Cohomology Homework: Chapter 3

Daniel J. Cross

April 12, 2007

Problem 3.1 *Show for an open set in $\mathbb{R}^2$ that the de Rham complex*

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to 0,$$

is isomorphic to the complex

$$0 \to C^\infty(U,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U,\mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(U,\mathbb{R}) \to 0,$$

Analogously, show that for an open set in $\mathbb{R}^3$ that the de Rham complex is isomorphic to

$$0 \to C^\infty(U,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U,\mathbb{R}) \to 0,$$

defined in chapter 1.

We will do the $\mathbb{R}^3$ case first. We know that $\Omega^0(U) \simeq C^\infty(U,\mathbb{R})$. Furthermore, from exercise 2.3 we have

$$\begin{aligned} \Omega^1(\mathbb{R}^3) &\simeq \mathbb{R}^3 \\ \Omega^2(\mathbb{R}^3) &\simeq \mathbb{R}^3. \end{aligned}$$

Now, $\omega \in \Omega^1(U)$ is a smooth map $\omega : U \to \Omega^1(\mathbb{R}^3) \simeq \mathbb{R}^3$, in other words, a smooth vector field on U. Likewise $\omega \in \Omega^2(U)$ is a smooth map $\omega : U \to \Omega^2(\mathbb{R}^3) \simeq \mathbb{R}^3$, and so this too is a smooth vector field on U.

Now we need $\Omega^3(U)$. Any $\tau \in \Omega^3(\mathbb{R}^3)$ can be written as

$$\tau = \hat{\tau} e^1 \wedge e^2 \wedge e^3,$$

where $\hat{\tau} \in \mathbb{R}$. Thus we regard an $\omega \in \Omega^3(U)$ as a smooth map $\omega : U \to \Omega^3(\mathbb{R}^3) \simeq \mathbb{R}$, that is, a smooth function on U.

So, we've established the isomorphism on the vector spaces, next we need the maps. However, Theorem 3.7 and following establishes that the d operator acts as the differential operators grad, curl, and div on the appropriate spaces as indicated, and the result follows.

Next we take the $\mathbb{R}^2$ case. We again have $\Omega^0(U) \simeq C^\infty(U,\mathbb{R})$. We furthermore have $\Omega^1(\mathbb{R}^2) \simeq \mathbb{R}^2$ by problem 2.3 since the isomorphism depends only on the inner product which is defined in any dimension. The same argument as above then shows that $\Omega^1(U) \simeq C^\infty(U, \mathbb{R}^2)$.

Now, any $\tau \in \Omega^2(\mathbb{R})$ can be written as

$$\tau = \hat{\tau} e^1 \wedge e^2,$$

where $\hat{\tau} \in \mathbb{R}$. So any $\omega \in \Omega^2(U)$ is a smooth map $\omega \to \Omega^2(\mathbb{R}^2) \simeq \mathbb{R}$, so ω is a smooth function on U.

Now, we know that $d : \Omega^0 \to \Omega^1$ acts as div by theorem 3.7, so we only need $d : \Omega^1 \to \Omega^2$. We write $\omega \in \Omega^1(U)$ as $\omega_1 dx^1 + w_2 dx^2$. Then we have

$$\begin{aligned} d\omega &= d\omega_1 dx^1 + d\omega_2 dx^2 \\ &= \frac{\partial \omega_1}{\partial x^i} dx^i \wedge dx^1 + \frac{\partial \omega_2}{\partial x^i} dx^i \wedge dx^2 \\ &= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2 \\ &= \left(\frac{\partial \omega_1}{\partial x^2} - \frac{\partial \omega_2}{\partial x^1} \right) dx^1 \wedge dx^2 \\ &= (\operatorname{curl} \hat{\omega}) dx^1 \wedge dx^2 \end{aligned}$$

where $\hat{\omega} = \omega^1 dx_1 + \omega^2 dx_2 \in C^\infty(U, \mathbb{R}^2)$, and we have $\omega^i = \omega_i$.

Problem 3.2 *Let $U \subset \mathbb{R}^n$ be an open set and $\{dx^1, \ldots, dx^n\}$ the usual constant 1-forms. Let $\mathrm{vol} = dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(U)$. Use the star operator defined in 2.9 to define Hodge's star operator*

$$* : \Omega^p(U) \to \Omega^{n-p}(U),$$

and show that $*(dx^1 \wedge \ldots \wedge dx^p) = dx^{p+1} \wedge \ldots \wedge dx^n$ and $* \circ * = (-1)^{n(n-p)}$. Define $d^* : \Omega^p(U) \to \Omega^{p-1}(U)$ by

$$d^*(\omega) = (-1)^{np+n-1} * \circ\, d \circ * (\omega).$$

Show that $d^* \circ d^* = 0$. Verify the formula

$$d^*(f dx^{i_1} \wedge \ldots \wedge dx^{i_p}) = (-1)^j \frac{\partial f}{\partial x^{i_j}} dx^{i_1} \wedge \ldots \hat{dx}^{i_j} \ldots \wedge dx^{i_p}.$$

We will define the action of $*$ point-wise using the action of $*$ from 2.9 on basis elements:

$$\begin{aligned} *(\omega) &= *(\hat{\omega} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}) \\ &= \hat{\omega} * (e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}) \\ &= \operatorname{sgn}(\sigma) \hat{\omega} e^{\sigma(p+1)} \wedge \ldots \wedge e^{\sigma(n)}. \end{aligned}$$

We then extend this definition to arbitrary ω using linearity. Now we have (at each point)

$$\begin{aligned} *(dx^1 \wedge \ldots \wedge dx^p) &= *(e^1 \wedge \ldots \wedge e^p) \\ &= e^{p+1} \wedge \ldots \wedge e^n \\ &= dx^{p+1} \wedge \ldots \wedge dx^n. \end{aligned}$$

We now demonstrate that this result is the same as the previous definition extended to differential forms using this last result on the dx^i and the definition of $*$ from 2.9:

$$\begin{aligned}\langle *\omega, \tau\rangle &= \langle *(\hat{\omega} dx^1 \wedge \ldots \wedge dx^p), \tau\rangle \mathrm{vol} \\ &= \hat{\omega}\langle *(dx^1 \wedge \ldots \wedge dx^p), \tau\rangle \mathrm{vol} \\ &= \hat{\omega}(dx^1 \wedge \ldots \wedge dx^p \wedge \tau) \\ &= \omega \wedge \tau.\end{aligned}$$

Now, since $*$ as defined above only acts on the dx^i and no the component functions, the calculation reduces to the one in 2.9 giving the same result, that $* \circ * = (-1)^{n(n-p)}$.

Next we want to calculate $d^* \circ d^*$ on a p-form ω. We can write the operator as

$$d^* \circ d^* = \big((-1)^{np-1} * \circ\, d \circ *\big) \circ \big((-1)^{np+n-1} * \circ\, d \circ *\big)\circ,$$

since the second d^* acts on a $(p-1)$-form. We then write

$$d^* \circ d^* = (-1)^n * \circ\, d \circ * \circ * \circ\ d \circ *,$$

and the succession of the orders of the form in each map is:

$$(p) \xrightarrow{*} (n-p) \xrightarrow{d} (n-p+1) \xrightarrow{*} (p-1) \xrightarrow{*} (n-p+1) \xrightarrow{d} (n-p+2) \xrightarrow{*} (p-2).$$

Now we have

$$d^* \circ d^* = (-1)^n * \circ\, d \circ \underbrace{* \ \circ \ *}_{(-1)^{n(p-1)}} \circ\, d \circ *,$$

since the bracketed operator acts on an $(n-p+1)$-form. Then we have

$$d^* \circ d^* = (-1)^{np} * \circ \underbrace{d \circ d}_{0} \circ * = 0.$$

Finally, we calculate the action of d^* on a p-form:

$$\begin{aligned}d^*(f dx^1 \wedge \ldots \wedge dx^p) &= (-1)^{np+n-1} * \circ\, d \circ *(f dx^1 \wedge \ldots \wedge dx^p) \\ &= (-1)^{np+n-1} * \circ\, d\left(f * (dx^1 \wedge \ldots \wedge dx^p)\right) \\ &= (-1)^{np+n-1} * \circ\, df\left(*(dx^1 \wedge \ldots \wedge dx^p)\right).\end{aligned}$$

Next, we note that df can be though of here as a map whose action is given by

$$df(\omega) = df \wedge \omega,$$

so that we can apply the result of 2.11, that is, we can rewrite our last line (ignoring sign for now) as

$$* \circ\ df\left(*(dx^1 \wedge \ldots \wedge dx^p)\right) = (-1)^{n(p-1)}(df^*)(dx^1 \wedge \ldots \wedge dx^p),$$

where the sign is because this acts on a p-form. But then

$$\begin{aligned}(df^*)(dx^1\wedge\ldots\wedge dx^p) &= (-1)^{i+1}\langle df, dx^i\rangle dx^1\wedge\ldots\hat{dx}^i\ldots\wedge dx^p\\ &= (-1)^{i+1}\left(\frac{\partial f}{\partial x^i}\right)dx^1\wedge\ldots\hat{dx}^i\ldots\wedge dx^p,\end{aligned}$$

using 2.11 again. the sign factor, including all contributions, becomes

$$i+1+n(p-q)+np+n-1 = i+1-1+2np+n-n\to i,$$

which establishes the result. We note that our derivation didn't depend on which p of the dx^i's we picked, but only that there were p of them. so we can replace $dx^1\wedge\ldots\wedge dx^p$ with any permutation of the p indices and result will holds. If the indices are labeled $j_1,\ldots,j_p$, then we replace i in our formula with j_i, that is

$$(df^*)(dx^{j_1}\wedge\ldots\wedge dx^{j_p}) = (-1)^{j_i}\frac{\partial f}{\partial x^{j_i}}dx^{j_1}\wedge\ldots\hat{dx}^{j_i}\ldots\wedge dx^{j_p}.$$

Problem 3.3 *With the notation of the previous problem, the Laplace operator* $\Delta:\Omega^p(U)\to\Omega^p(U)$ *is defined by*

$$\Delta = d\circ\ d^* + d^*\circ\ d.$$

Let $f\in\Omega^0(U)$. *Show that* $\Delta(f dx^1\wedge\ldots\wedge dx^p) = \Delta(f)dx^1\wedge\ldots\wedge dx^p$, *where*

$$-\Delta(f) = \frac{\partial^2 f}{(\partial x^1)^2}+\ldots+\frac{\partial^2 f}{(\partial x^n)^2}.$$

A p*-form* $\omega\in\Omega^p(U)$ *is said to be harmonic if* $\Delta(\omega)=0$. *Show that* $*$ *maps harmonic forms to harmonic forms.*

We will take a p-form and first compute the action of $d^*\circ d$. First the action of d:

$$\begin{aligned}d(f dx^1\wedge\ldots\wedge dx^p) &= df\wedge dx^1\wedge\ldots\wedge dx^p\\ &= \frac{\partial f}{\partial x^i}dx^i\wedge dx^1\wedge\ldots\wedge dx^p,\end{aligned}$$

where we note that we must have $i>p$ for this to be nonzero.

Next we have d^* acting on a $(p+q)$-form:

$$d^* = (-1)^{np-1} *\circ d\circ *,$$

which we will take one mapping at a time. First

$$\begin{aligned}*\left(\frac{\partial f}{\partial x^i}dx^i\wedge dx^1\wedge\ldots\wedge dx^p\right) &= \frac{\partial f}{\partial x^i} *(dx^i\wedge dx^1\wedge\ldots\wedge dx^p)\\ &= \operatorname{sgn}(\sigma)\frac{\partial f}{\partial x^i}dx^{p+1}\wedge\ldots\hat{dx}^i\ldots\wedge dx^n,\end{aligned}$$

where σ is the permutation taking

$$(1, \ldots, n) \to (i, 1, \ldots, p, p+1, \ldots, \hat{i}, \ldots, n),$$

which has sign $i-1$.

Next we have d which gives

$$(-1)^{i-1} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

where we note that either $j \leq p$ or $j = i$ for this not to be zero. We will take these as two cases. If $j \neq i$ (the same as $j \leq p$ since $i > p$), we get for $*$:

$$(-1)^{i-1} \mathrm{sgn}\tau \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

where τ is the permutation taking

$$(1, \ldots, n) \to (j, p+1, \ldots, \hat{i}, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p),$$

which we will reduce in steps - first move i, then j, then swap the first $n-p$ with the last p:

$$\begin{array}{cl}
(j, p+1, \ldots, \hat{i}, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p) & \\
\downarrow & (-1)^{n-i} \\
(j, p+1, \ldots, i, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p) & \\
\downarrow & (-1)^{n-p+j-1} \\
(p+1, \ldots, i, \ldots, n, i, 1, \ldots, j, \ldots, p) & \\
\downarrow & (-1)^{p(n-p)} \\
(1, \ldots, j, \ldots, p, p+1, \ldots, i, \ldots, n) &
\end{array}$$

which shows that $\mathrm{sgn}(\tau)$ is given by

$$n - i + n - p + j - 1 + pn - p^2 \to j - i - 1 + pn.$$

So, if we take all our cumulative signs this gives

$$(np_1) + (i-1) + (j - i - 1 + pn) \to j - 1,$$

so that our sum is

$$-\sum_{\substack{i>p \\ j \leq p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p.$$

In the case $i = j$ the expression that $*$ acts on will now be

$$(-1)^{i-1} \frac{\partial^2 f}{(\partial x^i)^2} dx^i \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

which can be written as

$$(-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^{p+1} \wedge \ldots \wedge dx^i \wedge \ldots \wedge dx^n = (-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^{p+1} \wedge \ldots \wedge dx^n,$$

since there are $i-(p+1)$ interchanges to move i to its proper spot, and $(i-i)+(i-p+1) = 2i-2-p\top$. so, now $*$ gives

$$\operatorname{sgn}(\zeta)(-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^1 \wedge \ldots \wedge dx^p,$$

where ζ is the permutation that takes

$$(1, \ldots, n) \to (p+1, \ldots, m, 1, \ldots, p),$$

which has sign $p(n-p)$. So, putting it all together, the total sigh will be

$$(np-1)+(p)+p(n-p) = 2np = p(1-p) - 1 \to -1,$$

since $p(1-p)$ is always even, and the sum becomes

$$-\sum_{i>p} \frac{\partial^2 f}{(\partial x^i)^2} dx^1 \wedge \ldots \wedge dx^p.$$

Whew. Now we have to do the operator $d \circ d^*$. In this case we can use our previous result to simply write

$$d^*(f dx^1 \wedge \ldots \wedge dx^p) = (-1)^i \frac{\partial f}{\partial x^i} dx^1 \wedge \ldots \hat{dx}^{i} \ldots \wedge dx^p,$$

where we know that here $i \leq p$. Then d gives

$$(-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^{i} \ldots \wedge dx^p,$$

where now $j > p$ or $j = i$ (the two cases are mutually exclusive as before) or else we get zero, and will consider the two cases separately again.

First, if $j > p$ we have

$$\sum_{\substack{i \leq p \\ j>p}} (-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^{i} \ldots \wedge dx^p,$$

whereas when $i = j$ we have

$$(-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^{i} \ldots \wedge dx^p = -\frac{\partial^2 f}{\partial x^i \partial x^j} dx^1 \wedge \ldots \wedge dx^p,$$

since there are $i-1$ interchanges to move i to its proper spot.

Now, if take our two sums when $i = j$ and combine them, we see that the summands are the same, and they only differ in the range of i, which between the two if all of $1, \ldots, n$. Thus these two give

$$-\sum_{1 \le i \le n} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^1 \wedge \ldots \wedge dx^p,$$

which we will then write as

$$\Delta(f) dx^1 \wedge \ldots \wedge dx^p,$$

following the notation in the problem statement. So, we will be done if we can show that the other two sums cancel each other out. We will reproduce those sums here:

$$-\sum_{\substack{i>p\\ j\le p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p,$$

and

$$\sum_{\substack{i\le p\\ j>p}} (-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p.$$

But, we note that the i and j are just dummy indices, so we can exchange the two in the second sum, which yields

$$\sum_{\substack{j\le p\\ i>p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p,$$

which is now exactly the same as the first sum (since partial derivatives commute on smooth functions) except that they have opposite sign, so the two sums cancel.

Problem 3.4 *Let $\Omega^p(\mathbb{R}^n, \mathbb{C})$ be the $\mathbb{C}$-vector space of alternating $\mathbb{R}$-multilinear maps*

$$\omega : \mathbb{R}^n \underbrace{\times \cdots \times}_{p} \mathbb{R}^n \to \mathbb{C}.$$

Note that ω can be written uniquely as

$$\omega = \omega_R + i\omega_I,$$

where ω_R is the real part, ω_I is the imaginary part, and both are real-valued p-forms. Extend $\wedge$ to a $\mathbb{C}$-linear map

$$\Omega^p(\mathbb{R}^n, \mathbb{C}) \times \Omega^q(\mathbb{R}^n, \mathbb{C}) \xrightarrow{\wedge} \Omega^{p+q}(\mathbb{R}^n, \mathbb{C}),$$

and show that we obtain a graded anti-commutative $\mathbb{C}$-algebra $\Omega^*(\mathbb{R}^n, \mathbb{C})$.

The most straight-forward thing to do is to expand each complex form into its real-valued constituents and apply the usual wedge on these:

$$\begin{aligned}\omega\wedge\tau &= (\omega_R+i\omega_I)\wedge(\tau_R+i\tau_I)\\ &= (\omega_R\wedge\tau_R+i\omega_I)\wedge\tau_R+i\omega_I\wedge i\tau_I+\omega_R\wedge i\tau_I\\ &= (\omega_R\wedge\tau_R-\omega_I\wedge\tau_I)+i(\omega_I\wedge\tau_R-\omega_R\wedge\tau_I)\\ &= (\omega\wedge\tau)_R+i(\omega\wedge\tau)_I,\end{aligned}$$

which makes sense since the wedge products on the real-valued forms are always of a p and q form, resulting in a real-valued $(p+q)$-form.

To obtain the graded algebra we need to check associativity and the (anti)-commutativity, since we already have a mapping between the grades of the algebra. Associativity follow from that of the usual wedge product:

$$\begin{aligned}\alpha\wedge[\beta\wedge\gamma] &= (\alpha_R+i\alpha_I)\wedge[(\beta_R+i\beta_I)\wedge(\gamma_R+i\gamma_I)]\\ &= [(\alpha_R+i\alpha_I)\wedge(\beta_R+i\beta_I)]\wedge(\gamma_R+i\gamma_I)\\ &= [\alpha\wedge\beta]\wedge\gamma\end{aligned}$$

Now, let α be a p-form and β a q-form. Then we have

$$\alpha\wedge\beta=(\alpha_R\wedge\beta_R-\alpha_I\wedge\beta_I)+i(\alpha_I\wedge\beta_R+\alpha_R\wedge\beta_I),$$

but it costs a $(-1)^{pq}$ to flip the order of the wedge product in each term since each is a product of a real-valued p and q form. We then factor out the common sign factor to get

$$\begin{aligned}\alpha\wedge\beta &= (-1)^{pq}(\beta_R\wedge\alpha_R-\beta_I\wedge\alpha_I)+i(\beta_I\wedge\alpha_R+\beta_R\wedge\alpha_I)\\ &= (-1)^{pq}\beta\wedge\alpha,\end{aligned}$$

and these properties establish $\Omega^*\mathbb{R}^n,\mathbb{C}$ as a graded anti-commutative $\mathbb{C}$-algebra.

Problem 3.5 *Introduce $\mathbb{C}$-valued differential p-forms on an open set $U\subset\mathbb{R}^n$ by setting*

$$\Omega^p(U,\mathbb{C})=C^\infty(U,\Omega^p(\mathbb{R}^n,\mathbb{C})).$$

Extend d to a $\mathbb{C}$-linear operator

$$d:\Omega^p(\mathbb{R}^n,\mathbb{C})\to\Omega^{p+1}(\mathbb{R}^n,\mathbb{C}),$$

and show that theorem 3.7 holds for this case, and generalize theorem 3.12 to this case.

We extend the definition as in the previous problem, by having the usual d operator act on the real valued forms:

$$d(\omega)=d(\omega_R+i\omega_I)=d\omega_r+id\omega_I.$$

Now, we need to establish the following properties of d:

$$\begin{array}{rl} (i) & f \in \Omega^0(U,\mathbb{C}),\ df = (\partial_i f)dx^i \\ (ii) & d \circ d = 0 \\ (iii) & d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^p \omega \wedge d\tau,\ \omega \in \Omega^p(U,\mathbb{C}), \end{array}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}.$$

Now, an $f \in \Omega^0(U,\mathbb{C})$ is a smooth map $f : U \to \mathbb{C}$, that is, a smooth $\mathbb{C}$-valued function, so we can write f as $f_R + if_R$, so we have

$$\begin{aligned} df &= df_r + idf_I \\ &= \partial_j f_R dx^j + i\partial_j f_I dx^j \\ &= \partial_j (f_r + if_I) dx^j \\ &= \partial_j f dx^j, \end{aligned}$$

which establishes the first property.

Next we have

$$\begin{aligned} (d \circ\ d) f &= d(df_R + idf_I) \\ &= d^2 f_R + id^2 f_I \\ &= 0, \end{aligned}$$

which establishes the second. And finally we have

$$\begin{aligned} d(\omega \wedge \tau) &= d[(\omega_R \wedge \tau_R - \omega_I \wedge \tau_I) + i((\omega_R \wedge \tau_I - \omega_I \wedge \tau_R)] \\ &= d(\omega_R \wedge \tau_R) - d(\omega_I \wedge \tau_I) + id(\omega_R \wedge \tau_I) + id(\omega_I \wedge \tau_R) \\ &= d\omega_R \wedge \tau_R + (-1)^p \omega_R \wedge d\tau_R - d\omega_I \wedge \tau_I + (-1)^p \omega_I \wedge d\tau_I \\ &\quad + i(d\omega_R \wedge \tau_I + (-1)^p \omega_R \wedge d\tau_I) + i(d\omega_I \wedge \tau_R + (-1)^p \omega_I \wedge d\tau_R) \\ &= d\omega_R \wedge \tau_R + id\omega_R \wedge \tau_I + id\omega_I \wedge \tau_R - d\omega_I \wedge \tau_I + \\ &\quad (-1)^p (\omega_R \wedge d\tau_R + i\omega_R \wedge d\tau_I + i\omega_I \wedge d\tau_R - \omega_I \wedge d\tau_I) \\ &= (d\omega_R + id\omega_I) \wedge (\tau_R + i\tau_I) + (-1)^p (\omega_R + i\omega_I) \wedge (d\tau_R + id\tau_I) \\ &= d(\omega_R + i\omega_I) \wedge (\tau_R + i\tau_I) + (-1)^p (\omega_R + i\omega_I) \wedge d(\tau_R + i\tau_I) \\ &= d\omega \wedge \tau + (-1)^p \omega \wedge d\tau, \end{aligned}$$

where in the fourth line we simply rearranged the entries. This proves the result, and shows the existence of the operator. The argument for uniqueness follows exactly as the one given in the book for the real-valued case by distributing $\wedge$ across the real and imaginary parts of each form.

Next we generalize theorem 3.12: if we have a map $\varphi : U \to V$, then the induced map $\varphi^* : \Omega^p(V,\mathbb{C}) \to \Omega^p(U,\mathbb{C})$ will have the properties:

$$\begin{array}{rl} (i) & \varphi^*(\omega \wedge \tau) = \varphi^*(\omega) \wedge \varphi^*(\tau) \\ (ii) & \varphi^*(f) = f \circ \varphi,\ f \in C^\infty(V,\mathbb{C}) \\ (iii) & d\varphi^*(\omega) = \varphi^*(d\omega) \end{array}$$

First we show that $(\varphi^*\omega)_R = (\varphi^*\omega_R)$, and similarly for the imaginary parts. Using the calculational formula, we have

$$\begin{aligned}
\varphi^*(\omega)(x) &= \omega_k(\varphi(x))d\varphi^k \\
&= [\Re(\omega_k(\varphi(x))) + i\Im(\omega_k(\varphi(x))]d\varphi^k \\
&= [\omega_R(\varphi(x))_k]d\varphi^k + i[\omega_I(\varphi(x))_k]d\varphi^k \\
&= \varphi^*(\omega_R(x)) + i\varphi^*(\omega_I(x)),
\end{aligned}$$

which shows the result. Now we have

$$\begin{aligned}
\varphi^*(\omega \wedge \tau) &= \varphi^*(\omega_R \wedge \tau_R - \omega_I \wedge \tau_I + i\omega_R \wedge \tau_I + i\omega_I \wedge \tau_R) \\
&= \varphi^*(\omega_R \wedge \tau_R - \varphi^*(\omega_I \wedge \tau_I) + i\varphi^*(\omega_R \wedge \tau_I) + i\varphi^*(\omega_I \wedge \tau_R) \\
&= \varphi(\omega_R) \wedge \varphi^*(\tau_R) - \varphi^*(\omega_I) \wedge \varphi * (\tau_I) + i\varphi^*(\omega_R) \wedge \varphi^*(\tau_I) + i\varphi^*(\omega_I) \wedge \varphi^*(\tau_R) \\
&= \varphi(\omega)_R \wedge \varphi^*(\tau)_R - \varphi^*(\omega)_I \wedge \varphi * (\tau)_I + i\varphi^*(\omega)_R \wedge \varphi^*(\tau)_I + i\varphi^*(\omega)_I \wedge \varphi^*(\tau)_R \\
&= (\varphi^*\omega) \wedge (\varphi^*\tau).
\end{aligned}$$

For the second property we have

$$\begin{aligned}
\varphi^*(f) &= \varphi^*(f_R + if_I) \\
&= f_R \circ \varphi + if_i\varphi \\
&= (f_r + if_I)(\varphi) \\
&= f \circ \varphi.
\end{aligned}$$

And for the third property we have

$$\begin{aligned}
d\varphi^*(\omega) &= d\varphi^*(\omega_R + i\omega_I) \\
&= d\varphi^*(\omega_R) + d\varphi^*(i\omega_I) \\
&= \varphi^*(d\omega_R) + \varphi^*(id\omega_I) \\
&= \varphi(d\omega_R + id\omega_I) \\
&= \varphi * (d(\omega_R + i\omega_I)) \\
&= \varphi^*(d\omega).
\end{aligned}$$

Now we prove the uniqueness of our pullback map. Suppose that φ' were another map satisfying the above three properties. First we note that

$$\varphi'(f) = f \circ \varphi = \varphi^*(f),$$

so that φ' and φ^* agree on $C^\infty(U, \mathbb{C})$. Now, by linearity it is enough to look at a basis p-form:

$$\begin{aligned}
\varphi'(fdx^J) &= \varphi(f \wedge dx^J) \\
&= (\varphi' f) \wedge (\varphi' dx^J) \\
&= (\varphi' f) \wedge (\varphi'(dx^{j_1} \wedge \ldots \wedge dx^{j_p})) \\
&= (\varphi' f) \wedge (\varphi' dx^{j_1}) \wedge \ldots \wedge (\varphi' dx^{j_p}) \\
&= (\varphi' f) \wedge d(\varphi' x^{j_1}) \wedge \ldots \wedge d(\varphi' x^{j_p}) \\
&= (f \circ \varphi) \wedge d(x^{j_1} \circ \varphi) \wedge \ldots \wedge d(x^{j_p} \circ \varphi) \\
&= (\varphi^* f) \wedge d(\varphi^* x^{j_1}) \wedge \ldots \wedge d(\varphi^* x^{j_p}),
\end{aligned}$$

where we have applied our previous result. At this point, we just follow the same steps backwards to get $\varphi * (f dx^J)$, which establishes the uniqueness.

Problem 3.6 *Take* $U = \mathbb{C} - \{0\} \simeq \mathbb{R}^2 - \{0\}$ *and let* $z \in \mathbb{C}^\infty(U, \mathbb{C})$ *be the inclusion* $U \hookrightarrow \mathbb{C}$. *Write* $z = x + iy$. *Show that*

$$\Re(z^{-1}dz) = d\log r,$$

where $r : U \to \mathbb{R}$ *is defined by* $r(z) = |z| = \sqrt{x^2+y^2}$. *Show that*

$$\Im(z^{-1}dz) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Since we have $z = x + iy$ we can write $dz = dx + idy$. Thus we have

$$\begin{aligned} z^{-1}dz &= (x+iy)^{-1}(dx+idy) \\ &= \frac{dx+idy}{x+iy}\left(\frac{x-iy}{x-iy}\right) \\ &= \frac{xdx - iydx + ixdy + ydy}{x^2+y^2} \\ &= \frac{xdx+ydy}{x^2+y^2} + i\frac{xdy-ydx}{x^2+y^2}, \end{aligned}$$

which gives the required equality on the imaginary part. Now we have

$$d\log\sqrt{x^2+y^2} = \frac{d\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}},$$

but

$$d\sqrt{x^2+y^2} = \frac{2xdx+2ydy}{2\sqrt{x^2+y^2}} = \frac{xdx+ydy}{\sqrt{x^2+y^2}},$$

which proves the claim.

Problem 3.7 *Prove for the complex exponential map* $\exp : \mathbb{C} \to \mathbb{C}^*$ *that*

$$d_z \exp = \exp(z)dz, \text{ and } \exp^*(z^{-1}dz) = dz.$$

We begin by writing the exponential map as

$$\exp(z) = \exp(x + iy_= \exp(x) \cdot \exp(iy),$$

so we get

$$\begin{aligned} d(\exp(z)) &= \frac{\partial \exp(x)\exp(iy)}{\partial x}dx + \frac{\partial \exp(x)\exp(iy)}{\partial y}dy \\ &= \exp(iy)\frac{\partial \exp(x)}{\partial x}dx + \exp(x)\frac{\partial \exp(iy)}{\partial y}dy \\ &= \exp(iy)\exp(x)dx + i\exp(x)\exp(iy)dy \\ &= \exp(z)(dx+idy) \\ &= \exp(z)dz, \end{aligned}$$

which proves the first part.

Next, for clarity, we will write $\exp : \mathbb{C} \to \mathbb{C}^*$ as $\exp : w \mapsto z$, by $z = \exp(w)$. Then we have (using the calculational formula)

$$\begin{aligned}\exp^*(z^{-1}dz) &= (\exp(w))^{-1} d\exp(w) \\ &= (\exp(w))^{-1}\exp(w)dw \\ &= dw,\end{aligned}$$

which proves the second part.