Cohomology Homework: Chapter 7

Daniel J. Cross

December 13, 2006

Problem 7.1 *Show that $\mathbb{R}^n$ does not contain a subset homeomorphic to D^m when $m > n$.*

Let $\Delta \subseteq \mathbb{R}^n$ and $\phi : D^m \to \Delta$ a homeomorphism with $D^m \subset \mathbb{R}^m$. Then the restriction of ϕ to the interior of D^m is a homeomorphism onto the interior of Δ. But the interior of D^m is homeomorphic to $\mathbb{R}^m$, so we have $\overset{\circ}{\Delta} \cong \mathbb{R}^m$.

However, we can consider the inclusion $\iota : \overset{\circ}{\Delta} \hookrightarrow \mathbb{R}^m$, which maps $\overset{\circ}{\Delta}$ homeomorphically onto the proper subspace $\mathbb{R}^n \subset \mathbb{R}^m$. But this set is homeomorphic to $\mathbb{R}^m$ so must be open by invariance of domain, which is a contradiction since it is a subset of a proper subspace.

Problem 7.2 *Let $\Sigma \subseteq \mathbb{R}^n$ be homeomorphic to S^k $(1 \leq k \leq n-2)$. Show that*

$$H^p(\mathbb{R}^n - \Sigma) \cong \begin{cases} \mathbb{R} & \text{for } p = 0, n-k-1, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 7.8 we have the isomorphisms

$$H^p(\mathbb{R}^n - \Sigma) \cong H^p(\mathbb{R}^n - S^k),$$

so we will compute the latter groups. We will further consider $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$. First we have

$$H^p(\mathbb{R}^n - S^{n-1}) \cong H^p(\overset{\circ}{D^n}) \oplus H^p(\mathbb{R}^n - D^n).$$

The first set is star shaped, so we get $\mathbb{R}$ for $p = 0$ and 0 otherwise, while the second set is homeomorphic to $\mathbb{R}^n - \{0\}$, so we get $\mathbb{R}$ for $p = 0, n-1$ and 0 otherwise, so all together we get

$$H^p(\mathbb{R}^n - S^{n-1}) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & p = 0 \\ \mathbb{R} & p = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Next will iteratively apply Proposition 6.11 to extend this result. First we get

$$H^p(\mathbb{R}^{n+1} - S^{n-1}) \cong \begin{cases} \mathbb{R} & p = 0, 1, n \\ 0 & \text{otherwise,} \end{cases}$$

and more generally,

$$H^p(\mathbb{R}^{n+a} - S^{n-1}) \cong \begin{cases} \mathbb{R} & p = 0, a, n+a-1 \\ 0 & otherwise. \end{cases}$$

In the present case we set $n - 1 = k$ and $m = n + a = n + k - 1$. Thus $a = k - 1$ and $n + a - 1 = (m - a) + a - 1 = m - 1$, so we get

$$H^p(\mathbb{R}^m - S^k) \cong \begin{cases} \mathbb{R} & for\ p = 0, m-k-1, m-1 \\ 0 & otherwise. \end{cases}$$

Problem 7.3 *Show that there is no continuous map $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} \simeq id_{S^{n-1}}$.*

We follow the proof of Lemma 7.2 except that the function g satisfies $g|_{S^{n-1}} \simeq \mathrm{id}_{S^{n-1}}$. Then the function $g(tr(x))$ defines a homotopy between $g(r(x))$ and $g(0)$, a constant map. Thus we have

$$g(0) \simeq g(r(x)) \simeq \mathrm{id} \circ r(x) = r(x),$$

so that we still obtain a homotopy between $r(x)$ and a constant map, so the rest of the argument still holds.

Problem 7.4 *Let $f : D^n \to \mathbb{R}^n$ be a continuous map and let $r \in (0, 1)$ be given. Suppose for all $x \in S^{n-1}$ that $||f(x) - x|| \leq 1 - r$. Show that $f(D^n)$ contains the closed disc with radius r and center 0.*

Suppose the conclusion false, then there exists a point x_0 with $||x_0|| \leq r$ and $x_0 \neq f(x)$ for any $x \in D^n$. As in the Brouwer Fixed Point Theorem, we we wish to define a function $g(x)$ to be the intersection of the half line from x_0 to $f(x)$ with S^{n-1}, which is well-defined as x_0 and $f(x)$ are always distinct. The function $g(x)$ is defined as

$$g(x) = x_0 + t\frac{x_0 - f(x)}{||x_0 - f(x)||},$$

with t given so that $||g(x)|| = 1$, that is

$$t = -x_0 \cdot u + \sqrt{1 - ||x_0||^2 + (x_0 \cdot u)^2},$$

where

$$u = \frac{x_0 - f(x)}{||x_0 - f(x)||},$$

and thus $g(x)$ is continuous.

If we can show that the restriction of $g(x)$ to S^{n-1} is homotopic to the identity then by problem 7.3 we have a contradiction, so that $g(x)$ cannot exist and neither the point x_0. We prove this next.

Let $x \in S^{n-1}$ and let D_x be the solid ball of radius $1 - r$ centered on x, and let D_0 be the solid ball of radius r centered on 0. Then D_0 and D_x intersect in an unique point p, and moreover this intersection defines a $n-1$ dimensional tangent hyperplane in $\mathbb{R}^n \simeq \mathbb{R}^{n-1} \times \mathbb{R}$, and let z be the coordinate function in the direction orthogonal to the hyperplane oriented so that $z(p) < 0$ ($z(p) \neq 0$ since D_0 has radius $r > 0$). Then for every $y \in D_x$, $y \neq p$, $z(y) < z(p) < 0$, in particular $z(x) < 0$ and $z(f(x)) < 0$.

Now, if $x_0 = p$ then $f(x) \neq p$ and $z(x_0) = z(p) > z(f(x))$. Likewise if $x_0 \neq p$ then $z(x_0) > z(p) \geq z(f(x))$. Thus in either case we will have $z(x_0) > z(f(x))$, but $z(f(x)) < 0$, so that $z(g(x)) < 0$ as well. Thus x and $g(x)$ are always on the same side of the hyperplane, so, in particular, they are never antipodal points. Thus by problem 6.4 $g(x)$ is homotopic to the identity on S^{n-1}.

Problem 7.5 *Assume given two injective continuous maps $\alpha, \beta : [0,1] \to D^2$ such that*

$$\begin{aligned} \alpha(0) &= (-1,0), & \alpha(1) &= (1,0), \\ \beta(0) &= (0,-1), & \beta(1) &= (0,1). \end{aligned}$$

Prove that the two curves α and β intersect.

We will use α and β to denote the maps and their images in D^2. α may be in the boundary of D^2 at places but we can assume that $\alpha \cap \beta(1) = \emptyset$. Thus there must exist a neighborhood N of $\beta(1)$ with $B \cap \alpha = \emptyset$. Moreover there are points $t_1, t_2 \in [0,1]$ with the property that $\alpha(t_1), \alpha(t_2) \in S^1$, $\alpha(t) \notin S^1$ for $t_1 < t < t_2$, and $\beta(1)$ is between $\alpha(t_1)$ and $\alpha(t_2)$. We have, at the least, that $t_1 = 0$ and $t_2 = 1$.

We consider S^1 to be parametrized by angle in the usual way and let θ_1 and θ_2 be angles corresponding to the points $\alpha(t_1)$ and $\alpha(t_2)$ respectively. We note that $\theta_1 > \theta_2$.

Now we wish to define a map $\phi : S^1 \to D^2$ by

$$\phi(\theta) = \begin{cases} \alpha\left(T_1 t_2 + (1-T_1)t_1\right) & 0 \leq \theta \leq \theta_2 \\ (\cos(\theta), \sin(\theta)) & \theta_2 \leq \theta \leq \theta_1 \\ \alpha\left(T_2 t_2 + (1-T_2)t_1\right) & \theta_1 \leq \theta \leq 2\pi, \end{cases}$$

where we have

$$\begin{aligned} T_1 &= \frac{\theta - \theta_1 + 2\pi}{\theta_2 - \theta_1 + 2\pi} \\ T_2 &= \frac{\theta - \theta_1}{\theta_2 - \theta_1 + 2\pi}. \end{aligned}$$

This function is certainly piecewise continuous, but it's easy to see that the pieces agree at their points of overlap (including $\phi(0) = \phi(2\pi)$, so ϕ is continuous. Moreover, it is injective, so the image, Σ, of ϕ is homeomorphic to S^1 (I suppose this needs further justification - such as showing that ϕ is an open map, sending open sets to (relatively) open sets in the image).

Thus, by the Jordan-Brouwer Separation Theorem, $\mathbb{R}^n - \Sigma$ has two connected components U_1 and U_2, the former bounded, the latter unbounded, and Σ is their common boundary. Moreover we have that $\Sigma \cap \partial D^2 = \phi(\theta)$ for $\theta_2 \leq \theta \leq \theta_1$.

We have $\beta(1) \in \Sigma$ and $\beta(0) \in U_2$. There is a $t_0 \in (0,1)$ such that for every $t < t_0$, $\beta(t) \cap \Sigma = \emptyset$. We can suppose that $\beta(t_0) \neq \alpha(t_1), \alpha(t_2)$, so that for some $t' < t_0$ we have $\beta(t') \in U_1$. Thus we have a curve from U_1 to U_2 which must intersect the boundary Σ along $\phi(\theta)$ for $0 \leq \theta \leq \theta_1$ or $\theta_1 \leq \theta \leq 2\pi$, that is, it must intersect along α.