$$(x, y) \mapsto (x, y, \sqrt{1-x^2-y^2}),$$

which in polar is

$$(r, \phi) \mapsto (r, \phi, \sqrt{1-r^2}),$$

so that $(0, \phi) \mapsto (0, \phi, 1)$ and $(1, \phi) \mapsto (1, \phi, 0)$. We want to consider the geodesic along $\phi = 0$, so that

$$\begin{aligned} L &= \int ds \\ &= \int_0^1 \frac{dr}{\sqrt{1-r^2}} \\ &= \int_0^{\pi/2} \frac{d(\sin\theta)}{\sqrt{1-\sin^2\theta}} \\ &= \int_0^{\pi/2} d\theta \\ &= \pi/2. \end{aligned}$$

The volume of the hemisphere is the volume of the volume form in these local coordinates

$$\begin{aligned}\int \text{vol} &= \int_o^1 \int_o^{2\pi} \frac{r dr \wedge d\phi}{\sqrt{1-r^2}} \\ &= \int_o^1 \frac{r dr}{\sqrt{1-r^2}} \int_o^{2\pi} d\phi \\ &= \int_0^{\pi/2} \sin\theta d\theta \int_o^{2\pi} d\phi \\ &= 2\pi.\end{aligned}$$