

Cohomology Homework: Chapter 4

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Problem 4.1 Consider a commutative diagram of vector spaces and linear maps with exact rows

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{d_A^1} & A_2 & \xrightarrow{d_A^2} & A_3 & \xrightarrow{d_A^3} & A_4 & \xrightarrow{d_A^4} & A_5 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 B_1 & \xrightarrow{d_B^1} & B_2 & \xrightarrow{d_B^2} & B_3 & \xrightarrow{d_B^3} & B_4 & \xrightarrow{d_B^4} & B_5
 \end{array}$$

Suppose that f_2 and f_4 are injective and f_1 surjective. Show that f_3 is injective. Similarly show that if f_2 and f_4 are surjective and f_5 injective that f_3 is surjective. Thus when $f_1, f_2, f_4,$ and f_5 are isomorphisms, so is f_3 .

We will show injectivity by showing that f_3 has trivial kernel. Let $a_3 \in \ker(f_3)$ so that $f_3(a_3) = 0$. We have $d_B^3(0) = 0$, so that $d_B^3(f_3(a_3)) = 0 = f_4(d_A^3(a_3))$. f_4 is injective so we have $d_A^3(a_3) = 0$ and $a_3 \in \text{Ker } d_A^3 = \text{Im } d_A^2$. Thus there exists an $a_2 \in A_2$ such that $d_A^2(a_2) = a_3$.

Now we have $f_3(a_3) = f_3(d_A^2(a_2)) = 0 = d_B^2(f_2(a_2))$ so that $f_2(a_2) = b_2 \in \text{Ker } d_B^2 = \text{Im } d_B^1$. Thus there exists a $b_1 \in B_1$ with $d_B^1(b_1) = b_2$. Surjectivity of f_1 thus gives an $a_1 \in A_1$ with $f_1(a_1) = b_1$ and $d_B^1(f_1(a_1)) = b_2 = f_2(d_A^1(a_1))$. Thus we have $f_2(d_A^1(a_1)) = f_2(a_2)$, but f_2 is injective so we have $d_A^1(a_1) = a_2$. Thus $a_3 = d_A^2(d_A^1(a_1)) = 0$.

Next we prove surjectivity. Let $b_3 \in B_3$. Then $d_B^3(b_3) = b_4 \in B_4$ and $d_B^4(b_4) = 0$. f_4 is surjective so there exists an $a_4 \in A_4$ with $f_4(a_4) = b_4$ and we have $f_5(d_A^4(a_4)) = d_B^4(f_4(a_4)) = 0$. But f_5 is injective so we have $d_A^4(a_4) = 0$ and thus $a_4 \in \text{Ker } d_A^4 = \text{Im } d_A^3$. This gives an $a_3 \in A_3$ with $a_4 = d_A^3(a_3)$ and $f_4(d_A^3(a_3)) = d_B^3(f_3(a_3)) = b_4$.

Next, let $f_3(a_3) = b'_3 \in B_3$. Then we have $d_B^3(b'_3) = d_B^3(b_3) = b_4$, or $d_B^3(b'_3 - b_3) = 0$. Thus $b'_3 - b_3 \in \text{Ker } d_B^3 = \text{Im } d_B^2$. So, there exists a $b_2 \in B_2$ with $d_B^2(b_2) = b'_3 - b_3$. f_2 is surjective so there exists an $a_2 \in A_2$ with $f_2(a_2) = b_2$. Then we have $d_B^2(f_2(a_2)) = b'_3 - b_3 = f_3(d_A^2(a_2))$. Thus we have

$$b_3 = b'_3 - f_3(d_A^2(a_2))$$

$$\begin{aligned}
&= f_3(a_3) - f_3(d_A^2(a_2)) \\
&= f_3(a_3 - d_A^2(a_2)),
\end{aligned}$$

which proves surjectivity.

Problem 4.2 Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
& & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0
\end{array}$$

where the rows are exact. Show that there exists an exact sequence

$$\begin{aligned}
0 &\rightarrow \text{Ker } f_1 \rightarrow \text{Ker } f_2 \rightarrow \text{Ker } f_3 \rightarrow \\
&\rightarrow \text{Cok } f_1 \rightarrow \text{Cok } f_2 \rightarrow \text{Cok } f_3 \rightarrow 0.
\end{aligned}$$

We can extend the columns by adding zeros before after, yielding the sequences

$$\begin{aligned}
C^* &: 0 \rightarrow A_1 \xrightarrow{f_1} B_1 \rightarrow 0 \\
D^* &: 0 \rightarrow A_2 \xrightarrow{f_2} B_2 \rightarrow 0 \\
E^* &: 0 \rightarrow A_3 \xrightarrow{f_3} B_3 \rightarrow 0.
\end{aligned}$$

since $f_i \circ \iota = f_1(0) = 0$ and $0 \circ f_i = 0$ these sequences are actually chain complexes, from which we wish to define the short sequence

$$0 \rightarrow C^* \xrightarrow{g} D^* \xrightarrow{h} E^* \rightarrow 0,$$

where the chain maps g and h are the maps between columns of the original diagram. This sequence is then exact since each row of the original diagram is exact. Thus we can form the long exact cohomology sequence

$$\begin{aligned}
0 &\rightarrow H^0(C^*) \xrightarrow{g^*} H^0(D^*) \xrightarrow{h^*} H^0(E^*) \xrightarrow{\partial^*} \\
&\rightarrow H^1(C^*) \xrightarrow{g^*} H^1(D^*) \xrightarrow{h^*} H^1(E^*) \xrightarrow{\partial^*} 0.
\end{aligned}$$

Now we have

$$\begin{aligned}
H^0(C^*) &= \text{Ker } f_1 \\
H^0(D^*) &= \text{Ker } f_2 \\
H^0(E^*) &= \text{Ker } f_3,
\end{aligned}$$

and likewise

$$\begin{aligned} H^1(C^*) &= B_1/\text{Im } f_1 = \text{Cok } f_1 \\ H^1(C^*) &= B_2/\text{Im } f_2 = \text{Cok } f_2 \\ H^1(C^*) &= B_3/\text{Im } f_3 = \text{Cok } f_3, \end{aligned}$$

which establishes the result.

Problem 4.3 *In the commutative diagram*

$$\begin{array}{cccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{0,0} & \longrightarrow & A^{1,0} & \longrightarrow & A^{2,0} & \longrightarrow & A^{3,0} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{0,1} & \longrightarrow & A^{1,1} & \longrightarrow & A^{2,1} & \longrightarrow & A^{3,1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{0,2} & \longrightarrow & A^{1,2} & \longrightarrow & A^{2,2} & \longrightarrow & A^{3,2} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{0,3} & \longrightarrow & A^{1,3} & \longrightarrow & A^{2,3} & \longrightarrow & A^{3,3} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

the horizontal $(A^{*,q})$ and the vertical $(A^{p,*})$ are chain complexes where $A^{p,q} = 0$ if either $p < 0$ or $q < 0$. Suppose that

$$\begin{aligned} H^p(A^{*,q}) &= 0 \quad \text{for } q \neq 0 \text{ and all } p. \\ H^q(A^{p,*}) &= 0 \quad \text{for } p \neq 0 \text{ and all } q. \end{aligned}$$

Construct isomorphisms $H^p(A^{*,0}) \rightarrow H^p(A^{0,*})$ for all p .

If we denote by d_{lm}^{ij} the map from $A^{i,j}$ to $A^{l,m}$, then commutativity of the diagram gives $d_{11}^{01} \circ d_{01}^{00} = d_{11}^{10} \circ d_{10}^{00}$. So these two maps have equal kernels but, since the second map of each composition is injective (H^0 of the first row and column are zero), the kernels of the first maps are isomorphic, and thus $H^0(A^{0,*}) = H^0(A^{*,0})$.

This is far as I've been able to get.

Problem 4.4 Let $0 \rightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} A^n \rightarrow 0$ be a chain complex and assume that $\dim A^i < \infty$. The Euler characteristic is defined by

$$\chi(A^*) = \sum_{i=0}^n (-1)^i \dim A^i.$$

Show that $\chi(A^*) = 0$ if A^* is exact. Show that the sequence

$$0 \rightarrow H^i(A^*) \rightarrow A^i / \text{Im } d^{i-1} \xrightarrow{d^i} \text{Im } d^i \rightarrow 0$$

is exact and conclude that

$$\dim A^i - \dim \text{Im } d^{i-1} = \dim H^i(A^*) + \dim \text{Im } d^i.$$

Show that

$$\chi(A^*) = \sum_{i=0}^n (-1)^i \dim H^i(A^*).$$

That $\chi(A^*) = 0$ when A^* is exact was proved in class - the alternating sum of dimensions of spaces in an exact sequence is zero (provided they are all finite).

Next we need to show that the sequence

$$0 \rightarrow \frac{\text{Ker } d^i}{\text{Im } d^{i-1}} \hookrightarrow \frac{A^i}{\text{Im } d^{i-1}} \xrightarrow{d^i} \text{Im } d^i \rightarrow 0,$$

is exact. We have $\text{Ker } d^i \subset A^i$ so that the corresponding quotients are subsets and we can include the former in the latter. The inclusion is injective so the first map is exact.

Next, let $[x] \in A^i / \text{Im } d^{i-1}$. We want to define $d^i([x]) = d^i(x)$. But we have

$$d^i(x) = d^i(y + d^{i-1}(z)) = d^i(y) + d^i(d^{i-1}(z)) = d^i(y),$$

so that the map does not depend on representative. So, let $x \in \text{Im } d^i$, then $x = d^i(y)$ for some $y \in A^i$. But then

$$d^i([y]) = d^i(y) = x,$$

so that the map d^i is surjective.

Finally we need to show exactness in the middle. Suppose $[x] \in \text{Im } f$. $\text{Im } d^{i-1} \subset \text{Ker } d^i$ so that $x \in \text{Ker } d^i$. Then

$$d^i([x]) = d^i(x) = 0,$$

so that $[x] \in \text{Ker } d^i$. Conversely, suppose $[x] \in \text{Ker } d^i$. Then $0 = d^i([x]) = d^i(x)$, so that $x \in \text{Ker } d^i$. But then

$$[x] \in \text{Ker } d^i / \text{Im } d^{i-1} = \text{Im } f.$$

Thus the sequence is exact.

Since the sequence above is short exact, we have immediately that

$$\begin{aligned}\dim(H^i(A^*)) + \dim(\operatorname{Im} d^i) &= \dim(A^i/\operatorname{Im} d^{i-1}) \\ &= \dim(A^i) - \dim(\operatorname{Im} d^{i-1}).\end{aligned}$$

Thus we can write

$$\sum_{i=0}^n (-1)^i \dim(A^i)$$

as

$$\begin{aligned}& \sum_{i=0}^n (-1)^i (\dim(H^i(A^*)) + \dim(\operatorname{Im} d^i) + \dim(\operatorname{Im} d^{i-1})) \\ &= \sum_{i=0}^n (-1)^i \dim(H^i(A^*)) + \sum_{i=0}^n (-1)^i \dim(\operatorname{Im} d^i) + \sum_{i=0}^n (-1)^i \dim(\operatorname{Im} d^{i-1}).\end{aligned}$$

But, we can write the last sum as

$$\begin{aligned}\sum_{i=0}^n (-1)^i \dim(\operatorname{Im} d^{i-1}) &= \sum_{i=1}^n (-1)^i \dim(\operatorname{Im} d^{i-1}) \\ &= \sum_{j=0}^{n-1} (-1)^{j+1} \dim(\operatorname{Im} d^j) \\ &= - \sum_{j=0}^{n-1} (-1)^j \dim(\operatorname{Im} d^j), \\ &= - \sum_{j=0}^n (-1)^j \dim(\operatorname{Im} d^j),\end{aligned}$$

where the last line follows since $\dim(\operatorname{Im} d^n) = 0$. Thus the final two sums above cancel and we are left with

$$\sum_{i=0}^n (-1)^i \dim(A^i) = \sum_{i=0}^n (-1)^i \dim(H^i(A^*)).$$

Problem 4.5 Associate to two composable linear maps

$$f : V_1 \rightarrow V_2, \quad g : V_2 \rightarrow V_3,$$

an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \operatorname{Ker} f & \rightarrow & \operatorname{Ker}(g \circ f) & \rightarrow & \operatorname{Ker} g \rightarrow \\ & & \operatorname{Cok} f & \rightarrow & \operatorname{Cok}(g \circ f) & \rightarrow & \operatorname{Cok} g \rightarrow 0. \end{array}$$

