Cohomology Homework: Chapters 5 & 6

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Problem 5.3 *Can* $\mathbb{R}^2$ *be written as* $\mathbb{R}^2 = U \cup V$ *where* U *and* V *are open connected sets such that* $U \cap V$ *is disconnected?*

We have part of the Mayer-Vietoris sequence

$$0 \quad\to\quad H^0(U \cup V) \quad\to\quad H^0(U) \oplus H^0(V) \quad\to\quad H^0(U \cap V) \quad\to\quad H^1(U \cup V),$$

which becomes

$$0 \quad\to\quad \mathbb{R} \quad\to\quad \mathbb{R} \oplus \mathbb{R} \quad\xrightarrow{f}\quad H^0(U \cap V) \quad\to\quad 0,$$

since U, V, and $\mathbb{R}^2$ are all connected, and we know $H^1(\mathbb{R}^2) = 0$. This sequence is exact so the map f must be onto, so $\dim H^0(U \cap V) \leq 2$, in particular it is finite. Thus we have

$$\dim \mathbb{R} - \dim \mathbb{R} \oplus \mathbb{R} + \dim H^0(U \cap V) = 0,$$

or $\dim H^0(U \cap V) = 1$, so that $U \cap V$ must be connected. This result holds for each $\mathbb{R}^n$, since their respective cohomology groups are isomorphic.

Problem 5.4 *Suppose* $p \neq q$ *belong to* $\mathbb{R}^n$. *A closed set* $A \subset \mathbb{R}^n$ *is said to separate* p *from* q *when* p *and* q *belong to two different connected components of* $\mathbb{R}^n - A$.

Let A *and* B *be two disjoint closed subsets of* $\mathbb{R}^n$. *Given two distinct points* p *and* q *in* $\mathbb{R}^n - (A \cup B)$, *show that if neither* A *or* B *separates* p *from* q, *then* $A \cup B$ *does not separate* p *from* q.

Denote the open complements by $\tilde{A} = \mathbb{R}^n - A$ and $\tilde{B} = \mathbb{R}^n - B$. We have

$$\tilde{A} \cup \tilde{B} = (\mathbb{R}^n - A) \cup (\mathbb{R}^n - B) = \mathbb{R}^n - (A \cap B) = \mathbb{R}^n.$$

Suppose that both consist of a single connected component. Then we have part of the Mayer-Vietoris sequence

$$0 \quad\to\quad H^0(\tilde{A} \cup \tilde{B}) \quad\to\quad H^0(\tilde{A}) \oplus H^0(\tilde{B}) \quad\to\quad H^0(\tilde{A} \cap \tilde{B}) \quad\to\quad H^1(\tilde{A} \cup \tilde{B}),$$

which becomes

$$0 \quad \to \quad \mathbb{R} \quad \to \quad \mathbb{R} \oplus \mathbb{R} \quad \to \quad H^0(\tilde{A} \cap \tilde{B}) \quad \to \quad 0,$$

which shows that $H^0(\tilde{A} \cap \tilde{B})$ is 1-dimensional and thus consists of a single connected component. Since all the complements consists of a single connected component, none of them separate points which proves the theorem in this case. The full result will follow by showing that it can always be reduced to the present special case.

For more general sets A and B we can write them as a (possibly uncountable) sum over connected components

$$\begin{aligned} A &= \bigcup A_i \\ B &= \bigcup B_j, \end{aligned}$$

and we can write the complement as a (countable) sum over connected components

$$\begin{aligned} \tilde{A} &= \bigcup \tilde{A}_i \\ \tilde{B} &= \bigcup \tilde{B}_j. \end{aligned}$$

Now, since $A \cap B = \emptyset$ each connected component of one must be contained within a connected component of the complement of the other, that is, for each j there exists and i such that

$$\begin{aligned} B_j &\subset \tilde{A}_i \\ A_j &\subset \tilde{B}_i. \end{aligned}$$

Now, suppose that the points p and q are contained in the sets $\tilde{A}_{i_1}$ and $\tilde{B}_{j_1}$ (they are not separated). Then both points are contained in the intersection $\tilde{A}_{i_1} \cap \tilde{B}_{j_1}$. We will now removed the extraneous components of the complements of the sets A and B without changing the relevant intersection of the connected components containing the points.

For every $i \neq i_1$, add the component $\tilde{A}_i$ to A, creating a new closed set which we will continue to call A. This does not change A_{i_1}, so $\tilde{A}_{i_1} \cap \tilde{B}_{j_1}$ remains unchanged. Now, such a component of $\tilde{A}$ may contain a component B_j of B. If so, remove this component from the set B, obtaining a new closed set which will continue to call B. This may change the set $\tilde{B}_{i_1}$, but not its intersection with $\tilde{A}_{i_1}$, since the change is happening in an open set $\tilde{A}_i$, $i \neq i_1$. Repeat this process until $\tilde{A}_{i_1}$ is the only remaining component of the complement of A.

Now we do the same procedure with the components of the complement of B which also doesn't change the desired intersection of sets by the same argument as for A. We continue until the complement has the one remaining component $\tilde{B}_{i_1}$. Then each complement has exactly one component and the result follows from the special case.

Problem 6.1 *Show that "homotopy equivalence" is an equivalence relation in the class of topological spaces.*

First we need to first show that for any topological space X, $X \simeq X$. Let $f = g = \mathrm{id}_X$, then

$$f \circ g = g \circ f = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \simeq \mathrm{id}_X.$$

Next we need to show that if $X \simeq Y$, then $Y \simeq X$, but this is obvious from the definition.

Finally we need to show that if $X \simeq Y$ and $Y \simeq Z$ that $X \simeq Z$. We first need the following result. Suppose we have $f \simeq f'$, then $g \circ f \circ h \simeq g \circ f' \circ h$, where g and h are continuous maps. Let $F(x,t)$ be the homotopy with $F(x,0) = f(x)$ and $F(x,1) = f'(x)$, then define a new homotopy $G(x,t)$ by $G(x,t) = g(x) \circ F(x,t) \circ h(x)$. Then we have $G(x,0) = g(x) \circ F(x,0) \circ h(x) = g(x) \circ f(x) \circ h(x)$ and $G(x,1) = g(x) \circ F(x,1) \circ h(x) = g(x) \circ f'(x) \circ h(x)$. $G(x,t)$ is continuous since it is the composition of continuous maps.

Now, we have the following maps

$$\begin{array}{rclcrcl} f & : & X \to Y & \qquad & g \circ f & \simeq & \mathrm{id}_X \\ g & : & Y \to X & & f \circ g & \simeq & \mathrm{id}_Y \\ f' & : & Y \to Z & & g' \circ f' & \simeq & \mathrm{id}_Y \\ g' & : & Z \to Y & & f' \circ g' & \simeq & \mathrm{id}_Z. \end{array}$$

We will define the maps $f'' : X \to Z$ and $g'' : Z \to X$ by

$$\begin{aligned} f'' &= f' \circ f \\ g'' &= g \circ g', \end{aligned}$$

then we have

$$\begin{aligned} g'' \circ f'' &= (g \circ g') \circ (f' \circ f) \\ &= g \circ (g' \circ f') \circ f \\ &\simeq g \circ \mathrm{id}_Y \circ f \\ &= g \circ f \\ &\simeq \mathrm{id}_X, \end{aligned}$$

$$\begin{aligned} f'' \circ g'' &= (f' \circ f) \circ (g \circ g') \\ &= f' \circ (f \circ g) \circ g' \\ &\simeq f' \circ \mathrm{id}_Y \circ g' \\ &= f' \circ g' \\ &\simeq \mathrm{id}_Z, \end{aligned}$$

where we have used our result above.

Problem 6.2 *Show that all continuous maps $f : U \to V$ that are homotopic to a constant map induce the 0-map $f^* : H^p(V) \to H^p(U)$ for $p > 0$.*

Since homotopic maps induce the same maps on cohomology groups (Thm. 6.8) we need only check the case when f is a constant map. So, let f be the constant map $f(x) = y_0$ for every x. The induced map on cohomology is given by

$$H^p(f) : [\omega] \to [f^*(\omega)],$$

where the induced map on (p-) forms is given by

$$(f^*\omega)_x(\xi_1, \ldots, \xi_p) = \omega_{f(x)}(D_x f(\xi_1), \ldots, D_x f(\xi_p)).$$

Now, if $p = 0$ this reduces to

$$(f^*\omega)_x = \omega_{f(x)},$$

the constant 0-form. However, if $p > 0$, then $D_x = 0$ since the map f is constant, and we have

$$(f^*\omega)_x(\xi_1, \ldots, \xi_p) = \omega_{f(x)}(0, \ldots, 0) = 0.$$

Thus $[f^*(\omega)] = [0] = 0$, so that $H^p(f)$ is the zero map for $p > 0$.

Problem 6.3 *Let $p_1, \ldots, p_k$ be k distinct points in $\mathbb{R}^n$, $n \geq 2$. Show that*

$$H^d(\mathbb{R}^n - \{p_1, \ldots, p_k\}) \cong \begin{cases} \mathbb{R}^k & \text{for } d = n-1 \\ \mathbb{R} & \text{for } d = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We will first take the case $n = 1$ and suppose one point p is missing ($k = 1$). Then we set $U = (-\infty, p)$ and $V = (p, \infty)$, which gives $\mathbb{R} - \{p\} = U \cup V$. The Mayer-Vietoris sequences gives

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U \cap V) & \to & \\ & \to & H^1(U \cup V) & \to & H^1(U) \oplus H^1(V) & \to & H^1(U \cap V) & \to & 0, \end{array}$$

which becomes, since U and V are star-shaped with empty intersection,

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & \mathbb{R} \oplus \mathbb{R} & \to & 0 & \to & \\ & \to & H^1(U \cup V) & \to & 0 & \to & 0 & \to & 0, \end{array}$$

so that $\dim H^0(U \cup V) = 2$ and $\dim H^1(U \cup V) = 0$.

Now we proceed by induction on k to show that $H^1 = 0$ for $\mathbb{R}$ minus a finite number of points (we already know $\dim H^0 = k + 1$, the number of connected components, but we'll get this too).

Let $X = \mathbb{R} - \{p_1, \ldots, p_k\}$. We can write X as the union of $k + 1$ disjoint open intervals separated by the $\{p_i\}$. In particular, we write $X = U \cup V$, where

$$U = (-\infty, p_1) \cup (p_1, p_2) \cup \cdots \cup (p_{k-1}, p_k),$$

and

$$V = (p_k, \infty),$$

where we have assumed without loss of generality that $p_i < p_j$ for $i < j$. But then Mayer Vietoris gives

$$\begin{array}{cccccccc} 0 & \to & H^0(U \cup V) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U \cap V) & \to \\ & \to & H^1(U \cup V) & \to & H^1(U) \oplus H^1(V) & \to & H^1(U \cap V) & \to & 0, \end{array}$$

which becomes

$$\begin{array}{cccccccc} 0 & \to & H^0(U \cup V) & \to & \mathbb{R} \oplus \mathbb{R}^k & \to & 0 & \to \\ & \to & H^1(U \cup V) & \to & 0 & \to & 0 & \to & 0, \end{array}$$

using the induction hypothesis on V which is a union of k disjoint intervals. Now we have the two exact sequences

$$0 \quad \to \quad H^0(U \cup V) \quad \to \quad \mathbb{R} \oplus \mathbb{R}^k \quad \to \quad 0,$$

and

$$0 \quad \to \quad H^1(U \cup V) \quad \to \quad 0.$$

Thus we get that $H^0(U \cup V) \cong \mathbb{R}^{k+1}$ and $H^1(U \cup V) \cong 0$.

Now that we know the cohomology groups of $\mathbb{R}$ minus a finite number of points, we will exploit Prop 6.11 to extend the result to $\mathbb{R}^n$. First, since the number of points is finite we can always find a diffeomorphism taking those points onto the subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, and we can calculate assuming this rearrangement since cohomology groups are diffeomorphism invariants. We will now assume our space to be replaced with this diffeomorphic image. Let A stand for the set up k points removed from the subspace $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$.

First we extend to $\mathbb{R}^2$. Prob 6.11 tells us that

$$\begin{aligned} H^2(\mathbb{R}^2 - A) &\cong H^1(\mathbb{R} - A) \cong 0 \\ H^1(\mathbb{R}^2 - A) &\cong H^0(\mathbb{R} - A)/\mathbb{R} \cong \mathbb{R}^{k+1}/\mathbb{R} \cong \mathbb{R}^k \\ H^0(\mathbb{R}^2 - A) &\cong \mathbb{R}, \end{aligned}$$

which is the intended result. Next we extend to $\mathbb{R}^3$

$$\begin{aligned} H^3(\mathbb{R}^3 - A) &\cong H^2(\mathbb{R}^2 - A) \cong 0 \\ H^2(\mathbb{R}^3 - A) &\cong H^1(\mathbb{R}^2 - A) \cong \mathbb{R}^k \\ H^1(\mathbb{R}^3 - A) &\cong H^0(\mathbb{R}^2 - A)/\mathbb{R} \cong \mathbb{R}/\mathbb{R} \cong 0 \\ H^0(\mathbb{R}^3 - A) &\cong \mathbb{R}, \end{aligned}$$

and at this point the further induction to $\mathbb{R}^n$ is clear: H^0 stays $\mathbb{R}$, H^1 will be $\mathbb{R}/\mathbb{R} \cong 0$, $H^{n-2}(\mathbb{R}^{n-1} - A) \to H^{n-1}(\mathbb{R}^n - A) \cong \mathbb{R}^k$, and the rest stay zero, which establishes the result.

Problem 6.4 *Suppose that* $f, g : X \to S^{n-1}$ *are two continuous maps, such that* $f(x)$ *and* $g(x)$ *are never antipodal. Show that* $f \simeq g$.

Show that every non-surjective map $f : X \to S^{n-1}$ *is homotopic to a constant map.*

Regard the $n-1$ sphere as a subspace of Euclidean space:

$$S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}.$$

We intend to construct a homotopy $F : X \times [0,1] \to S^{n-1}$ between f and g by

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}},$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. We then have

$$F(x,0) = \frac{f(x)}{\sqrt{1 + 2(0)(1)(\langle f(x), g(x)\rangle - 1)}} = f(x),$$

and

$$F(x,1) = \frac{g(x)}{\sqrt{1 + 2(2)(0)(\langle f(x), g(x)\rangle - 1)}} = g(x),$$

so we need to show the map is well defined.

First, we must have the expression under the square root be non-negative. We have

$$\begin{array}{rcccl} -1 & \leq & \langle f(x), g(x)\rangle & \leq & 1 \\ -2 & \leq & \langle f(x), g(x)\rangle - 1 & \leq & 0 \\ -1/2 & \leq & t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 0 \\ -1 & \leq & 2t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 0 \\ 0 & \leq & 1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 1, \end{array}$$

where the 3rd line follows since $0 \leq t(1-t) \leq 1/4$, since $0 \leq t \leq 1$. Thus we only have trouble when the expression is 0, which is when $\langle f(x), g(x)\rangle = -1$, that is, when $f(x)$ and $g(x)$ are antipodal, but by hypothesis this does not occur, so our map is continuous.

Second, we must have this map actually map into the sphere. We have for $|F(x,t)|$

$$\begin{array}{rl} & \left| \dfrac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}} \right| \\ = & \left\langle \dfrac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}}, \dfrac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}} \right\rangle \\ = & \dfrac{(1-t)^2\langle f(x), f(x)\rangle + 2t(1-t)\langle f(x), g(x)\rangle + t^2\langle g(x), g(x)\rangle}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\ = & \dfrac{(1-t)^2 + 2t(1-t)f(x)\cdot g(x) + t^2}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\ = & \dfrac{1 - 2t(1-t) + 2t(1-t)\langle f(x), g(x)\rangle}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\ = & \dfrac{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\ = & 1, \end{array}$$

where we used the fact that $\langle f(x), f(x)\rangle = \langle g(x), g(x)\rangle = 1$, since these maps are into the sphere. So our map does map into the sphere and is well defined.

As a corollary to this we prove the second part of the problem. If f is a non-surjective map into the sphere, then there exists a point $y_0 \in S^{n-1}$ such that $f^{-1}(y_0) = \emptyset$. Define the constant map $g : X \to S^{n-1}$ by $g(x) = y_1$ for every x, where y_1 is antipodal to y_0. Then $f(x)$ and $g(x)$ are never antipodal since y_0 isn't in the image of f, so we can construct a homotopy between them as outlined in the first half of the problem.

Problem 6.5 *Show that $S^{n-1} \simeq \mathbb{R}^n - \{0\}$. Show that two continuous maps*

$$f_0, f_1 : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\},$$

are homotopic iff their restrictions to S^{n-1} are.

That that these two spaces are homotopic was proved in class.

Now, suppose that the two maps f_0 and f_1 are homotopic. For clarity of notation let $X = \mathbb{R}^n - \{0\}$ and $Y = S^{n-1}$. We can regard the restrictions of these maps as being the the identity on X restricted to Y and the composed with the maps themselves:

$$f_0|_Y = f_0 \circ \mathrm{id}_X|_Y$$

and

$$f_1|_Y = f_1 \circ \mathrm{id}_X|_Y,$$

which are both compositions of continuous maps, so we have

$$f_0 \simeq f_1 \to f_0 \circ \mathrm{id}_X|_Y \simeq f_1 \circ \mathrm{id}_X|_Y,$$

or $f_0|_Y \simeq f_1|_Y$. There is nothing here particular to the case at hand - restrictions of homotopic maps are homotopic.

One the other hand, suppose the restricted maps are homotopic and let $g : Y \to X$ and $f : X \to Y$ be the maps defining the homotopy equivalence. Then we have

$$\begin{aligned} g \circ f &\simeq \mathrm{id}_X \\ f_i \circ (g \circ f) &\simeq f_i \end{aligned}$$

But, $g \circ f$ is in this case the map

$$x \mapsto \frac{x}{|x|},$$

mapping X onto Y. Thus we can regard $f_i \circ (g \circ f)$ as the restriction of f_i to Y. Thus $f_i|_Y \simeq f_i$, and we have

$$f_1 \simeq f_1|_Y \simeq f_2|_Y \simeq f_2,$$

since the restrictions are homotopic by hypothesis.

Problem 6.6 *Show that S^{n-1} is not contractible.*

A space is contractible if it has the same homotopy type as a point, or equivalently $\mathbb{R}^m$. Thus a space is contractible only if it has the same cohomology groups as $\mathbb{R}^m$. But we now know that $S^{n-1} \simeq \mathbb{R}^n - \{0\}$, so we have ($n \geq 2$)

$$H^p(\mathbb{R}^n - \{0\}) \cong H^p(S^{n-1}) \cong \left\{ \begin{array}{cl} \mathbb{R} & p = 0,\ n-1 \\ 0 & otherwise, \end{array} \right.$$

But, on the other hand, from Thm. 6.13 we have,

$$H^p(\mathbb{R}^m) \cong \left\{ \begin{array}{cl} \mathbb{R} & p = 0 \\ 0 & otherwise, \end{array} \right.$$

so these groups are not all isomorphic (there are two nontrivial groups for the first space, only one for the second).

Finally, for the case $n = 1$ we need to compute the groups for $\mathbb{R} - \{0\}$, which has two connected components, so $H^0(\mathbb{R} - \{0\}) = \mathbb{R} \oplus \mathbb{R}$, so the “0-sphere” is not contractible either.