Cohomology Homework: Chapters 1 & 2

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Problem 1.1 *Perform the calculations of Example 1.7.*

Let $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\right\}$ be the unit circle in the (x_1, x_2)-plane, let $U = \mathbb{R}^3 - S$, and consider the function $f : U \to \mathbb{R}$ given by

$$f : (x_1, x_2, x_3) \mapsto \left(\frac{-2x_1x_3}{\xi}, \frac{-2x_2x_3}{\xi}, \frac{x_1^2 + x_2^2 - 1}{\xi}\right),$$

where $\xi = x_3^2 + (x_1^2 + x_2^2 - 1)^2$. First we compute the derivative of ξ:

$$\begin{aligned}
\partial_1\xi &= 4x_1(x_1^2 + x_2^2 - 1) \\
\partial_2\xi &= 4x_2(x_1^2 + x_2^2 - 1) \\
\partial_3\xi &= 2x_3.
\end{aligned}$$

Now the components of the curl as evaluated as

$$\begin{aligned}
(\operatorname{curl} f)_1 &= \partial_2 f_3 - \partial_3 f_2 \\
&= \left(\frac{2x_2}{\xi} - \frac{x_1^2 + x_2^2 - 1}{\xi^2}\partial_2\xi\right) - \left(\frac{-2x_2}{\xi} + \frac{2x_2x_3}{\xi^2}\partial_3\xi\right) \\
&= \frac{4x_2\xi - (x_1^2 + x_2^2 - 1)\partial_2\xi - 2x_2x_3\partial_3\xi}{\xi^2} \\
&= \frac{4x_2\xi - (x_1^2 + x_2^2 - 1)^2(4x_2) - 4x_2x_3^2}{\xi^2} \\
&= \frac{4x_2}{\xi^2}(\xi - (x_3^2 + (x_1^2 + x_2^2 - 1)^2)) \\
&= \frac{4x_2}{\xi^2}(\xi - \xi) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(\operatorname{curl} f)_2 &= \partial_3 f_1 - \partial_1 f_3 \\
&= \left(\frac{-2x_1}{\xi} + \frac{2x_1x_3}{\xi^2}\partial_3\xi\right) - \left(\frac{2x_1}{\xi} - \frac{x_1^2 + x_2^2 - 1}{\xi^2}\partial_1\xi\right) \\
&= \frac{-4x_1\xi + 2x_1x_3\partial_3\xi + (x_1^2 + x_2^2 - 1)\partial_1\xi}{\xi^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-4x_1\xi + 4x_1x_3^2 + 4x_1(x_1^2+x_2^2-1)^2}{\xi^2} \\
&= \frac{4x_1}{\xi^2}(-\xi + 4x_3^2 + (x_1^2+x_2^2-1)^2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{curl} f)_3 &= \partial_1 f_2 - \partial_2 f_1 \\
&= \frac{2x_2x_3}{\xi}\partial_1\xi - \frac{2x_1x_3}{\xi}\partial_2\xi \\
&= \frac{2x_3}{\xi}(x_2\partial_1\xi - x_1\partial_2\xi) \\
&= \frac{2x_3}{\xi}(4x_1x_2(x_1^2+x_2^2-1) - 4x_1x_2(x_1^2+x_2^2-1)) \\
&= 0
\end{aligned}
$$

Thus we have $\operatorname{curl} f = 0$. So $f \in \ker(\operatorname{curl})$ and $[f] \in H^1(U)$. We will show, hwoever, that $[f] \neq 0$ by showing that there is no function F such that $f = \operatorname{grad} F$.

Suppose such an F existed and consider the integral of $\frac{d}{dt}F(\gamma(t))$ where the curve γ is given by

$$\gamma(t) = \left(\sqrt{1+\cos t}, 0, \sin t\right), \quad -\pi \le t \le \pi.$$

First we have

$$
\begin{aligned}
\int_{-\pi}^{\pi} \frac{d}{dt}F(\gamma(t))dt &= \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon}\frac{d}{dt}F(\gamma(t))dt \\
&= \lim_{\epsilon\to 0} F(\gamma(t))\Big|_{-\pi+\epsilon}^{\pi-\epsilon} \\
&= 0,
\end{aligned}
$$

where the limit was taken since the curve is not differentiable at its endpoints. One the other hand we have

$$
\begin{aligned}
\frac{d}{dt}F(\gamma(t))dt &= \partial_i F(\gamma(t))\dot\gamma_i(t) \\
&= f_i(\gamma(t))\dot\gamma_i(t)
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\dot\gamma_1(t) &= \frac{-\sin t}{2\sqrt{1+\cos t}} \\
\dot\gamma_2(t) &= 0 \\
\dot\gamma_3(t) &= \cos t,
\end{aligned}
$$

and $\xi(\gamma(t)) = \sin^2 t + (1+\cos t - 1)^2 = 1$. so that

$$f(\gamma(t)) = (-2\sin t\sqrt{1+\cos t}, 0, \cos t),$$

and finally

$$f_i(\gamma(t))\dot{\gamma}_i(t) = \sin^2 t + 0 + \cos^2 t = 1,$$

for all t, which yields a value of 2π for the integral over γ which gives a contradiction. Thus F cannot exist.

Problem 1.2 *Let W be the open set given by*

$$W = \left\{(x_1, x_2, x_3) \in \mathbb{R} : \text{ either } x_3 \neq 0 \text{ or } x_1^2 + x_2^2 < 1\right\}.$$

Prove the existance and uniqueness of a function $F \in C^\infty(W, \mathbb{R})$ such that grad (F) *is the vector field considered in Example 1.7 and $F(0) = 0$. Find a simple expression for F valid when $x_1^2 + x_2^2 < 1$.*

Existance: It suffices to show that W is star-shaped about the origin, since then the existance will follow from Theorem 1.6.

Let $x \in W$, $s = (s_1, s_2, s_3)$, and suppose that $s_3 = 0$. then s is contained in the unit circle in the (x_1, x_2)-plane which is star-shaped. Now suppose that $s_3 \neq 0$ and let p be a point on the line joining s to the origin. If $p_3 = 0$, then $p = 0$ and $p \in W$, while if $p_3 \neq 0$ then $p \in W$. Thus W is star-shaped.

Now we have some F which satisfies grad $(F) = f$. Consider a new function F' defined by

$$F'(x) = F(x) - F(0).$$

Then $F'(0) = 0$ and grad (F') = grad $(F) - 0 = f$ and thus satisfies the given condition.

Uniqueness: Suppose F_1 and F_2 both satisfy the given criteria, then we have grad $(F_1 - F_2) = 0$, so that the function $F_1 - F_2$ is constant on W (since W is connected). Moreover, $F_1(0) - F_2(0) = 0$ so that $F_1(x) = F_2(x)$ for all $x \in W$, so that the solution is unique.

Finally, we give an expression for F when $x_1^2 + x_2^2 < 1$. First, set $x_3 = 0$, then we have

$$f = \left(0, 0, \frac{1}{x_1^2 + x_2^2 - 1}\right),$$

that is, $\partial_1 F = \partial_2 F = 0$ whenever $x_3 = 0$. Thus we have

$$F(x_1, x_2, 0) = F(0, 0, 0) = 0,$$

for $x_1^2 + x_2^2 < 1$.

Now we have

$$F(x_1, x_2, x_3) = \int_0^1 f_i(\gamma(t))\dot{\gamma}_i(t)dt,$$

for any curve $\gamma(t)$ that connects the origin to (x_1, x_2, x_3). Due to the above we can take our curve as the straight line from the origin to the point $(x_1, x_2, 0)$, which doesn't contribute to the integral, and then as the line from there to the final point, which will be the curve

$$\gamma(t) = (x_1, x_2, tx_3),$$

so that our integral becomes

$$\begin{aligned} F(x_1,x_2,x_3) &= \int_0^1 f_3(x_1,x_2,tx_3)x_3 dt \\ &= x_3\int_0^1 \frac{x_1^2+x_2^2-1}{(tx_3)^2+(x_1^2+x_2^2-1)^2}dt \\ &= \frac{x_3}{\alpha}\int_0^1 \frac{dt}{(tx_3/\alpha)^2+1}, \end{aligned}$$

where we have set $\alpha = x_1^2+x_2^2-1$. Now if we make the substitution $y = tx_3/(x_1^2+x_2^2-1)$ we obtain

$$\begin{aligned} F(x_1,x_2,x_3) &= \int_0^{x_3/\alpha} \frac{dy}{y^2+1} \\ &= \arctan(y)\Big|_0^{x_3/\alpha} \\ &= \arctan\left(\frac{x_3}{x_1^2+x_2^2-1}\right). \end{aligned}$$

Problem 2.1 *Prove the formula in Remark 2.10.*

In the 'older' formula we sum over all permutations $\sigma \in S(p+q)$ of the vectors, not just the ordered ones. Let σ be such a permutation, then there are

$$(p+q)! = \frac{(p+q)!}{p}!q!p!q! = \left(\begin{array}{c} p+q \\ p \end{array}\right) p!q!,$$

arrangements, corresponding to permuting the first p objects among themselves, then the last q among themselves, and then shuffling the first p with the last q. We will write σ using this decomposition: $\sigma = \tau\circ\pi_p\circ\pi_q$, where $\tau \in S(p,q)$ is a (p,q)-shuffle, $\pi_p \in S(p,\bar{q})$ is a permutation among the first p objects, $\pi_q \in S(\bar{p},q)$ is a permutation among the remaining p. The our formula becomes

$$\frac{1}{p!q!}\sum_{\tau}\sum_{\pi_p}\sum_{\pi_q}\mathrm{sgn}(\sigma)\omega_1(\xi_{\sigma(1)}\ldots\xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}\ldots\xi_{\sigma(n)}),$$

where we have

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau\circ\pi_p\circ\pi_q) = \mathrm{sgn}(\tau)\cdot\mathrm{sgn}(\pi_p)\cdot\mathrm{sgn}(\pi_q).$$

Likewise, we have (since π_q leave the first p unchanged, π_p leave the last q unchanged, and therefore they commute)

$$\begin{aligned} \omega_1(\xi_{\sigma(1)}\ldots\xi_{\sigma(p)}) &= \omega_1(\xi_{\tau\circ\pi_p\circ\pi_q(1)}\ldots\xi_{\tau\circ\pi_p\circ\pi_q(p)}) \\ &= \omega_1(\xi_{\tau\circ\pi_p(1)}\ldots\xi_{\tau\circ\pi_p(p)}) \\ &= \mathrm{sgn}(\pi_p)\cdot\omega_1(\xi_{\tau(1)}\ldots\xi_{\tau(p)}), \end{aligned}$$

and

$$\begin{aligned}\omega_2(\xi_{\sigma(p+1)}\dots\xi_{\sigma(n)}) &= \omega_2(\xi_{\tau\circ\pi_p\circ\pi_q(p+1)}\dots\xi_{\tau\circ\pi_p\circ\pi_q(n)})\\ &= \omega_2(\xi_{\tau\circ\pi_q\circ\pi_p(p+1)}\dots\xi_{\tau\circ\pi_q\circ\pi_p(n)})\\ &= \omega_2(\xi_{\tau\circ\pi_q(p+1)}\dots\xi_{\tau\circ\pi_q(n)})\\ &= \mathrm{sgn}(\pi_q)\cdot\omega_2(\xi_{\tau(p+1)}\dots\xi_{\tau(n)}).\end{aligned}$$

This leave us with

$$\frac{1}{p!q!}\sum_{\tau}\sum_{\pi_p}\sum_{\pi_q}\mathrm{sgn}(\tau)\omega_1(\xi_{\tau(1)}\dots\xi_{\tau(p)})\omega_2(\xi_{\tau(p+1)}\dots\xi_{\tau(n)}),$$

since $\mathrm{sgn}^2 = 1$ for any permutation. The sums over π_p and pi_q just give $p!$ and $q!$ respectively, which cancel the factorials already present and thus establishes the result.

Problem 2.2 *Find an $\omega \in \Omega^2\mathbb{R}^4$ such that $\omega\wedge\omega \neq 0$.*

More generally, let V be a $2n$-dimensional vector space and α a $2-$form defined by the skew-symmetric $2n\times 2n$-matrix a_{ij}

$$\alpha = \alpha_{ij}e^i\wedge e^j,$$

in the orthonormal basis $\{e^i\}$,where it is understood that the sum is for $i<j$. First we verify that α is indeed a $2-$form.

$$\begin{aligned}\alpha(x,x) &= (\alpha_{ij}e^i\wedge e^j)(x^re_r, x^2e_s)\\ &= a_{ij}x^rx^se^i\wedge e^j(e_r,e_s)\\ &= a_{ij}x^rx^s(\delta^i_r\delta^j_s-\delta^i_s\delta^j_r)\\ &= a_{ij}x^ix^j - a_{ij}x^ix^j\\ &= 0.\end{aligned}$$

Now we define a $2n$-form β by

$$\beta = \alpha\underbrace{\wedge\dots\wedge}_{\text{n times}}\alpha.$$

Using our basis, we have

$$\begin{aligned}\beta &= (a_{i_1j_1}e^{i_1}\wedge e^{j_1})\wedge\dots\wedge(a_{i_nj_n}e^{i_n}\wedge e^{j_n})\\ &= a_{i_1j_1}\dots a_{i_nj_n}(e^{i_1}\wedge e^{j_1})\wedge\dots\wedge(e^{i_n}\wedge e^{j_n}).\end{aligned}$$

Now, there are only $2n$ basis elements, so a term in the sum will vanish unless the basis elements form a permutation of all $\{1,\dots,2n\}$. Moreover, we can swap products of pairs of basis elements without changing its value since this is always an even permutation, thus we can write

$$\beta = n!a_{i_1j_1}\dots a_{i_nj_n}(e^{i_1}\wedge e^{j_1})\wedge\dots\wedge(e^{i_n}\wedge e^{j_n}),$$

where we now have $i_l < i_k$ whenever $l < k$, that is, the sum is over all unordered pairs of basis elements. Since there were n pairs of basis elements, there are now $n!$ permutations of them, each contributing the same to the sum.

Now, we still have $i_l < i_k$, so our sum is over all unordered partitions of the integers $\{1, \ldots, 2n\}$ into pairs. We now rewrite our sum as

$$\beta = n! \left(\sum_{\sigma} \operatorname{sgn}(\sigma) \cdot a_{i_1 j_1} \ldots a_{i_n j_n} \right) e^1 \wedge \ldots \wedge e^{2n},$$

where σ is a permutation that gives the desired partition, and we sum over only partitions. We can rewrite the sum this way since any such permutation of the basis elements changes the term by exactly $\operatorname{sgn}(\sigma)$.

But, we notice that

$$\sum_{\sigma} \operatorname{sgn}(\sigma) \cdot a_{i_1 j_1} \ldots a_{i_n j_n} = \operatorname{pf}(A),$$

the Pfaffian of A, and for a $2n \times 2n$ skew-symmetric matrix we have

$$\operatorname{pf}^2(A) = \det(A).$$

Thus we see that $\beta = 0$ iff $\det(A) = 0$. (We note that this holds as long as A has an even number of rows, as it does here. For an odd number of rows, the determinant is always zero.)

For the problem at hand it suffices to pick any 4×4 matrix with non-vanishing determinant, such as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which has determinant 1.

Problem 2.3 *Show that there exist isomorphisms*

$$i : \mathbb{R}^3 \to \Omega^1\mathbb{R}^3, \;\; j : \mathbb{R}^3 \to \Omega^2\mathbb{R}^3,$$

given by

$$i(v)(w) = \langle v, w, \rangle, \;\; j(v)(w_1, w_2) = \det(v, w_1, w_2),$$

where $\langle , \rangle$ *is the usual inner product. Show that for* $v_1, v_2 \in \mathbb{R}^3$*, we have*

$$i(v_1) \wedge i(v_2) = j(v_1 \times v_2).$$

Define the map $i : \mathbb{R}^3 \to \Omega^1\mathbb{R}^3$ between vector spaces by

$$i(v)(w) = \langle v, w \rangle.$$

Then we have

$$\begin{aligned} i(\alpha u+\beta v)(w) &= \langle \alpha u+\beta v, w\rangle \\ &= \alpha\langle u,w\rangle+\beta\langle v,w\rangle \\ &= \alpha i(u)(w)+\beta i(v)(w), \end{aligned}$$

which shows that i is a linear map.

Now, let $u \in \Omega^1\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ an orthonormal basis for $\mathbb{R}^3$ and $\{e^1, e^2, e^3\}$ the dual basis. Then we have

$$\begin{aligned} u(w) &= u_i e^i(w^j e_j) \\ &= u_i w^j e^i(e_j) \\ &= u_i w^j \delta^i_j \\ &= u_i w^i \\ &= \langle u, w\rangle, \end{aligned}$$

which shows that i is surjective. But the vector spaces have the same dimension, so we conlude that i is injective and thus an isomorphism.

Now we define the map $j : \mathbb{R}^3 \to \Omega^2\mathbb{R}^3$ by

$$j(v)(w_1, w_2) = \det(v, w_1, w_2).$$

Then we have

$$\begin{aligned} j(\alpha u+\beta v)(w_1,w_2) &= \det(\alpha u+\beta v, w_1, w_2) \\ &= \det(\alpha u, w_1, w_2)+\det(\beta v, w_1, w_2) \\ &= \alpha\det(u, w_1, w_2)+\beta\det(v, w_1, w_2) \\ &= \alpha j(u)(w_1,w_2)+\beta j(v)(w_1,w_2), \end{aligned}$$

which shows that j is linear.

Let $u \in \Omega^2\mathbb{R}^3$, then we can write

$$u = \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} e^{\sigma(1)} \wedge e^{\sigma(2)}.$$

The action of the dual basis on the basis of $\mathbb{R}^3$ is given by

$$\begin{aligned} (e^a \wedge e^b)(e_i, e_j) &= \delta^a_i\delta^b_j - \delta^a_j\delta^b_i \\ &= \epsilon^{abk}\epsilon_{ijk}, \end{aligned}$$

where ϵ_{ijk} is the Levi-civita symbol. So we have

$$\begin{aligned} u(x,y) &= \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} x^i y^j e^{\sigma(1)} \wedge e^{\sigma(2)}(e_i, e_j) \\ &= \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} x^i y^j \epsilon^{\sigma(1)\sigma(2)k}\epsilon_{ijk}. \end{aligned}$$

So, let us define

$$u^k = \sum_{\sigma \in S(2,1)} u_{\sigma(1),\sigma(2)} \epsilon^{\sigma(1)\sigma(2)k},$$

which is a triple of numbers, so a vector in $\mathbb{R}^3$. In particular, we get $u^1 = u_{23}$, $u^2 = u_{13}$, and $u_3 = u_{12}$. Now we have

$$\begin{aligned} u(x,y) &= \epsilon_{ijk} u^k x^i y^j \\ &= \epsilon_{kij} u^k x^i y^j \\ &= \det(u, x, y), \end{aligned}$$

and our mapping is surjective. Again, since the dimensions are the same, j is automatically injective and hence an isomorphism.

Next, consider $i(v_1) \wedge i(v_2)$, which is given by

$$\begin{aligned} i(v_1) \wedge i(v_2)(w_1, w_2) &= \det \left| \begin{array}{cc} i(v_1)(w_1) & i(v_1)(w_2) \\ i(v_2)(w_1) & i(v_2)(w_2) \end{array} \right| \\ &= \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \cdot \langle v_2, w_1 \rangle. \end{aligned}$$

On the other hand consider $j(v_1 \times v_2)$, which is given by

$$\begin{aligned} j(v_1 \times v_2)(w_1, w_2) &= \det(v_1 \times v_2, w_1, w_2) \\ &= \epsilon_{ijk} (v_1 \times v_2)^i w_1^j w_2^k \\ &= \epsilon_{ijk} \epsilon^i_{ab} v_1^a v_2^b w_1^j w_2^k \\ &= (\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) v_1^a v_2^b w_1^j w_2^k \\ &= (v_1^a w_1^a) \cdot (v_2^b w_2^b) - (v_1^a w_2^a) \cdot (v_2^b w_1^b) \\ &= \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \cdot \langle v_2, w_1 \rangle. \end{aligned}$$

Comparing these expressinos establishes the result.

Problem 2.4 *Let V be a finite-dimensional vector space over $\mathbb{R}$ with inner product $\langle , \rangle$, and let*

$$i : V \to V^* = \Omega^1(V),$$

be the linear map given by

$$i(v)(w) = \langle v, w \rangle.$$

Show that if $\{e_1, \ldots, e_n\}$ is an an orthonormal basis of V, then

$$i(e_k) = e^k,$$

where $\{e^1, \ldots, e^n\}$ is the dual basis.

Let $i : V \to V^*$ be given by $i(v)(w) = \langle v, w \rangle$. The dual basis is defined by the relations $e^i(e_j) = \delta^i_j$. We have

$$i(e_k)(w^i e_i) = \langle w, e_k \rangle = w^k.$$

So $i(e_k)$ is the functional that picks out the k-th component of a vecotr, that is, the functional e^k:

$$e^k(w^i e_i) = w^i e^k(e_i) = w^i \delta^k_i = w^k.$$

Problem 2.5 *With the assumptions of the previous problem, show the existance of an inner product on $\Omega^p(V)$ sich that*

$$\langle w_1 \wedge \ldots \wedge \omega_p, \tau_1 \wedge \ldots \wedge \tau_p \rangle = \det(\langle \omega_i, \tau_j \rangle),$$

whenever $\omega_i, \tau_j \in \Omega^1(V)$, and

$$\langle \omega, tau \rangle = \langle i^{-1}(\omega), i^{-1}(\tau) \rangle.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V, and let $\beta^j = i(e_j)$. Show that

$$\left\{ \beta^{\sigma(1)} \wedge \ldots \wedge \beta^{\sigma(p)} : \sigma \in S(p, n-p) \right\},$$

is an orthonormal basis of $\Omega^p(V)$.

We define the map $\langle , \rangle : \Omega^p(V) \times \Omega^p(V) \to \mathbb{R}$ by

$$\begin{aligned} \langle \omega, \tau \rangle &= \langle \omega_\sigma e^\sigma, \tau_\pi e^\pi \rangle \\ &= \omega_\sigma \tau_\pi \langle e^\sigma, e^\pi \rangle, \end{aligned}$$

where σ and τ stand for all the p-tuples of indices that are $(p, m-p$-shuffles, and the inner product on basis elements is defined as in the statement of the problem. The inner product on the basis elements is well-defined because i is an isomorphism and the Euclidean inner product is well-defined.

Exchanging the factors makes the exchange $\langle e^\sigma, e^\pi \rangle \to \langle e^\pi, e^\sigma \rangle$, but leaves the matrix $\langle e^i, e^j \rangle$ invariant because the Euclidean inner product is symmetric.

Next we have

$$\begin{aligned} \langle \alpha\omega + \beta\rho, \tau \rangle &= \langle (\alpha\omega_\sigma + \beta\rho_\sigma) e^\sigma, \tau_\pi e^\pi \rangle \\ &= (\alpha\omega_\sigma + \beta\rho_\sigma) \tau_\pi \langle e^\sigma, e^\pi \rangle, \end{aligned}$$

and the properties of an inner product are satisfied.

The elements form a basis by Theorem 2.15. Orthonormality follows from the previous exercise.

Problem 2.6 *Suppose $\omega \in \Omega^p(V)$. Let $v_1, \ldots, v_p$ be vectors in V and let $A = (a_{ij})$ be a $p \times p$ matrix. Show that for $w_i = a_{ij} w_j$ we have*

$$\omega(w_1, \ldots, w_p) = \det(A) \omega(v_1, \ldots, v_p).$$

We have

$$\omega(w_1, \ldots, w_p) = \sum_{\sigma \in S(p, n-p)} \omega_{\sigma(1) \ldots \sigma(p)} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}(w_1, \ldots, w_p),$$

where

$$\begin{aligned} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}(w_1, \ldots, w_p) &= \det \begin{vmatrix} e^{\sigma(1)} w_1 & \cdots & e^{\sigma(1)} w_p \\ \vdots & \ddots & \vdots \\ e^{\sigma(p)} w_1 & \cdots & e^{\sigma(p)} w_p \end{vmatrix} \\ &= \det \begin{vmatrix} e^{\sigma(1)} a_{1j} v_j & \cdots & e^{\sigma(1)} a_{pj} v_j \\ \vdots & \ddots & \vdots \\ e^{\sigma(p)} a_{1j} v_j & \cdots & e^{\sigma(p)} a_{pj} v_j \end{vmatrix}, \end{aligned}$$

since $w_i = a_{ij} v_j$. Now consider the matrix N given by $N_{ij} = e^{\sigma(i)} v_j$. Then the matrix

$$M_{ij} = N_{ik} a_{kj} = e^{\sigma(i)} v_k a_{kj},$$

is exactly the matrix in the above equation. Since the determinant of a product is the product of the determinants, the formula is established.

Problem 2.7 *Show for* $f : V \to W$ *that*

$$\Omega^{p+q}(f)(\omega_1 \wedge \omega_2) = \Omega^p(f)(\omega_1) \wedge \Omega^q(f)(\omega_2),$$

where $\omega_1 \in \Omega^p(W)$ *and* $\omega_2 \in \Omega^q(W)$.

On the one hand for $\Omega^p(f)(\omega_1) \wedge \Omega^q(f)(\omega_2)(\xi_1, \ldots, \xi_{p+q})$

$$\begin{aligned} & \sum_\sigma \Omega^p(f)(\omega_1)(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) \cdot \Omega^q(f)(\omega_2)(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\ = & \sum_\sigma \omega_1(f(\xi_{\sigma(1)}), \ldots, f(\xi_{\sigma(p)})) \cdot \omega_2(f(\xi_{\sigma(p+1)}), \ldots, f(\xi_{\sigma(p+q)})), \end{aligned}$$

while on the other hand $\Omega^{p+q}(f)(\omega_1 \wedge \omega_2)(\xi_1, \ldots, \xi_{p+q})$ is

$$\begin{aligned} & (\omega_1 \wedge \omega_2)(f(\xi_1), \ldots, f(\xi_{p+q})) \\ = & \sum_\sigma \omega_1(f(\xi_{\sigma(1)}), \ldots, f(\xi_{\sigma(p)})) \cdot \omega_2(f(\xi_{\sigma(p+1)}), \ldots, f(\xi_{\sigma(p+q)})), \end{aligned}$$

which proves the formula.

Problem 2.8 *Show that the set*

$$\left\{ f \in \mathrm{End}(V) : \exists g \in GL(V) : gfg^{-1} \text{ is diagonal} \right\},$$

is everywhere dense in $\mathrm{End}(V)$*, assuming* V *is a finite-dimensional complex vector space.*

Such an f is represented by an $n \times n$ complex matrix once a basis is chosen for V, and we will denote this matrix representation by f as well. If f has n distinct eigenvalues then it is obviously diagonalizable, so we will show that there exists complex matrices with distinct eigenvalues arbitrarily close to every given matrix f. We will consider these matrices as points in $\mathbb{C}^{n^2} \simeq \mathbb{R}^{4n^2}$, and use the standard topology in Euclidean space.

Given an f, denote its characteristic polynomial by

$$P_f(\lambda) = \det(f - \lambda I) = a_o + a_i \lambda^i,$$

where I is the unit matrix and $i \in (1, \ldots, n)$. Next we will define, for a given f and characteristic polynomial $P_f(\lambda)$, a function $F : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ defined by

$$F : (z_0, \ldots, z_n, \lambda) \mapsto z_0 + a_0 + (z_i + a_i)\lambda^i,$$

which is polynomial and therefore smooth (analytic).

The partial map

$$F_\lambda : (z_0, \ldots, z_n) \mapsto z_0 + z_i\lambda_i + P_f(\lambda),$$

has a (complex) differential given by

$$DF_\lambda = (1, \lambda^1, \ldots, \lambda^n),$$

so that

$$DF_\lambda \cdot DF_\lambda^\dagger = 1 + \sum \lambda^i \bar{\lambda}^i \geq 1,$$

so that the differential is surjective and thus F intersects every submanifold of $\mathbb{C}$ transversely, in particular $\{0\}$. Thus, by the transversality theorem, for almost every $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, the map $F_z : \lambda \to z_0 + a_0 + (a_i + z_i)\lambda^i$ intersects $\{0\}$ transversely.

Now, by the fundamental theorem of algebra this polynomial must take the value 0, so that the perturbed characteristic polynomial has distinct root (ie, if the roots were not distinct, the map would intersect $\{0\}$ tangentially rather than transversely). Since this holds for almost all $(z_0, \ldots, z_n)$, we can choose such a point within any δ-ball about the origin. Thus, this polynomial can be made arbitrarily close to our original one and our result will follow if a nearby polynomial is associated with a nearby matrix, which we show next.

We note that the maps $\varphi_k : f_{ij} \to b_k$, which give the coefficients of the characteristic polynomial from the matrix are analytic maps (they are polynomial themselves). Each of these function will possess a smooth local inverse whenever the image point is regular, but almost every such point is regular by Sard. Since there are only finitely many functions, almost every $(n+1)$-tuple $(b_0, \ldots, b_n)$ is regular for $\{\varphi_0, \ldots, \varphi_n\}$.

Now, almost every $z \in \mathbb{C}^{n+1}$ will satisfy both conditions above, ie, be regular for the maps φ_k and give distinct zeros for the perturbed characteristic polynomial. Thus, for every ϵ-ball about f we can find a corresponding δ-ball about the origin in $\mathbb{C}^{n+1}$ that contains a point z such that the characteristic polynomial

$$z_0 + a_0 + (a_i + z_i)\lambda^i,$$

has n distinct roots and is the characteristic polynomial of some matrix in V, which establishes the result. We note that it is essential to consider complex matrices since the proof depends on the fundamental theorem of algebra, which doesn't gaurantee roots over $\mathbb{R}$.

Problem 2.9 *Let V be an n-dimensional vector space with inner product $\langle , \rangle$. From Exercise 2.5 we obtain an inner product on $\Omega^p(V)$ for all p, in particular for $p = n$.*

A volume element on V is an unit vector vol $\in \Omega^n(V)$. *Hodge's star operator*

$$* : \Omega^p(V) \to \Omega^{n-p}(V),$$

is defined by the equation $\langle *\omega, \tau\rangle \mathrm{vol} = \omega \wedge \tau$ *for all* $\tau \in \Omega^p(V)$. *Show that* $*$ *is well-defined and linear.*

Let $\{e_1 \dots e_n\}$ *be a basis of* V *with* $\mathrm{vol}(e_1, \dots, e_n) = 1$ *and* $\{e^1 \dots e^n\}$ *the dual basis. Show that*

$$*(e^1 \wedge \dots \wedge e^p) = e^{p+1} \wedge \dots \wedge e^n,$$

and in general that

$$*(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) = \mathrm{sgn}(\sigma) e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)},$$

with $\sigma \in S(p, n-p)$. *Show that* $* \circ * = (-1)^{p(n-p)}$ *on* $\Omega^p(V)$.

Suppose that u_1 and u_2 are two forms that satisfy the above equation for every τ, then we have

$$\begin{aligned} 0 = \langle u_1, \tau\rangle \mathrm{vol} - \langle u_2, \tau\rangle \mathrm{vol} &= (\langle u_1, \tau\rangle - \langle u_2, \tau\rangle)\mathrm{vol} \\ &= (\langle u_1 - u_2, \tau\rangle \mathrm{vol}, \end{aligned}$$

which shows that $u_1 = u_2$, so that the map is well-defined. For linearity we have

$$\begin{aligned} \langle *(\omega + \pi), \tau\rangle \mathrm{vol} &= (\omega + \pi) \wedge \tau \\ &= \omega \wedge \tau + \pi \wedge \tau \\ &= \langle *\omega, \tau\rangle \mathrm{vol} + \langle *\pi, \tau\rangle \mathrm{vol} \end{aligned}$$

from linearity of the wedge product.

Next, suppose that we have

$$\langle *(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}), \tau\rangle \mathrm{vol} = (e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) \wedge \tau.$$

Using linearity we will let τ be a basis element

$$\tau = e^{i_1} \wedge \dots \wedge e^{i_{n-p}}.$$

Thus the right hand side of our formula yields

$$e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)} \wedge e^{i_1} \wedge \dots \wedge e^{i_{n-p}},$$

which is zero unless τ is a permutation of the remaining basis vectors:

$$\tau = e^{\sigma(p+q)} \wedge \dots \wedge e^{\sigma(n)}.$$

Thus we have

$$\langle *(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}), e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)}\rangle \mathrm{vol} = \mathrm{sgn}(\sigma)\mathrm{vol},$$

which yields $*(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) = \mathrm{sgn}(\sigma) \cdot e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)})$.

Now we wish to evaluate $* \circ * : \Omega^p(V) \to \Omega^p(V)$. By the linearity of $*$ it suffices to consider its action on basis elements. We have

$$* : e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)} \mapsto \mathrm{sgn}(\sigma) e^{\sigma(p+1)} \wedge \ldots \wedge e^{\sigma(n)},$$

and likewise

$$* : e^{\pi(1)} \wedge \ldots \wedge e^{\pi(n-p)} \mapsto \mathrm{sgn}(\pi) e^{\pi(n-p+1)} \wedge \ldots \wedge e^{\pi(n)},$$

where the permutations are related by

$$(\pi(1), \ldots, \pi(n-p), \pi(n-p+1), \ldots, \pi(n)) = (\sigma(p+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(p)),$$

that is, π is a cyclic (*cyc*) permutation of σ, thus $\mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(cyc)$. Since we go from σ to π by moving the last p elements across the first $n-p$ there are $p(n-p)$ interchanges, which is the sign of *cyc*, thus we have

$$\begin{aligned} * : e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)} &\mapsto \mathrm{sgn}(cyc) \cdot e^{\pi(n-p+1)} \wedge \ldots \wedge e^{\pi(n)} \\ &= (-1)^{p(n-p)} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}, \end{aligned}$$

which establishes the formula.

Problem 2.10 *Let V be a 4-dimensional vector space and $\{e^1, \ldots, e^4\}$ the dual basis. Let $A = (a_{ij})$ be a skew-symmetric matrix and define*

$$\alpha = \sum_{i<j} a_{ij} e^i \wedge e^j.$$

Show that

$$\alpha \wedge \alpha = 0 \leftrightarrow \det(A) = 0.$$

Say $\alpha \wedge \alpha = \lambda e^1 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4$. What is the relation between λ and $\det(A)$?

The first part follows immediately from proof of exercise 2.2 with $n = 2$. Also, according to that exercise we have $\lambda = n!\mathrm{pf}(A)$, or $\lambda = n!\sqrt{\det(A)}$.

Problem 2.11 *Let V be an n-dimensional vector space with inner product $\langle , \rangle$ and volume element $\mathrm{vol} \in \Omega^n(V)$, as in Exercise 2.9. Let $v \in \Omega^1(V)$ and*

$$F_v : \Omega^p \to \Omega^{p+1}(V),$$

be the map

$$F_v(\omega) = v \wedge \omega.$$

Show that the map

$$F_v^* = (-1)^{np} * \circ F_v \circ * : \Omega^{p+1}(V) \to \Omega^p(V),$$

is adjoint to F_v, that is, $\langle F_v \omega, \tau \rangle = \langle \omega, F_v^* \tau \rangle$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V with $\mathrm{vol}(e^1, \ldots, e^n) = 1$. Show that

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \langle v, e^i \rangle e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1}.$$

Show that $F_v F_v^* + F_v^* F_v : \Omega^p(V) \to \Omega^p(V)$ is multiplication by $||v||^2$.

In this problem v is a 1-form, ω a p-form, and τ a $p+q$-form. First we make the observation that (for forms of appropriate rank)

$$\langle *(a \wedge b), c\rangle \mathrm{vol} = (a \wedge b) \wedge c = a \wedge (b \wedge c) = \langle *a, b \wedge c\rangle \mathrm{vol},$$

so that

$$\langle *(a \wedge b), c\rangle = \langle *a, b \wedge c\rangle.$$

Now, on one hand we have

$$\langle F_v \omega, \tau\rangle = \langle v \wedge \omega, \tau\rangle,$$

while on the other

$$\begin{aligned}
\langle F_v^* \tau, \omega\rangle &= (-1)^{np} \langle *F_v(*\tau), \omega\rangle \\
&= (-1)^{np} \langle *(v \wedge (*\tau), \omega\rangle \\
&= (-1)^{np+p} \langle *(*\tau \wedge v), \omega\rangle \\
&= (-1)^{p(n+1)} \langle * * \tau, v \wedge \omega\rangle \\
&= (-1)^{p(n+1)} (-1)^{p(n-p)} \langle \tau, v \wedge \omega\rangle \\
&= (-1)^{p-p^2} \langle v \wedge \omega, \tau\rangle,
\end{aligned}$$

but $p - p^2 = p(1-p)$ is always even, which establishes the result.

Next, we have $F_v^*() = (-1)^{np} * F_v * ()$, so we'll take the component mappings in sequence:

$$* : e^1 \wedge \ldots \wedge e^{p+1} \mapsto e^{p+2} \wedge \ldots \wedge e^n.$$

Next is

$$F_v : e^{p+2} \wedge \ldots \wedge e^n \mapsto v \wedge e^{p+2} \wedge \ldots \wedge e^n,$$

and finally

$$* : v \wedge e^{p+2} \wedge \ldots \wedge e^n \mapsto \mathrm{sgn}(\sigma) v_i e^1 \wedge \ldots \wedge \bar{e}^i \wedge \ldots \wedge e^{p+1},$$

where the hat indicated that term is missing and σ is the permutation that takes $\{1, \ldots, n\} \to \{i, p+2, \ldots, n, 1, 2, \ldots, i-1, \ldots, p+1\}$. This permutation is obtained by first moving the $n-p$ elements $\{i, p+2, \ldots, n\}$ past the remaining p elements which requires $p(n-p)$ interchanges, and then moving i to its proper place, which requires an additional $p+1-i$, which yeilds

$$\mathrm{sgn}(\sigma) = (-1)^{p(n-p)+p+1-i}.$$

The exponent can be rewritten as $pn + (i+1) + p(1-p)$, since adding 2 doesn't change the parity. But $p(1-p)$ is always even, and the pn will cancel with the pn from the map definition, yeilding

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = (-1)^{i+1} v_i e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1},$$

which can also be written as

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = (-1)^{i+1} \langle v, e^i\rangle \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1},$$

proving the formula.

Next, consider the operator $F_v F_v^* + F_v^* F_V : \Omega^p(V) \to \Omega^p(V)$ with $v = \lambda e^1$. We have two cases. The first is the action of this operator on a basis element containing e^1. We have

$$F_v^* : e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^{i_2} \wedge \ldots \wedge e^{i_p},$$

followed by

$$F_v : \lambda e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda^2 e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p}.$$

However, the second term in the operator gives zero because

$$F_v : e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^1 \wedge e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which is zero, thus the operator is multiplication by $\lambda^2 = ||v||^2$.

The second case is the action of the operator on a basis element not containing e^1. The first term gives zero in this case since

$$* : e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \operatorname{sgn}(\sigma) e^1 \wedge \ldots \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n},$$

and then

$$F_v : e^1 \wedge \ldots \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \mapsto \lambda e^1 \wedge e^1 \wedge \ldots \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which is zero. Now, the second term yields

$$F_v : e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^1 \wedge e^2 \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n},$$

and then

$$F_v^* : \lambda e^1 \wedge e^2 \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \mapsto \lambda^2 e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which again shwos that the operator is multiplication by $||v||^2$.

Finally, for arbitrary v, we are free to choose our basis so that e^1 is in the direction of v, so the problem reduces to the particular case above with the same result, since it was written in a coordinate independent form.

Problem 2.12 *Let V be an n-dimensional vector space. Show for a linear map $f : V \to V$ the existance of a number $d(f)$ such that*

$$\Omega^n(f)(\omega) = d(f)\omega,$$

for $\omega \in \Omega^n(V)$. Verify the product rule

$$d(g \circ f) = d(g)d(f),$$

for linear maps $f, g : V \to V$ using the functoriality of $\Omega^n()$. Prove that $d(f) = \det(f)$.

Let $\{e_1 \dots e_n\}$ be a basis of V and $\{e^1, \dots, e^n\}$ the dual. There is only one basis element in $\Omega^n(V)$, so we have $\omega = \hat{\omega} e^1 \wedge \dots \wedge e^n$, where $\hat{\omega}$ is the component of ω in this basis. The action of $\Omega^n(f)(\omega)$ on the basis vectors of V is given by

$$\begin{aligned}
\Omega^n(f)(\omega)(e_1, \dots, e_n) &= \omega(f(e_1), \dots, f(e_n)) \\
&= \hat{\omega} e^1 \wedge \dots \wedge e^n (f_1^k e_k, \dots, f_n^k e_k) \\
&= \hat{\omega} \det \begin{pmatrix} e^1(f_1^k e_k) & \cdots & e^1(f_n^k e_k) \\ \vdots & \ddots & \dots \\ e^n(f_1^k e_k) & \cdots & e^n(f_n^k e_k) \end{pmatrix} \\
&= \hat{\omega} \det(e^i f_j^k e_k).
\end{aligned}$$

The matrix whose determinant we want is simply the product of two matrices

$$e^i f_j^k e_k = (f_j^k)(e_k e^i),$$

and so we have

$$\begin{aligned}
\det(e^i f_j^k e_k) &= \det(f_j^k) \det(e_k e^i) \\
&= \det(f_j^k) \det(\delta_k^i) \\
&= \det(f).
\end{aligned}$$

This establishes the formula with $d(f) = \det(f)$. Next, a commutative diagram of maps on V gives rise to a commutative diagrams of maps on $\Omega^n(V)$:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[ddr, "g \circ f"'] & V \arrow[dd, "g"] & & & \Omega^n(V) & \Omega^n(V) \arrow[l, "\Omega^n(f)"'] \\
& & {} \arrow[r] & {} & & \\
& V & & & & \Omega^n(V) \arrow[uu, "\Omega^n(g)"'] \arrow[uul, "\Omega^n(g \circ f)"]
\end{tikzcd}$$

Given the first result of the problem we can rewrite the second diaram as

$$\begin{tikzcd}
\Omega^n(V) & \Omega^n(V) \arrow[l, "d(f)"'] \\
& \Omega^n(V) \arrow[u, "d(g)"'] \arrow[ul, "d(g \circ f)"]
\end{tikzcd}$$

which shows that $(d \circ f) = d(f)d(g)$.