Cohomology Homework: Chapters 1 & 2

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Problem 1.1 *Perform the calculations of Example 1.7.*

Let $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\right\}$ be the unit circle in the (x_1, x_2)-plane, let $U = \mathbb{R}^3 - S$, and consider the function $f : U \to \mathbb{R}$ given by

$$f : (x_1, x_2, x_3) \mapsto \left(\frac{-2x_1x_3}{\xi}, \frac{-2x_2x_3}{\xi}, \frac{x_1^2 + x_2^2 - 1}{\xi}\right),$$

where $\xi = x_3^2 + (x_1^2 + x_2^2 - 1)^2$. First we compute the derivative of ξ:

$$\begin{aligned}
\partial_1\xi &= 4x_1(x_1^2 + x_2^2 - 1) \\
\partial_2\xi &= 4x_2(x_1^2 + x_2^2 - 1) \\
\partial_3\xi &= 2x_3.
\end{aligned}$$

Now the components of the curl as evaluated as

$$\begin{aligned}
(\operatorname{curl} f)_1 &= \partial_2 f_3 - \partial_3 f_2 \\
&= \left(\frac{2x_2}{\xi} - \frac{x_1^2 + x_2^2 - 1}{\xi^2}\partial_2\xi\right) - \left(\frac{-2x_2}{\xi} + \frac{2x_2x_3}{\xi^2}\partial_3\xi\right) \\
&= \frac{4x_2\xi - (x_1^2 + x_2^2 - 1)\partial_2\xi - 2x_2x_3\partial_3\xi}{\xi^2} \\
&= \frac{4x_2\xi - (x_1^2 + x_2^2 - 1)^2(4x_2) - 4x_2x_3^2}{\xi^2} \\
&= \frac{4x_2}{\xi^2}(\xi - (x_3^2 + (x_1^2 + x_2^2 - 1)^2)) \\
&= \frac{4x_2}{\xi^2}(\xi - \xi) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(\operatorname{curl} f)_2 &= \partial_3 f_1 - \partial_1 f_3 \\
&= \left(\frac{-2x_1}{\xi} + \frac{2x_1x_3}{\xi^2}\partial_3\xi\right) - \left(\frac{2x_1}{\xi} - \frac{x_1^2 + x_2^2 - 1}{\xi^2}\partial_1\xi\right) \\
&= \frac{-4x_1\xi + 2x_1x_3\partial_3\xi + (x_1^2 + x_2^2 - 1)\partial_1\xi}{\xi^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-4x_1\xi + 4x_1x_3^2 + 4x_1(x_1^2+x_2^2-1)^2}{\xi^2} \\
&= \frac{4x_1}{\xi^2}(-\xi + 4x_3^2 + (x_1^2+x_2^2-1)^2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{curl} f)_3 &= \partial_1 f_2 - \partial_2 f_1 \\
&= \frac{2x_2x_3}{\xi}\partial_1\xi - \frac{2x_1x_3}{\xi}\partial_2\xi \\
&= \frac{2x_3}{\xi}(x_2\partial_1\xi - x_1\partial_2\xi) \\
&= \frac{2x_3}{\xi}(4x_1x_2(x_1^2+x_2^2-1) - 4x_1x_2(x_1^2+x_2^2-1)) \\
&= 0
\end{aligned}
$$

Thus we have $\operatorname{curl} f = 0$. So $f \in \ker(\operatorname{curl})$ and $[f] \in H^1(U)$. We will show, hwoever, that $[f] \neq 0$ by showing that there is no function F such that $f = \operatorname{grad} F$.

Suppose such an F existed and consider the integral of $\frac{d}{dt}F(\gamma(t))$ where the curve γ is given by

$$
\gamma(t) = \left(\sqrt{1+\cos t}, 0, \sin t\right), \quad -\pi \leq t \leq \pi.
$$

First we have

$$
\begin{aligned}
\int_{-\pi}^{\pi} \frac{d}{dt}F(\gamma(t))dt &= \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon}\frac{d}{dt}F(\gamma(t))dt \\
&= \lim_{\epsilon\to 0} F(\gamma(t))\Big|_{-\pi+\epsilon}^{\pi-\epsilon} \\
&= 0,
\end{aligned}
$$

where the limit was taken since the curve is not differentiable at its endpoints. One the other hand we have

$$
\begin{aligned}
\frac{d}{dt}F(\gamma(t))dt &= \partial_i F(\gamma(t))\dot{\gamma}_i(t) \\
&= f_i(\gamma(t))\dot{\gamma}_i(t)
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\dot{\gamma}_1(t) &= \frac{-\sin t}{2\sqrt{1+\cos t}} \\
\dot{\gamma}_2(t) &= 0 \\
\dot{\gamma}_3(t) &= \cos t,
\end{aligned}
$$

and $\xi(\gamma(t)) = \sin^2 t + (1+\cos t - 1)^2 = 1.$ so that

$$
f(\gamma(t)) = (-2\sin t\sqrt{1+\cos t}, 0, \cos t),
$$

and finally

$$f_i(\gamma(t))\dot{\gamma}_i(t) = \sin^2 t + 0 + \cos^2 t = 1,$$

for all t, which yields a value of 2π for the integral over γ which gives a contradiction. Thus F cannot exist.

Problem 1.2 *Let W be the open set given by*

$$W = \left\{(x_1, x_2, x_3) \in \mathbb{R} : \text{ either } x_3 \neq 0 \text{ or } x_1^2 + x_2^2 < 1\right\}.$$

Prove the existance and uniqueness of a function $F \in C^\infty(W, \mathbb{R})$ such that $\operatorname{grad}(F)$ *is the vector field considered in Example 1.7 and $F(0) = 0$. Find a simple expression for F valid when $x_1^2 + x_2^2 < 1$.*

Existance: It suffices to show that W is star-shaped about the origin, since then the existance will follow from Theorem 1.6.

Let $x \in W$, $s = (s_1, s_2, s_3)$, and suppose that $s_3 = 0$. then s is contained in the unit circle in the (x_1, x_2)-plane which is star-shaped. Now suppose that $s_3 \neq 0$ and let p be a point on the line joining s to the origin. If $p_3 = 0$, then $p = 0$ and $p \in W$, while if $p_3 \neq 0$ then $p \in W$. Thus W is star-shaped.

Now we have some F which satisfies $\operatorname{grad}(F) = f$. Consider a new function F' defined by

$$F'(x) = F(x) - F(0).$$

Then $F'(0) = 0$ and $\operatorname{grad}(F') = \operatorname{grad}(F) - 0 = f$ and thus satisfies the given condition.

Uniqueness: Suppose F_1 and F_2 both satisfy the given criteria, then we have $\operatorname{grad}(F_1 - F_2) = 0$, so that the function $F_1 - F_2$ is constant on W (since W is connected). Moreover, $F_1(0) - F_2(0) = 0$ so that $F_1(x) = F_2(x)$ for all $x \in W$, so that the solution is unique.

Finally, we give an expression for F when $x_1^2 + x_2^2 < 1$. First, set $x_3 = 0$, then we have

$$f = \left(0, 0, \frac{1}{x_1^2 + x_2^2 - 1}\right),$$

that is, $\partial_1 F = \partial_2 F = 0$ whenever $x_3 = 0$. Thus we have

$$F(x_1, x_2, 0) = F(0, 0, 0) = 0,$$

for $x_1^2 + x_2^2 < 1$.

Now we have

$$F(x_1, x_2, x_3) = \int_0^1 f_i(\gamma(t))\dot{\gamma}_i(t)dt,$$

for any curve $\gamma(t)$ that connects the origin to (x_1, x_2, x_3). Due to the above we can take our curve as the straight line from the origin to the point $(x_1, x_2, 0)$, which doesn't contribute to the integral, and then as the line from there to the final point, which will be the curve

$$\gamma(t) = (x_1, x_2, tx_3),$$

so that our integral becomes

$$\begin{aligned} F(x_1,x_2,x_3) &= \int_0^1 f_3(x_1,x_2,tx_3)x_3 dt \\ &= x_3 \int_0^1 \frac{x_1^2+x_2^2-1}{(tx_3)^2+(x_1^2+x_2^2-1)^2} dt \\ &= \frac{x_3}{\alpha}\int_0^1 \frac{dt}{(tx_3/\alpha)^2+1}, \end{aligned}$$

where we have set $\alpha = x_1^2+x_2^2-1$. Now if we make the substitution $y = tx_3/(x_1^2+x_2^2-1)$ we obtain

$$\begin{aligned} F(x_1,x_2,x_3) &= \int_0^{x_3/\alpha} \frac{dy}{y^2+1} \\ &= \arctan(y)\Big|_0^{x_3/\alpha} \\ &= \arctan\left(\frac{x_3}{x_1^2+x_2^2-1}\right). \end{aligned}$$

Problem 2.1 *Prove the formula in Remark 2.10.*

In the 'older' formula we sum over all permutations $\sigma \in S(p+q)$ of the vectors, not just the ordered ones. Let σ be such a permutation, then there are

$$(p+q)! = \frac{(p+q)!}{p}!q!p!q! = \left(\begin{array}{c} p+q \\ p \end{array}\right) p!q!,$$

arrangements, corresponding to permuting the first p objects among themselves, then the last q among themselves, and then shuffling the first p with the last q. We will write σ using this decomposition: $\sigma = \tau \circ \pi_p \circ \pi_q$, where $\tau \in S(p,q)$ is a (p,q)-shuffle, $\pi_p \in S(p,\bar{q})$ is a permutation among the first p objects, $\pi_q \in S(\bar{p},q)$ is a permutation among the remaining p. The our formula becomes

$$\frac{1}{p!q!}\sum_{\tau}\sum_{\pi_p}\sum_{\pi_q} \operatorname{sgn}(\sigma)\omega_1(\xi_{\sigma(1)}\dots\xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}\dots\xi_{\sigma(n)}),$$

where we have

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau\circ\pi_p\circ\pi_q) = \operatorname{sgn}(\tau)\cdot\operatorname{sgn}(\pi_p)\cdot\operatorname{sgn}(\pi_q).$$

Likewise, we have (since π_q leave the first p unchanged, π_p leave the last q unchanged, and therefore they commute)

$$\begin{aligned} \omega_1(\xi_{\sigma(1)}\dots\xi_{\sigma(p)}) &= \omega_1(\xi_{\tau\circ\pi_p\circ\pi_q(1)}\dots\xi_{\tau\circ\pi_p\circ\pi_q(p)}) \\ &= \omega_1(\xi_{\tau\circ\pi_p(1)}\dots\xi_{\tau\circ\pi_p(p)}) \\ &= \operatorname{sgn}(\pi_p)\cdot\omega_1(\xi_{\tau(1)}\dots\xi_{\tau(p)}), \end{aligned}$$

and

$$\begin{aligned}
\omega_2(\xi_{\sigma(p+1)}\ldots\xi_{\sigma(n)}) &= \omega_2(\xi_{\tau\circ\pi_p\circ\pi_q(p+1)}\ldots\xi_{\tau\circ\pi_p\circ\pi_q(n)}) \\
&= \omega_2(\xi_{\tau\circ\pi_q\circ\pi_p(p+1)}\ldots\xi_{\tau\circ\pi_q\circ\pi_p(n)}) \\
&= \omega_2(\xi_{\tau\circ\pi_q(p+1)}\ldots\xi_{\tau\circ\pi_q(n)}) \\
&= \operatorname{sgn}(\pi_q)\cdot\omega_2(\xi_{\tau(p+1)}\ldots\xi_{\tau(n)}).
\end{aligned}$$

This leave us with

$$\frac{1}{p!q!}\sum_{\tau}\sum_{\pi_p}\sum_{\pi_q}\operatorname{sgn}(\tau)\omega_1(\xi_{\tau(1)}\ldots\xi_{\tau(p)})\omega_2(\xi_{\tau(p+1)}\ldots\xi_{\tau(n)}),$$

since $\operatorname{sgn}^2 = 1$ for any permutation. The sums over π_p and pi_q just give $p!$ and $q!$ respectively, which cancel the factorials already present and thus establishes the result.

Problem 2.2 *Find an $\omega \in \Omega^2\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.*

More generally, let V be a $2n$-dimensional vector space and α a $2-$form defined by the skew-symmetric $2n \times 2n$-matrix a_{ij}

$$\alpha = \alpha_{ij}e^i \wedge e^j,$$

in the orthonormal basis $\{e^i\}$,where it is understood that the sum is for $i < j$. First we verify that α is indeed a $2-$form.

$$\begin{aligned}
\alpha(x,x) &= (\alpha_{ij}e^i \wedge e^j)(x^r e_r, x^2 e_s) \\
&= a_{ij}x^r x^s e^i \wedge e^j(e_r, e_s) \\
&= a_{ij}x^r x^s(\delta^i_r\delta^j_s - \delta^i_s\delta^j_r) \\
&= a_{ij}x^i x^j - a_{ij}x^i x^j \\
&= 0.
\end{aligned}$$

Now we define a $2n$-form β by

$$\beta = \alpha\underbrace{\wedge\ldots\wedge}_{\text{n times}}\alpha.$$

Using our basis, we have

$$\begin{aligned}
\beta &= (a_{i_1j_1}e^{i_1}\wedge e^{j_1})\wedge\ldots\wedge(a_{i_nj_n}e^{i_n}\wedge e^{j_n}) \\
&= a_{i_1j_1}\ldots a_{i_nj_n}(e^{i_1}\wedge e^{j_1})\wedge\ldots\wedge(e^{i_n}\wedge e^{j_n}).
\end{aligned}$$

Now, there are only $2n$ basis elements, so a term in the sum will vanish unless the basis elements form a permutation of all $\{1,\ldots,2n\}$. Moreover, we can swap products of pairs of basis elements without changing its value since this is always an even permutation, thus we can write

$$\beta = n!a_{i_1j_1}\ldots a_{i_nj_n}(e^{i_1}\wedge e^{j_1})\wedge\ldots\wedge(e^{i_n}\wedge e^{j_n}),$$

where we now have $i_l < i_k$ whenever $l < k$, that is, the sum is over all unordered pairs of basis elements. Since there were n pairs of basis elements, there are now $n!$ permutations of them, each contributing the same to the sum.

Now, we still have $i_l < i_k$, so our sum is over all unordered partitions of the integers $\{1, \ldots, 2n\}$ into pairs. We now rewrite our sum as

$$\beta = n! \left(\sum_{\sigma} \operatorname{sgn}(\sigma) \cdot a_{i_1 j_1} \ldots a_{i_n j_n} \right) e^1 \wedge \ldots \wedge e^{2n},$$

where σ is a permutation that gives the desired partition, and we sum over only partitions. We can rewrite the sum this way since any such permutation of the basis elements changes the term by exactly $\operatorname{sgn}(\sigma)$.

But, we notice that

$$\sum_{\sigma} \operatorname{sgn}(\sigma) \cdot a_{i_1 j_1} \ldots a_{i_n j_n} = \operatorname{pf}(A),$$

the Pfaffian of A, and for a $2n \times 2n$ skew-symmetric matrix we have

$$\operatorname{pf}^2(A) = \det(A).$$

Thus we see that $\beta = 0$ iff $\det(A) = 0$. (We note that this holds as long as A has an even number of rows, as it does here. For an odd number of rows, the determinant is always zero.)

For the problem at hand it suffices to pick any 4×4 matrix with non-vanishing determinant, such as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which has determinant 1.

Problem 2.3 *Show that there exist isomorphisms*

$$i : \mathbb{R}^3 \to \Omega^1\mathbb{R}^3, \;\; j : \mathbb{R}^3 \to \Omega^2\mathbb{R}^3,$$

given by

$$i(v)(w) = \langle v, w, \rangle, \;\; j(v)(w_1, w_2) = \det(v, w_1, w_2),$$

where $\langle , \rangle$ *is the usual inner product. Show that for* $v_1, v_2 \in \mathbb{R}^3$*, we have*

$$i(v_1) \wedge i(v_2) = j(v_1 \times v_2).$$

Define the map $i : \mathbb{R}^3 \to \Omega^1\mathbb{R}^3$ between vector spaces by

$$i(v)(w) = \langle v, w \rangle.$$

Then we have

$$\begin{aligned} i(\alpha u + \beta v)(w) &= \langle \alpha u + \beta v, w\rangle \\ &= \alpha\langle u, w\rangle + \beta\langle v, w\rangle \\ &= \alpha i(u)(w) + \beta i(v)(w), \end{aligned}$$

which shows that i is a linear map.

Now, let $u \in \Omega^1\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ an orthonormal basis for $\mathbb{R}^3$ and $\{e^1, e^2, e^3\}$ the dual basis. Then we have

$$\begin{aligned} u(w) &= u_i e^i(w^j e_j) \\ &= u_i w^j e^i(e_j) \\ &= u_i w^j \delta^i_j \\ &= u_i w^i \\ &= \langle u, w\rangle, \end{aligned}$$

which shows that i is surjective. But the vector spaces have the same dimension, so we conlude that i is injective and thus an isomorphism.

Now we define the map $j : \mathbb{R}^3 \to \Omega^2\mathbb{R}^3$ by

$$j(v)(w_1, w_2) = \det(v, w_1, w_2).$$

Then we have

$$\begin{aligned} j(\alpha u + \beta v)(w_1, w_2) &= \det(\alpha u + \beta v, w_1, w_2) \\ &= \det(\alpha u, w_1, w_2) + \det(\beta v, w_1, w_2) \\ &= \alpha \det(u, w_1, w_2) + \beta \det(v, w_1, w_2) \\ &= \alpha j(u)(w_1, w_2) + \beta j(v)(w_1, w_2), \end{aligned}$$

which shows that j is linear.

Let $u \in \Omega^2\mathbb{R}^3$, then we can write

$$u = \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} e^{\sigma(1)} \wedge e^{\sigma(2)}.$$

The action of the dual basis on the basis of $\mathbb{R}^3$ is given by

$$\begin{aligned} (e^a \wedge e^b)(e_i, e_j) &= \delta^a_i \delta^b_j - \delta^a_j \delta^b_i \\ &= \epsilon^{abk}\epsilon_{ijk}, \end{aligned}$$

where ϵ_{ijk} is the Levi-civita symbol. So we have

$$\begin{aligned} u(x, y) &= \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} x^i y^j e^{\sigma(1)} \wedge e^{\sigma(2)}(e_i, e_j) \\ &= \sum_{\sigma\in S(2,1)} u_{\sigma(1),\sigma(2)} x^i y^j \epsilon^{\sigma(1)\sigma(2)k}\epsilon_{ijk}. \end{aligned}$$

So, let us define

$$u^k = \sum_{\sigma \in S(2,1)} u_{\sigma(1),\sigma(2)} \epsilon^{\sigma(1)\sigma(2)k},$$

which is a triple of numbers, so a vector in $\mathbb{R}^3$. In particular, we get $u^1 = u_{23}$, $u^2 = u_{13}$, and $u_3 = u_{12}$. Now we have

$$\begin{aligned} u(x,y) &= \epsilon_{ijk} u^k x^i y^j \\ &= \epsilon_{kij} u^k x^i y^j \\ &= \det(u,x,y), \end{aligned}$$

and our mapping is surjective. Again, since the dimensions are the same, j is automatically injective and hence an isomorphism.

Next, consider $i(v_1) \wedge i(v_2)$, which is given by

$$\begin{aligned} i(v_1) \wedge i(v_2)(w_1,w_2) &= \det \begin{vmatrix} i(v_1)(w_1) & i(v_1)(w_2) \\ i(v_2)(w_1) & i(v_2)(w_2) \end{vmatrix} \\ &= \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \cdot \langle v_2, w_1 \rangle. \end{aligned}$$

On the other hand consider $j(v_1 \times v_2)$, which is given by

$$\begin{aligned} j(v_1 \times v_2)(w_1,w_2) &= \det(v_1 \times v_2, w_1, w_2) \\ &= \epsilon_{ijk}(v_1 \times v_2)^i w_1^j w_2^k \\ &= \epsilon_{ijk}\epsilon^i_{ab} v_1^a v_2^b w_1^j w_2^k \\ &= (\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) v_1^a v_2^b w_1^j w_2^k \\ &= (v_1^a w_1^a) \cdot (v_2^b w_2^b) - (v_1^a w_2^a) \cdot (v_2^b w_1^b) \\ &= \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \cdot \langle v_2, w_1 \rangle. \end{aligned}$$

Comparing these expressinos establishes the result.

Problem 2.4 *Let V be a finite-dimensional vector space over $\mathbb{R}$ with inner product $\langle , \rangle$, and let*

$$i : V \to V^* = \Omega^1(V),$$

be the linear map given by

$$i(v)(w) = \langle v, w \rangle.$$

Show that if $\{e_1, \ldots, e_n\}$ is an an orthonormal basis of V, then

$$i(e_k) = e^k,$$

where $\{e^1, \ldots, e^n\}$ is the dual basis.

Let $i : V \to V^*$ be given by $i(v)(w) = \langle v, w \rangle$. The dual basis is defined by the relations $e^i(e_j) = \delta^i_j$. We have

$$i(e_k)(w^i e_i) = \langle w, e_k \rangle = w^k.$$

So $i(e_k)$ is the functional that picks out the k-th component of a vecotr, that is, the functional e^k:

$$e^k(w^i e_i) = w^i e^k(e_i) = w^i \delta^k_i = w^k.$$

Problem 2.5 *With the assumptions of the previous problem, show the existance of an inner product on $\Omega^p(V)$ sich that*

$$\langle w_1 \wedge \ldots \wedge \omega_p, \tau_1 \wedge \ldots \wedge \tau_p \rangle = \det(\langle \omega_i, \tau_j \rangle),$$

whenever $\omega_i, \tau_j \in \Omega^1(V)$, and

$$\langle \omega, tau \rangle = \langle i^{-1}(\omega), i^{-1}(\tau) \rangle.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V, and let $\beta^j = i(e_j)$. Show that

$$\left\{ \beta^{\sigma(1)} \wedge \ldots \wedge \beta^{\sigma(p)} : \sigma \in S(p, n-p) \right\},$$

is an orthonormal basis of $\Omega^p(V)$.

We define the map $\langle , \rangle : \Omega^p(V) \times \Omega^p(V) \to \mathbb{R}$ by

$$\begin{aligned} \langle \omega, \tau \rangle &= \langle \omega_\sigma e^\sigma, \tau_\pi e^\pi \rangle \\ &= \omega_\sigma \tau_\pi \langle e^\sigma, e^\pi \rangle, \end{aligned}$$

where σ and τ stand for all the p-tuples of indices that are $(p, m-p$-shuffles, and the inner product on basis elements is defined as in the statement of the problem. The inner product on the basis elements is well-defined because i is an isomorphism and the Euclidean inner product is well-defined.

Exchanging the factors makes the exchange $\langle e^\sigma, e^\pi \rangle \to \langle e^\pi, e^\sigma \rangle$, but leaves the matrix $\langle e^i, e^j \rangle$ invariant because the Euclidean inner product is symmetric.

Next we have

$$\begin{aligned} \langle \alpha\omega + \beta\rho, \tau \rangle &= \langle (\alpha\omega_\sigma + \beta\rho_\sigma) e^\sigma, \tau_\pi e^\pi \rangle \\ &= (\alpha\omega_\sigma + \beta\rho_\sigma) \tau_\pi \langle e^\sigma, e^\pi \rangle, \end{aligned}$$

and the properties of an inner product are satisfied.

The elements form a basis by Theorem 2.15. Orthonormality follows from the previous exercise.

Problem 2.6 *Suppose $\omega \in \Omega^p(V)$. Let $v_1, \ldots, v_p$ be vectors in V and let $A = (a_{ij})$ be a $p \times p$ matrix. Show that for $w_i = a_{ij} w_j$ we have*

$$\omega(w_1, \ldots, w_p) = \det(A) \omega(v_1, \ldots, v_p).$$

We have

$$\omega(w_1, \ldots, w_p) = \sum_{\sigma \in S(p, n-p)} \omega_{\sigma(1) \ldots \sigma(p)} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}(w_1, \ldots, w_p),$$

where

$$\begin{aligned} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}(w_1, \ldots, w_p) &= \det \begin{vmatrix} e^{\sigma(1)} w_1 & \cdots & e^{\sigma(1)} w_p \\ \vdots & \ddots & \vdots \\ e^{\sigma(p)} w_1 & \cdots & e^{\sigma(p)} w_p \end{vmatrix} \\ &= \det \begin{vmatrix} e^{\sigma(1)} a_{1j} v_j & \cdots & e^{\sigma(1)} a_{pj} v_j \\ \vdots & \ddots & \vdots \\ e^{\sigma(p)} a_{1j} v_j & \cdots & e^{\sigma(p)} a_{pj} v_j \end{vmatrix}, \end{aligned}$$

since $w_i = a_{ij}v_j$. Now consider the matrix N given by $N_{ij} = e^{\sigma(i)}v_j$. Then the matrix

$$M_{ij} = N_{ik}a_{kj} = e^{\sigma(i)}v_k a_{kj},$$

is exactly the matrix in the above equation. Since the determinant of a product is the product of the determinants, the formula is established.

Problem 2.7 *Show for $f : V \to W$ that*

$$\Omega^{p+q}(f)(\omega_1 \wedge \omega_2) = \Omega^p(f)(\omega_1) \wedge \Omega^q(f)(\omega_2),$$

where $\omega_1 \in \Omega^p(W)$ and $\omega_2 \in \Omega^q(W)$.

On the one hand for $\Omega^p(f)(\omega_1) \wedge \Omega^q(f)(\omega_2)(\xi_1, \ldots, \xi_{p+q})$

$$\begin{aligned} & \sum_\sigma \Omega^p(f)(\omega_1)(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) \cdot \Omega^q(f)(\omega_2)(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\ = & \sum_\sigma \omega_1(f(\xi_{\sigma(1)}), \ldots, f(\xi_{\sigma(p)})) \cdot \omega_2(f(\xi_{\sigma(p+1)}), \ldots, f(\xi_{\sigma(p+q)})), \end{aligned}$$

while on the other hand $\Omega^{p+q}(f)(\omega_1 \wedge \omega_2)(\xi_1, \ldots, \xi_{p+q})$ is

$$\begin{aligned} & (\omega_1 \wedge \omega_2)(f(\xi_1), \ldots, f(\xi_{p+q})) \\ = & \sum_\sigma \omega_1(f(\xi_{\sigma(1)}), \ldots, f(\xi_{\sigma(p)})) \cdot \omega_2(f(\xi_{\sigma(p+1)}), \ldots, f(\xi_{\sigma(p+q)})), \end{aligned}$$

which proves the formula.

Problem 2.8 *Show that the set*

$$\left\{ f \in \mathrm{End}(V) : \exists g \in GL(V) : gfg^{-1} \text{ is diagonal} \right\},$$

is everywhere dense in $\mathrm{End}(V)$*, assuming V is a finite-dimensional complex vector space.*

Such an f is represented by an $n \times n$ complex matrix once a basis is chosen for V, and we will denote this matrix representation by f as well. If f has n distinct eigenvalues then it is obviously diagonalizable, so we will show that there exists complex matrices with distinct eigenvalues arbitrarily close to every given matrix f. We will consider these matrices as points in $\mathbb{C}^{n^2} \simeq \mathbb{R}^{4n^2}$, and use the standard topology in Euclidean space.

Given an f, denote its characteristic polynomial by

$$P_f(\lambda) = \det(f - \lambda I) = a_o + a_i\lambda^i,$$

where I is the unit matrix and $i \in (1, \ldots, n)$. Next we will define, for a given f and characteristic polynomial $P_f(\lambda)$, a function $F : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ defined by

$$F : (z_0, \ldots, z_n, \lambda) \mapsto z_0 + a_0 + (z_i + a_i)\lambda^i,$$

which is polynomial and therefore smooth (analytic).

The partial map

$$F_\lambda : (z_0, \ldots, z_n) \mapsto z_0 + z_i \lambda_i + P_f(\lambda),$$

has a (complex) differential given by

$$DF_\lambda = (1, \lambda^1, \ldots, \lambda^n),$$

so that

$$DF_\lambda \cdot DF_\lambda^\dagger = 1 + \sum \lambda^i \bar{\lambda}^i \geq 1,$$

so that the differential is surjective and thus F intersects every submanifold of $\mathbb{C}$ transversely, in particular $\{0\}$. Thus, by the transversality theorem, for almost every $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, the map $F_z : \lambda \to z_0 + a_0 + (a_i + z_i)\lambda^i$ intersects $\{0\}$ transversely.

Now, by the fundamental theorem of algebra this polynomial must take the value 0, so that the perturbed characteristic polynomial has distinct root (ie, if the roots were not distinct, the map would intersect $\{0\}$ tangentially rather than transversely). Since this holds for almost all $(z_0, \ldots, z_n)$, we can choose such a point within any δ-ball about the origin. Thus, this polynomial can be made arbitrarily close to our original one and our result will follow if a nearby polynomial is associated with a nearby matrix, which we show next.

We note that the maps $\varphi_k : f_{ij} \to b_k$, which give the coefficients of the characteristic polynomial from the matrix are analytic maps (they are polynomial themselves). Each of these function will possess a smooth local inverse whenever the image point is regular, but almost every such point is regular by Sard. Since there are only finitely many functions, almost every $(n+1)$-tuple $(b_0, \ldots, b_n)$ is regular for $\{\varphi_0, \ldots, \varphi_n\}$.

Now, almost every $z \in \mathbb{C}^{n+1}$ will satisfy both conditions above, ie, be regular for the maps φ_k and give distinct zeros for the perturbed characteristic polynomial. Thus, for every ϵ-ball about f we can find a corresponding δ-ball about the origin in $\mathbb{C}^{n+1}$ that contains a point z such that the characteristic polynomial

$$z_0 + a_0 + (a_i + z_i)\lambda^i,$$

has n distinct roots and is the characteristic polynomial of some matrix in V, which establishes the result. We note that it is essential to consider complex matrices since the proof depends on the fundamental theorem of algebra, which doesn't gaurantee roots over $\mathbb{R}$.

Problem 2.9 *Let V be an n-dimensional vector space with inner product $\langle , \rangle$. From Exercise 2.5 we obtain an inner product on $\Omega^p(V)$ for all p, in particular for $p = n$.*

A volume element on V is an unit vector $\mathrm{vol} \in \Omega^n(V)$. *Hodge's star operator*

$$* : \Omega^p(V) \to \Omega^{n-p}(V),$$

is defined by the equation $\langle *\omega, \tau\rangle \mathrm{vol} = \omega \wedge \tau$ *for all* $\tau \in \Omega^p(V)$*. Show that* $*$ *is well-defined and linear.*

Let $\{e_1 \dots e_n\}$ *be a basis of* V *with* $\mathrm{vol}(e_1, \dots, e_n) = 1$ *and* $\{e^1 \dots e^n\}$ *the dual basis. Show that*

$$*(e^1 \wedge \dots \wedge e^p) = e^{p+1} \wedge \dots \wedge e^n,$$

and in general that

$$*(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) = \mathrm{sgn}(\sigma) e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)},$$

with $\sigma \in S(p, n-p)$*. Show that* $* \circ * = (-1)^{p(n-p)}$ *on* $\Omega^p(V)$*.*

Suppose that u_1 and u_2 are two forms that satisfy the above equation for every τ, then we have

$$\begin{aligned} 0 = \langle u_1, \tau\rangle \mathrm{vol} - \langle u_2, \tau\rangle \mathrm{vol} &= (\langle u_1, \tau\rangle - \langle u_2, \tau\rangle)\mathrm{vol} \\ &= (\langle u_1 - u_2, \tau\rangle \mathrm{vol}, \end{aligned}$$

which shows that $u_1 = u_2$, so that the map is well-defined. For linearity we have

$$\begin{aligned} \langle *(\omega + \pi), \tau\rangle \mathrm{vol} &= (\omega + \pi) \wedge \tau \\ &= \omega \wedge \tau + \pi \wedge \tau \\ &= \langle *\omega, \tau\rangle \mathrm{vol} + \langle *\pi, \tau\rangle \mathrm{vol} \end{aligned}$$

from linearity of the wedge product.

Next, suppose that we have

$$\langle *(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}), \tau\rangle \mathrm{vol} = (e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) \wedge \tau.$$

Using linearity we will let τ be a basis element

$$\tau = e^{i_1} \wedge \dots \wedge e^{i_{n-p}}.$$

Thus the right hand side of our formula yields

$$e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)} \wedge e^{i_1} \wedge \dots \wedge e^{i_{n-p}},$$

which is zero unless τ is a permutation of the remaining basis vectors:

$$\tau = e^{\sigma(p+q)} \wedge \dots \wedge e^{\sigma(n)}.$$

Thus we have

$$\langle *(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}), e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)}\rangle \mathrm{vol} = \mathrm{sgn}(\sigma)\mathrm{vol},$$

which yields $*(e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(p)}) = \mathrm{sgn}(\sigma) \cdot e^{\sigma(p+1)} \wedge \dots \wedge e^{\sigma(n)})$.

Now we wish to evaluate $* \circ * : \Omega^p(V) \to \Omega^p(V)$. By the linearity of $*$ it suffices to consider its action on basis elements. We have

$$* : e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)} \mapsto \mathrm{sgn}(\sigma) e^{\sigma(p+1)} \wedge \ldots \wedge e^{\sigma(n)},$$

and likewise

$$* : e^{\pi(1)} \wedge \ldots \wedge e^{\pi(n-p)} \mapsto \mathrm{sgn}(\pi) e^{\pi(n-p+1)} \wedge \ldots \wedge e^{\pi(n)},$$

where the permutations are related by

$$(\pi(1), \ldots, \pi(n-p), \pi(n-p+1), \ldots, \pi(n)) = (\sigma(p+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(p)),$$

that is, π is a cyclic (cyc) permutation of σ, thus $\mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(cyc)$. Since we go from σ to π by moving the last p elements across the first $n-p$ there are $p(n-p)$ interchanges, which is the sign of cyc, thus we have

$$\begin{aligned} * : e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)} &\mapsto \mathrm{sgn}(cyc) \cdot e^{\pi(n-p+1)} \wedge \ldots \wedge e^{\pi(n)} \\ &= (-1)^{p(n-p)} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}, \end{aligned}$$

which establishes the formula.

Problem 2.10 *Let V be a 4-dimensional vector space and $\{e^1, \ldots, e^4\}$ the dual basis. Let $A = (a_{ij})$ be a skew-symmetric matrix and define*

$$\alpha = \sum_{i<j} a_{ij} e^i \wedge e^j.$$

Show that

$$\alpha \wedge \alpha = 0 \leftrightarrow \det(A) = 0.$$

Say $\alpha \wedge \alpha = \lambda e^1 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4$. What is the relation between λ and $\det(A)$?

The first part follows immediately from proof of exercise 2.2 with $n = 2$. Also, according to that exercise we have $\lambda = n!\mathrm{pf}(A)$, or $\lambda = n!\sqrt{\det(A)}$.

Problem 2.11 *Let V be an n-dimensional vector space with inner product $\langle , \rangle$ and volume element $\mathrm{vol} \in \Omega^n(V)$, as in Exercise 2.9. Let $v \in \Omega^1(V)$ and*

$$F_v : \Omega^p \to \Omega^{p+1}(V),$$

be the map

$$F_v(\omega) = v \wedge \omega.$$

Show that the map

$$F_v^* = (-1)^{np} * \circ F_v \circ * : \Omega^{p+1}(V) \to \Omega^p(V),$$

is adjoint to F_v, that is, $\langle F_v \omega, \tau \rangle = \langle \omega, F_v^* \tau \rangle$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V with $\mathrm{vol}(e^1, \ldots, e^n) = 1$. Show that

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \langle v, e^i \rangle e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1}.$$

Show that $F_v F_v^* + F_v^* F_v : \Omega^p(V) \to \Omega^p(V)$ is multiplication by $||v||^2$.

In this problem v is a 1-form, ω a p-form, and τ a $p+q$-form. First we make the observation that (for forms of appropriate rank)

$$\langle *(a \wedge b), c\rangle \mathrm{vol} = (a \wedge b) \wedge c = a \wedge (b \wedge c) = \langle *a, b \wedge c\rangle \mathrm{vol},$$

so that

$$\langle *(a \wedge b), c\rangle = \langle *a, b \wedge c\rangle.$$

Now, on one hand we have

$$\langle F_v \omega, \tau\rangle = \langle v \wedge \omega, \tau\rangle,$$

while on the other

$$\begin{aligned}
\langle F_v^* \tau, \omega\rangle &= (-1)^{np} \langle * F_v(*\tau), \omega\rangle \\
&= (-1)^{np} \langle *(v \wedge (*\tau), \omega\rangle \\
&= (-1)^{np+p} \langle *(*\tau \wedge v), \omega\rangle \\
&= (-1)^{p(n+1)} \langle * * \tau, v \wedge \omega\rangle \\
&= (-1)^{p(n+1)} (-1)^{p(n-p)} \langle \tau, v \wedge \omega\rangle \\
&= (-1)^{p-p^2} \langle v \wedge \omega, \tau\rangle,
\end{aligned}$$

but $p - p^2 = p(1-p)$ is always even, which establishes the result.

Next, we have $F_v^*() = (-1)^{np} * F_v * ()$, so we'll take the component mappings in sequence:

$$* : e^1 \wedge \ldots \wedge e^{p+1} \mapsto e^{p+2} \wedge \ldots \wedge e^n.$$

Next is

$$F_v : e^{p+2} \wedge \ldots \wedge e^n \mapsto v \wedge e^{p+2} \wedge \ldots \wedge e^n,$$

and finally

$$* : v \wedge e^{p+2} \wedge \ldots \wedge e^n \mapsto \mathrm{sgn}(\sigma) v_i e^1 \wedge \ldots \wedge \bar{e}^i \wedge \ldots \wedge e^{p+1},$$

where the hat indicated that term is missing and σ is the permutation that takes $\{1, \ldots, n\} \to \{i, p+2, \ldots, n, 1, 2, \ldots, i-1, \ldots, p+1\}$. This permutation is obtained by first moving the $n-p$ elements $\{i, p+2, \ldots, n\}$ past the remaining p elements which requires $p(n-p)$ interchanges, and then moving i to its proper place, which requires an additional $p+1-i$, which yeilds

$$\mathrm{sgn}(\sigma) = (-1)^{p(n-p)+p+1-i}.$$

The exponent can be rewritten as $pn + (i+1) + p(1-p)$, since adding 2 doesn't change the parity. But $p(1-p)$ is always even, and the pn will cancel with the pn from the map definition, yeilding

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = (-1)^{i+1} v_i e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1},$$

which can also be written as

$$F_v^*(e^1 \wedge \ldots \wedge e^{p+1}) = (-1)^{i+1} \langle v, e^i\rangle \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{p+1},$$

proving the formula.

Next, consider the operator $F_v F_v^* + F_v^* F_V : \Omega^p(V) \to \Omega^p(V)$ with $v = \lambda e^1$. We have two cases. The first is the action of this operator on a basis element containing e^1. We have

$$F_v^* : e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^{i_2} \wedge \ldots \wedge e^{i_p},$$

followed by

$$F_v : \lambda e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda^2 e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p}.$$

However, the second term in the operator gives zero because

$$F_v : e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^1 \wedge e^1 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which is zero, thus the operator is multiplication by $\lambda^2 = ||v||^2$.

The second case is the action of the operator on a basis element not containing e^1. The first term gives zero in this case since

$$* : e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \mathrm{sgn}(\sigma) e^1 \wedge \ldots \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n},$$

and then

$$F_v : e^1 \wedge \ldots \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \mapsto \lambda e^1 \wedge e^1 \wedge \ldots \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which is zero. Now, the second term yields

$$F_v : e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \mapsto \lambda e^1 \wedge e^2 \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n},$$

and then

$$F_v^* : \lambda e^1 \wedge e^2 \wedge e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \mapsto \lambda^2 e^2 \wedge e^{i_2} \wedge \ldots \wedge e^{i_p},$$

which again shwos that the operator is multiplication by $||v||^2$.

Finally, for arbitrary v, we are free to choose our basis so that e^1 is in the direction of v, so the problem reduces to the particular case above with the same result, since it was written in a coordinate independent form.

Problem 2.12 *Let V be an n-dimensional vector space. Show for a linear map $f : V \to V$ the existance of a number $d(f)$ such that*

$$\Omega^n(f)(\omega) = d(f)\omega,$$

for $\omega \in \Omega^n(V)$. Verify the product rule

$$d(g \circ f) = d(g)d(f),$$

for linear maps $f, g : V \to V$ using the functoriality of $\Omega^n()$. Prove that $d(f) = \det(f)$.

Let $\{e_1 \ldots e_n\}$ be a basis of V and $\{e^1, \ldots, e^n\}$ the dual. There is only one basis element in $\Omega^n(V)$, so we have $\omega = \hat{\omega} e^1 \wedge \ldots \wedge e^n$, where $\hat{\omega}$ is the component of ω in this basis. The action of $\Omega^n(f)(\omega)$ on the basis vectors of V is given by

$$\begin{aligned}
\Omega^n(f)(\omega)(e_1, \ldots, e_n) &= \omega(f(e_1), \ldots, f(e_n)) \\
&= \hat{\omega} e^1 \wedge \ldots \wedge e^n (f_1^k e_k, \ldots, f_n^k e_k) \\
&= \hat{\omega} \det \begin{pmatrix} e^1(f_1^k e_k) & \cdots & e^1(f_n^k e_k) \\ \vdots & \ddots & \ldots \\ e^n(f_1^k e_k) & \cdots & e^n(f_n^k e_k) \end{pmatrix} \\
&= \hat{\omega} \det(e^i f_j^k e_k).
\end{aligned}$$

The matrix whose determinant we want is simply the product of two matrices

$$e^i f_j^k e_k = (f_j^k)(e_k e^i),$$

and so we have

$$\begin{aligned}
\det(e^i f_j^k e_k) &= \det(f_j^k) \det(e_k e^i) \\
&= \det(f_j^k) \det(\delta_k^i) \\
&= \det(f).
\end{aligned}$$

This establishes the formula with $d(f) = \det(f)$. Next, a commutative diagram of maps on V gives rise to a commutative diagrams of maps on $\Omega^n(V)$:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "g \circ f"'] & V \arrow[d, "g"] \\ & V \end{tikzcd} \longrightarrow \begin{tikzcd} \Omega^n(V) & \Omega^n(V) \arrow[l, "\Omega^n(f)"'] \\ & \Omega^n(V) \arrow[ul, "\Omega^n(g \circ f)"] \arrow[u, "\Omega^n(g)"'] \end{tikzcd}$$

Given the first result of the problem we can rewrite the second diaram as

$$\begin{tikzcd} \Omega^n(V) & \Omega^n(V) \arrow[l, "d(f)"'] \\ & \Omega^n(V) \arrow[ul, "d(g \circ f)"] \arrow[u, "d(g)"'] \end{tikzcd}$$

which shows that $(d \circ f) = d(f)d(g)$.

Cohomology Homework: Chapter 3

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Problem 3.1 *Show for an open set in $\mathbb{R}^2$ that the de Rham complex*

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to 0,$$

is isomorphic to the complex

$$0 \to C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R}) \to 0,$$

Analogously, show that for an open set in $\mathbb{R}^3$ that the de Rham complex is isomorphic to

$$0 \to C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U, \mathbb{R}) \to 0,$$

defined in chapter 1.

We will do the $\mathbb{R}^3$ case first. We know that $\Omega^0(U) \simeq C^\infty(U, \mathbb{R})$. Furthermore, from exercise 2.3 we have

$$\begin{aligned} \Omega^1(\mathbb{R}^3) &\simeq \mathbb{R}^3 \\ \Omega^2(\mathbb{R}^3) &\simeq \mathbb{R}^3. \end{aligned}$$

Now, $\omega \in \Omega^1(U)$ is a smooth map $\omega : U \to \Omega^1(\mathbb{R}^3) \simeq \mathbb{R}^3$, in other words, a smooth vector field on U. Likewise $\omega \in \Omega^2(U)$ is a smooth map $\omega : U \to \Omega^2(\mathbb{R}^3) \simeq \mathbb{R}^3$, and so this too is a smooth vector field on U.

Now we need $\Omega^3(U)$. Any $\tau \in \Omega^3(\mathbb{R}^3)$ can be written as

$$\tau = \hat{\tau} e^1 \wedge e^2 \wedge e^3,$$

where $\hat{\tau} \in \mathbb{R}$. Thus we regard an $\omega \in \Omega^3(U)$ as a smooth map $\omega : U \to \Omega^3(\mathbb{R}^3) \simeq \mathbb{R}$, that is, a smooth function on U.

So, we've established the isomorphism on the vector spaces, next we need the maps. However, Theorem 3.7 and following establishes that the d operator acts as the differential operators grad, curl, and div on the appropriate spaces as indicated, and the result follows.

Next we take the $\mathbb{R}^2$ case. We again have $\Omega^0(U) \simeq C^\infty(U, \mathbb{R})$. We furthermore have $\Omega^1(\mathbb{R}^2) \simeq \mathbb{R}^2$ by problem 2.3 since the isomorphism depends only on the inner product which is defined in any dimension. The same argument as above then shows that $\Omega^1(U) \simeq C^\infty(U, \mathbb{R}^2)$.

Now, any $\tau \in \Omega^2(\mathbb{R})$ can be written as

$$\tau = \hat{\tau} e^1 \wedge e^2,$$

where $\hat{\tau} \in \mathbb{R}$. So any $\omega \in \Omega^2(U)$ is a smooth map $\omega \to \Omega^2(\mathbb{R}^2) \simeq \mathbb{R}$, so ω is a smooth function on U.

Now, we know that $d : \Omega^0 \to \Omega^1$ acts as div by theorem 3.7, so we only need $d : \Omega^1 \to \Omega^2$. We write $\omega \in \Omega^1(U)$ as $\omega_1 dx^1 + w_2 dx^2$. Then we have

$$\begin{aligned}
d\omega &= d\omega_1 dx^1 + d\omega_2 dx^2 \\
&= \frac{\partial \omega_1}{\partial x^i} dx^i \wedge dx^1 + \frac{\partial \omega_2}{\partial x^i} dx^i \wedge dx^2 \\
&= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2 \\
&= \left(\frac{\partial \omega_1}{\partial x^2} - \frac{\partial \omega_2}{\partial x^1} \right) dx^1 \wedge dx^2 \\
&= (\operatorname{curl} \hat{\omega}) dx^1 \wedge dx^2
\end{aligned}$$

where $\hat{\omega} = \omega^1 dx_1 + \omega^2 dx_2 \in C^\infty(U, \mathbb{R}^2)$, and we have $\omega^i = \omega_i$.

Problem 3.2 *Let $U \subset \mathbb{R}^n$ be an open set and $\{dx^1, \ldots, dx^n\}$ the usual constant 1-forms. Let $\mathrm{vol} = dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(U)$. Use the star operator defined in 2.9 to define Hodge's star operator*

$$* : \Omega^p(U) \to \Omega^{n-p}(U),$$

and show that $*(dx^1 \wedge \ldots \wedge dx^p) = dx^{p+1} \wedge \ldots \wedge dx^n$ and $* \circ * = (-1)^{n(n-p)}$. Define $d^* : \Omega^p(U) \to \Omega^{p-1}(U)$ by

$$d^*(\omega) = (-1)^{np+n-1} * \circ\, d \circ * (\omega).$$

Show that $d^* \circ d^* = 0$. Verify the formula

$$d^*(f dx^{i_1} \wedge \ldots \wedge dx^{i_p}) = (-1)^j \frac{\partial f}{\partial x^{i_j}} dx^{i_1} \wedge \ldots \hat{dx}^{i_j} \ldots \wedge dx^{i_p}.$$

We will define the action of $*$ point-wise using the action of $*$ from 2.9 on basis elements:

$$\begin{aligned}
*(\omega) &= *(\hat{\omega} e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}) \\
&= \hat{\omega} * (e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(p)}) \\
&= \operatorname{sgn}(\sigma) \hat{\omega} e^{\sigma(p+1)} \wedge \ldots \wedge e^{\sigma(n)}.
\end{aligned}$$

We then extend this definition to arbitrary ω using linearity. Now we have (at each point)

$$\begin{aligned}
*(dx^1 \wedge \ldots \wedge dx^p) &= *(e^1 \wedge \ldots \wedge e^p) \\
&= e^{p+1} \wedge \ldots \wedge e^n \\
&= dx^{p+1} \wedge \ldots \wedge dx^n.
\end{aligned}$$

We now demonstrate that this result is the same as the previous definition extended to differential forms using this last result on the dx^i and the definition of $*$ from 2.9:

$$\begin{aligned}\langle *\omega, \tau\rangle &= \langle *(\hat{\omega} dx^1 \wedge \ldots \wedge dx^p), \tau\rangle \mathrm{vol} \\ &= \hat{\omega}\langle *(dx^1 \wedge \ldots \wedge dx^p), \tau\rangle \mathrm{vol} \\ &= \hat{\omega}(dx^1 \wedge \ldots \wedge dx^p \wedge \tau) \\ &= \omega \wedge \tau.\end{aligned}$$

Now, since $*$ as defined above only acts on the dx^i and no the component functions, the calculation reduces to the one in 2.9 giving the same result, that $* \circ * = (-1)^{n(n-p)}$.

Next we want to calculate $d^* \circ d^*$ on a p-form ω. We can write the operator as

$$d^* \circ d^* = \big((-1)^{np-1} * \circ\, d \circ *\big) \circ \big((-1)^{np+n-1} * \circ\, d \circ *\big) \circ,$$

since the second d^* acts on a $(p-1)$-form. We then write

$$d^* \circ d^* = (-1)^n * \circ\, d \circ * \circ * \circ\; d \circ *,$$

and the succession of the orders of the form in each map is:

$$(p) \xrightarrow{*} (n-p) \xrightarrow{d} (n-p+1) \xrightarrow{*} (p-1) \xrightarrow{*} (n-p+1) \xrightarrow{d} (n-p+2) \xrightarrow{*} (p-2).$$

Now we have

$$d^* \circ d^* = (-1)^n * \circ\, d \circ \underbrace{* \;\circ\; *}_{(-1)^{n(p-1)}} \circ\, d \circ *,$$

since the bracketed operator acts on an $(n-p+1)$-form. Then we have

$$d^* \circ d^* = (-1)^{np} * \circ \underbrace{d \circ d}_{0} \circ * = 0.$$

Finally, we calculate the action of d^* on a p-form:

$$\begin{aligned}d^*(f dx^1 \wedge \ldots \wedge dx^p) &= (-1)^{np+n-1} * \circ\, d \circ *(f dx^1 \wedge \ldots \wedge dx^p) \\ &= (-1)^{np+n-1} * \circ\, d\left(f *(dx^1 \wedge \ldots \wedge dx^p)\right) \\ &= (-1)^{np+n-1} * \circ\, df\left(*(dx^1 \wedge \ldots \wedge dx^p)\right).\end{aligned}$$

Next, we note that df can be though of here as a map whose action is given by

$$df(\omega) = df \wedge \omega,$$

so that we can apply the result of 2.11, that is, we can rewrite our last line (ignoring sign for now) as

$$* \circ\; df\left(*(dx^1 \wedge \ldots \wedge dx^p)\right) = (-1)^{n(p-1)}(df^*)(dx^1 \wedge \ldots \wedge dx^p),$$

where the sign is because this acts on a p-form. But then

$$\begin{aligned}(df^*)(dx^1 \wedge \ldots \wedge dx^p) &= (-1)^{i+1} \langle df, dx^i \rangle dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p \\ &= (-1)^{i+1} \left(\frac{\partial f}{\partial x^i} \right) dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p,\end{aligned}$$

using 2.11 again. the sign factor, including all contributions, becomes

$$i + 1 + n(p - q) + np + n - 1 = i + 1 - 1 + 2np + n - n \to i,$$

which establishes the result. We note that our derivation didn't depend on which p of the dx^i's we picked, but only that there were p of them. so we can replace $dx^1 \wedge \ldots \wedge dx^p$ with any permutation of the p indices and result will holds. If the indices are labeled $j_1, \ldots, j_p$, then we replace i in our formula with j_i, that is

$$(df^*)(dx^{j_1} \wedge \ldots \wedge dx^{j_p}) = (-1)^{j_i} \frac{\partial f}{\partial x^{j_i}} dx^{j_1} \wedge \ldots \hat{dx}^{j_i} \ldots \wedge dx^{j_p}.$$

Problem 3.3 *With the notation of the previous problem, the Laplace operator* $\Delta : \Omega^p(U) \to \Omega^p(U)$ *is defined by*

$$\Delta = d \circ d^* + d^* \circ d.$$

Let $f \in \Omega^0(U)$. *Show that* $\Delta(f dx^1 \wedge \ldots \wedge dx^p) = \Delta(f) dx^1 \wedge \ldots \wedge dx^p$, *where*

$$-\Delta(f) = \frac{\partial^2 f}{(\partial x^1)^2} + \ldots + \frac{\partial^2 f}{(\partial x^n)^2}.$$

A p*-form* $\omega \in \Omega^p(U)$ *is said to be harmonic if* $\Delta(\omega) = 0$. *Show that* $*$ *maps harmonic forms to harmonic forms.*

We will take a p-form and first compute the action of $d^* \circ d$. First the action of d:

$$\begin{aligned}d(f dx^1 \wedge \ldots \wedge dx^p) &= df \wedge dx^1 \wedge \ldots \wedge dx^p \\ &= \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \ldots \wedge dx^p,\end{aligned}$$

where we note that we must have $i > p$ for this to be nonzero.

Next we have d^* acting on a $(p+q)$-form:

$$d^* = (-1)^{np-1} * \circ\, d \circ *,$$

which we will take one mapping at a time. First

$$\begin{aligned}* \left(\frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \ldots \wedge dx^p \right) &= \frac{\partial f}{\partial x^i} * (dx^i \wedge dx^1 \wedge \ldots \wedge dx^p) \\ &= \operatorname{sgn}(\sigma) \frac{\partial f}{\partial x^i} dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,\end{aligned}$$

where σ is the permutation taking

$$(1, \ldots, n) \to (i, 1, \ldots, p, p+1, \ldots, \hat{i}, \ldots, n),$$

which has sign $i-1$.

Next we have d which gives

$$(-1)^{i-1} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

where we note that either $j \leq p$ or $j = i$ for this not to be zero. We will take these as two cases. If $j \neq i$ (the same as $j \leq p$ since $i > p$), we get for $*$:

$$(-1)^{i-1} \mathrm{sgn}\tau \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

where τ is the permutation taking

$$(1, \ldots, n) \to (j, p+1, \ldots, \hat{i}, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p),$$

which we will reduce in steps - first move i, then j, then swap the first $n-p$ with the last p:

$$\begin{array}{cl}
(j, p+1, \ldots, \hat{i}, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p) & \\
\downarrow & (-1)^{n-i} \\
(j, p+1, \ldots, i, \ldots, n, i, 1, \ldots, \hat{j}, \ldots, p) & \\
\downarrow & (-1)^{n-p+j-1} \\
(p+1, \ldots, i, \ldots, n, i, 1, \ldots, j, \ldots, p) & \\
\downarrow & (-1)^{p(n-p)} \\
(1, \ldots, j, \ldots, p, p+1, \ldots, i, \ldots, n) &
\end{array}$$

which shows that $\mathrm{sgn}(\tau)$ is given by

$$n - i + n - p + j - 1 + pn - p^2 \to j - i - 1 + pn.$$

So, if we take all our cumulative signs this gives

$$(np_1) + (i-1) + (j - i - 1 + pn) \to j - 1,$$

so that our sum is

$$-\sum_{\substack{i>p \\ j \leq p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p.$$

In the case $i = j$ the expression that $*$ acts on will now be

$$(-1)^{i-1} \frac{\partial^2 f}{(\partial x^i)^2} dx^i \wedge dx^{p+1} \wedge \ldots \hat{dx}^i \ldots \wedge dx^n,$$

which can be written as

$$(-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^{p+1} \wedge \ldots \wedge dx^i \wedge \ldots \wedge dx^n = (-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^{p+1} \wedge \ldots \wedge dx^n,$$

since there are $i - (p+1)$ interchanges to move i to its proper spot, and $(i - i) + (i - p + 1) = 2i - 2 - p\top$. so, now $*$ gives

$$\operatorname{sgn}(\zeta)(-1)^p \frac{\partial^2 f}{(\partial x^i)^2} dx^1 \wedge \ldots \wedge dx^p,$$

where ζ is the permutation that takes

$$(1, \ldots, n) \to (p+1, \ldots, m, 1, \ldots, p),$$

which has sign $p(n-p)$. So, putting it all together, the total sigh will be

$$(np - 1) + (p) + p(n-p) = 2np = p(1-p) - 1 \to -1,$$

since $p(1-p)$ is always even, and the sum becomes

$$-\sum_{i>p} \frac{\partial^2 f}{(\partial x^i)^2} dx^1 \wedge \ldots \wedge dx^p.$$

Whew. Now we have to do the operator $d \circ d^*$. In this case we can use our previous result to simply write

$$d^*(f dx^1 \wedge \ldots \wedge dx^p) = (-1)^i \frac{\partial f}{\partial x^i} dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p,$$

where we know that here $i \leq p$. Then d gives

$$(-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p,$$

where now $j > p$ or $j = i$ (the two cases are mutually exclusive as before) or else we get zero, and will consider the two cases separately again.

First, if $j > p$ we have

$$\sum_{\substack{i \leq p \\ j > p}} (-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p,$$

whereas when $i = j$ we have

$$(-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p = -\frac{\partial^2 f}{\partial x^i \partial x^j} dx^1 \wedge \ldots \wedge dx^p,$$

since there are $i - 1$ interchanges to move i to its proper spot.

Now, if take our two sums when $i = j$ and combine them, we see that the summands are the same, and they only differ in the range of i, which between the two if all of $1, \ldots, n$. Thus these two give

$$-\sum_{1 \leq i \leq n} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^1 \wedge \ldots \wedge dx^p,$$

which we will then write as

$$\Delta(f) dx^1 \wedge \ldots \wedge dx^p,$$

following the notation in the problem statement. So, we will be done if we can show that the other two sums cancel each other out. We will reproduce those sums here:

$$-\sum_{\substack{i>p \\ j \leq p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p,$$

and

$$\sum_{\substack{i \leq p \\ j>p}} (-1)^i \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \ldots \hat{dx}^i \ldots \wedge dx^p.$$

But, we note that the i and j are just dummy indices, so we can exchange the two in the second sum, which yields

$$\sum_{\substack{j \leq p \\ i>p}} (-1)^j \frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^1 \wedge \ldots \hat{dx}^j \ldots \wedge dx^p,$$

which is now exactly the same as the first sum (since partial derivatives commute on smooth functions) except that they have opposite sign, so the two sums cancel.

Problem 3.4 *Let $\Omega^p(\mathbb{R}^n, \mathbb{C})$ be the $\mathbb{C}$-vector space of alternating $\mathbb{R}$-multilinear maps*

$$\omega : \mathbb{R}^n \underbrace{\times \cdots \times}_{p} \mathbb{R}^n \to \mathbb{C}.$$

Note that ω can be written uniquely as

$$\omega = \omega_R + i\omega_I,$$

where ω_R is the real part, ω_I is the imaginary part, and both are real-valued p-forms. Extend $\wedge$ to a $\mathbb{C}$-linear map

$$\Omega^p(\mathbb{R}^n, \mathbb{C}) \times \Omega^q(\mathbb{R}^n, \mathbb{C}) \xrightarrow{\wedge} \Omega^{p+q}(\mathbb{R}^n, \mathbb{C}),$$

and show that we obtain a graded anti-commutative $\mathbb{C}$-algebra $\Omega^*(\mathbb{R}^n, \mathbb{C})$.

The most straight-forward thing to do is to expand each complex form into its real-valued constituents and apply the usual wedge on these:

$$\begin{aligned}\omega\wedge\tau &= (\omega_R + i\omega_I)\wedge(\tau_R + i\tau_I)\\ &= (\omega_R\wedge\tau_R + i\omega_I)\wedge\tau_R + i\omega_I\wedge i\tau_I + \omega_R\wedge i\tau_I\\ &= (\omega_R\wedge\tau_R - \omega_I\wedge\tau_I) + i(\omega_I\wedge\tau_R - \omega_R\wedge\tau_I)\\ &= (\omega\wedge\tau)_R + i(\omega\wedge\tau)_I,\end{aligned}$$

which makes sense since the wedge products on the real-valued forms are always of a p and q form, resulting in a real-valued $(p+q)$-form.

To obtain the graded algebra we need to check associativity and the (anti)-commutativity, since we already have a mapping between the grades of the algebra. Associativity follow from that of the usual wedge product:

$$\begin{aligned}\alpha\wedge[\beta\wedge\gamma] &= (\alpha_R + i\alpha_I)\wedge[(\beta_R + i\beta_I)\wedge(\gamma_R + i\gamma_I)]\\ &= [(\alpha_R + i\alpha_I)\wedge(\beta_R + i\beta_I)]\wedge(\gamma_R + i\gamma_I)\\ &= [\alpha\wedge\beta]\wedge\gamma\end{aligned}$$

Now, let α be a p-form and β a q-form. Then we have

$$\alpha\wedge\beta = (\alpha_R\wedge\beta_R - \alpha_I\wedge\beta_I) + i(\alpha_I\wedge\beta_R + \alpha_R\wedge\beta_I),$$

but it costs a $(-1)^{pq}$ to flip the order of the wedge product in each term since each is a product of a real-valued p and q form. We then factor out the common sign factor to get

$$\begin{aligned}\alpha\wedge\beta &= (-1)^{pq}(\beta_R\wedge\alpha_R - \beta_I\wedge\alpha_I) + i(\beta_I\wedge\alpha_R + \beta_R\wedge\alpha_I)\\ &= (-1)^{pq}\beta\wedge\alpha,\end{aligned}$$

and these properties establish $\Omega^*\mathbb{R}^n, \mathbb{C}$ as a graded anti-commutative $\mathbb{C}$-algebra.

Problem 3.5 *Introduce $\mathbb{C}$-valued differential p-forms on an open set $U \subset \mathbb{R}^n$ by setting*

$$\Omega^p(U,\mathbb{C}) = C^\infty(U, \Omega^p(\mathbb{R}^n,\mathbb{C})).$$

Extend d to a $\mathbb{C}$-linear operator

$$d : \Omega^p(\mathbb{R}^n,\mathbb{C}) \to \Omega^{p+1}(\mathbb{R}^n,\mathbb{C}),$$

and show that theorem 3.7 holds for this case, and generalize theorem 3.12 to this case.

We extend the definition as in the previous problem, by having the usual d operator act on the real valued forms:

$$d(\omega) = d(\omega_R + i\omega_I) = d\omega_r + id\omega_I.$$

Now, we need to establish the following properties of d:

$$\begin{array}{rl} (i) & f \in \Omega^0(U,\mathbb{C}),\ df = (\partial_i f)dx^i \\ (ii) & d \circ\ d = 0 \\ (iii) & d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^p \omega \wedge d\tau,\ \omega \in \Omega^p(U,\mathbb{C}), \end{array}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}.$$

Now, an $f \in \Omega^0(U,\mathbb{C})$ is a smooth map $f : U \to \mathbb{C}$, that is, a smooth $\mathbb{C}$-valued function, so we can write f as $f_R + i f_R$, so we have

$$\begin{aligned} df &= df_r + i df_I \\ &= \partial_j f_R dx^j + i\partial_j f_I dx^j \\ &= \partial_j (f_r + i f_I) dx^j \\ &= \partial_j f dx^j, \end{aligned}$$

which establishes the first property.

Next we have

$$\begin{aligned} (d \circ\ d)f &= d(df_R + i df_I) \\ &= d^2 f_R + i d^2 f_I \\ &= 0, \end{aligned}$$

which establishes the second. And finally we have

$$\begin{aligned} d(\omega \wedge \tau) &= d[(\omega_R \wedge \tau_R - \omega_I \wedge \tau_I) + i((\omega_R \wedge \tau_I - \omega_I \wedge \tau_R)] \\ &= d(\omega_R \wedge \tau_R) - d(\omega_I \wedge \tau_I) + id(\omega_R \wedge \tau_I) + id(\omega_I \wedge \tau_R) \\ &= d\omega_R \wedge \tau_R + (-1)^p \omega_R \wedge d\tau_R - d\omega_I \wedge \tau_I + (-1)^p \omega_I \wedge d\tau_I \\ &\quad + i(d\omega_R \wedge \tau_I + (-1)^p \omega_R \wedge d\tau_I) + i(d\omega_I \wedge \tau_R + (-1)^p \omega_I \wedge d\tau_R) \\ &= d\omega_R \wedge \tau_R + i d\omega_R \wedge \tau_I + i d\omega_I \wedge \tau_R - d\omega_I \wedge \tau_I + \\ &\quad (-1)^p (\omega_R \wedge d\tau_R + i\omega_R \wedge d\tau_I + i\omega_I \wedge d\tau_R - \omega_I \wedge d\tau_I) \\ &= (d\omega_R + i d\omega_I) \wedge (\tau_R + i\tau_I) + (-1)^p (\omega_R + i\omega_I) \wedge (d\tau_R + i d\tau_I) \\ &= d(\omega_R + i\omega_I) \wedge (\tau_R + i\tau_I) + (-1)^p (\omega_R + i\omega_I) \wedge d(\tau_R + i\tau_I) \\ &= d\omega \wedge \tau + (-1)^p \omega \wedge d\tau, \end{aligned}$$

where in the fourth line we simply rearranged the entries. This proves the result, and shows the existence of the operator. The argument for uniqueness follows exactly as the one given in the book for the real-valued case by distributing $\wedge$ across the real and imaginary parts of each form.

Next we generalize theorem 3.12: if we have a map $\varphi : U \to V$, then the induced map $\varphi^* : \Omega^p(V,\mathbb{C}) \to \Omega^p(U,\mathbb{C})$ will have the properties:

$$\begin{array}{rl} (i) & \varphi^*(\omega \wedge \tau) = \varphi^*(\omega) \wedge \varphi^*(\tau) \\ (ii) & \varphi^*(f) = f \circ \varphi,\ f \in C^\infty(V,\mathbb{C}) \\ (iii) & d\varphi^*(\omega) = \varphi^*(d\omega) \end{array}$$

First we show that $(\varphi^*\omega)_R = (\varphi^*\omega_R)$, and similarly for the imaginary parts. Using the calculational formula, we have

$$\begin{aligned}
\varphi^*(\omega)(x) &= \omega_k(\varphi(x))d\varphi^k \\
&= [\Re(\omega_k(\varphi(x))) + i\Im(\omega_k(\varphi(x))]d\varphi^k \\
&= [\omega_R(\varphi(x))_k]d\varphi^k + i[\omega_I(\varphi(x))_k]d\varphi^k \\
&= \varphi^*(\omega_R(x)) + i\varphi^*(\omega_I(x)),
\end{aligned}$$

which shows the result. Now we have

$$\begin{aligned}
\varphi^*(\omega \wedge \tau) &= \varphi^*(\omega_R \wedge \tau_R - \omega_I \wedge \tau_I + i\omega_R \wedge \tau_I + i\omega_I \wedge \tau_R) \\
&= \varphi^*(\omega_R \wedge \tau_R - \varphi^*(\omega_I \wedge \tau_I) + i\varphi^*(\omega_R \wedge \tau_I) + i\varphi^*(\omega_I \wedge \tau_R) \\
&= \varphi(\omega_R) \wedge \varphi^*(\tau_R) - \varphi^*(\omega_I) \wedge \varphi * (\tau_I) + i\varphi^*(\omega_R) \wedge \varphi^*(\tau_I) + i\varphi^*(\omega_I) \wedge \varphi^*(\tau_R) \\
&= \varphi(\omega)_R \wedge \varphi^*(\tau)_R - \varphi^*(\omega)_I \wedge \varphi * (\tau)_I + i\varphi^*(\omega)_R \wedge \varphi^*(\tau)_I + i\varphi^*(\omega)_I \wedge \varphi^*(\tau)_R \\
&= (\varphi^*\omega) \wedge (\varphi^*\tau).
\end{aligned}$$

For the second property we have

$$\begin{aligned}
\varphi^*(f) &= \varphi^*(f_R + if_I) \\
&= f_R \circ \varphi + if_i\varphi \\
&= (f_r + if_I)(\varphi) \\
&= f \circ \varphi.
\end{aligned}$$

And for the third property we have

$$\begin{aligned}
d\varphi^*(\omega) &= d\varphi^*(\omega_R + i\omega_I) \\
&= d\varphi^*(\omega_R) + d\varphi^*(i\omega_I) \\
&= \varphi^*(d\omega_R) + \varphi^*(id\omega_I) \\
&= \varphi(d\omega_R + id\omega_I) \\
&= \varphi * (d(\omega_R + i\omega_I)) \\
&= \varphi^*(d\omega).
\end{aligned}$$

Now we prove the uniqueness of our pullback map. Suppose that φ' were another map satisfying the above three properties. First we note that

$$\varphi'(f) = f \circ \varphi = \varphi^*(f),$$

so that φ' and φ^* agree on $C^\infty(U, \mathbb{C})$. Now, by linearity it is enough to look at a basis p-form:

$$\begin{aligned}
\varphi'(fdx^J) &= \varphi(f \wedge dx^J) \\
&= (\varphi' f) \wedge (\varphi' dx^J) \\
&= (\varphi' f) \wedge (\varphi'(dx^{j_1} \wedge \ldots \wedge dx^{j_p})) \\
&= (\varphi' f) \wedge (\varphi' dx^{j_1}) \wedge \ldots \wedge (\varphi' dx^{j_p}) \\
&= (\varphi' f) \wedge d(\varphi' x^{j_1}) \wedge \ldots \wedge d(\varphi' x^{j_p}) \\
&= (f \circ \varphi) \wedge d(x^{j_1} \circ \varphi) \wedge \ldots \wedge d(x^{j_p} \circ \varphi) \\
&= (\varphi^* f) \wedge d(\varphi^* x^{j_1}) \wedge \ldots \wedge d(\varphi^* x^{j_p}),
\end{aligned}$$

where we have applied our previous result. At this point, we just follow the same steps backwards to get $\varphi * (f dx^J)$, which establishes the uniqueness.

Problem 3.6 *Take* $U = \mathbb{C} - \{0\} \simeq \mathbb{R}^2 - \{0\}$ *and let* $z \in \mathbb{C}^\infty(U, \mathbb{C})$ *be the inclusion* $U \hookrightarrow \mathbb{C}$. *Write* $z = x + iy$. *Show that*

$$\Re(z^{-1}dz) = d \log r,$$

where $r : U \to \mathbb{R}$ *is defined by* $r(z) = |z| = \sqrt{x^2 + y^2}$. *Show that*

$$\Im(z^{-1}dz) = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Since we have $z = x + iy$ we can write $dz = dx + idy$. Thus we have

$$\begin{aligned} z^{-1}dz &= (x + iy)^{-1}(dx + idy) \\ &= \frac{dx + idy}{x + iy} \left(\frac{x - iy}{x - iy} \right) \\ &= \frac{xdx - iydx + ixdy + ydy}{x^2 + y^2} \\ &= \frac{xdx + ydy}{x^2 + y^2} + i \frac{xdy - ydx}{x^2 + y^2}, \end{aligned}$$

which gives the required equality on the imaginary part. Now we have

$$d \log \sqrt{x^2 + y^2} = \frac{d\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}},$$

but

$$d\sqrt{x^2 + y^2} = \frac{2xdx + 2ydy}{2\sqrt{x^2 + y^2}} = \frac{xdx + ydy}{\sqrt{x^2 + y^2}},$$

which proves the claim.

Problem 3.7 *Prove for the complex exponential map* $\exp : \mathbb{C} \to \mathbb{C}^*$ *that*

$$d_z \exp = \exp(z)dz, \text{ and } \exp^*(z^{-1}dz) = dz.$$

We begin by writing the exponential map as

$$\exp(z) = \exp(x + iy_= \exp(x) \cdot \exp(iy),$$

so we get

$$\begin{aligned} d(\exp(z)) &= \frac{\partial \exp(x) \exp(iy)}{\partial x} dx + \frac{\partial \exp(x) \exp(iy)}{\partial y} dy \\ &= \exp(iy) \frac{\partial \exp(x)}{\partial x} dx + \exp(x) \frac{\partial \exp(iy)}{\partial y} dy \\ &= \exp(iy) \exp(x) dx + i \exp(x) \exp(iy) dy \\ &= \exp(z)(dx + idy) \\ &= \exp(z)dz, \end{aligned}$$

which proves the first part.

Next, for clarity, we will write $\exp : \mathbb{C} \to \mathbb{C}^*$ as $\exp : w \mapsto z$, by $z = \exp(w)$. Then we have (using the calculational formula)

$$\begin{aligned} \exp^*(z^{-1}dz) &= (\exp(w))^{-1} d \exp(w) \\ &= (\exp(w))^{-1} \exp(w) dw \\ &= dw, \end{aligned}$$

which proves the second part.

Cohomology Homework: Chapter 4

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Problem 4.1 *Consider a commutative diagram of vector spaces and linear maps with exact rows*

$$\begin{tikzcd}
A_1 \arrow[r, "d_A^1"] \arrow[d, "f_1"] & A_2 \arrow[r, "d_A^2"] \arrow[d, "f_2"] & A_3 \arrow[r, "d_A^3"] \arrow[d, "f_3"] & A_4 \arrow[r, "d_A^4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "d_B^1"] & B_2 \arrow[r, "d_B^2"] & B_3 \arrow[r, "d_B^3"] & B_4 \arrow[r, "d_B^4"] & B_5
\end{tikzcd}$$

Suppose that f_2 and f_4 are injective and f_1 surjective. Show that f_3 is injective. Similarly show that if f_2 and f_4 are surjective and f_5 injective that f_3 is surjective. Thus when f_1, f_2, f_4, and f_5 are isomorphisms, so is f_3.

We will show injectivity by showing that f_3 has trivial kernel. Let $a_3 \in \ker(f_3)$ so that $f_3(a_3) = 0$. We have $d_B^3(0) = 0$, so that $d_B^3(f_3(a_3)) = 0 = f_4(d_A^3(a_3))$. f_4 is injective so we have $d_A^3(a_3) = 0$ and $a_3 \in \operatorname{Ker} d_A^3 = \operatorname{Im} d_A^2$. Thus there exists an $a_2 \in A_2$ such that $d_A^2(a_2) = a_3$.

Now we have $f_3(a_3) = f_3(d_A^2(a_2)) = 0 = d_B^2(f_2(a_2))$ so that $f_2(a_2) = b_2 \in \operatorname{Ker} d_B^2 = \operatorname{Im} d_B^1$. Thus there exists a $b_1 \in B_1$ with $d_B^1(b_1) = b_2$. Surjectivity of f_1 thus gives an $a_1 \in A_1$ with $f_1(a_1) = b_1$ and $d_B^1(f_1(a_1)) = b_2 = f_2(d_A^1(a_1))$. Thus we have $f_2(d_A^1(a_1)) = f_2(a_2)$, but f_2 is injective so we have $d_A^1(a_1) = a_2$. Thus $a_3 = d_A^2(d_A^1(a_1)) = 0$.

Next we prove surjectivity. Let $b_3 \in B_3$. Then $d_B^3(b_3) = b_4 \in B_4$ and $d_B^4(b_4) = 0$. f_4 is surjective so there exists an $a_4 \in A_4$ with $f_4(a_4) = b_4$ and we have $f_5(d_a^4(a_4)) = d_B^4(f_4(a_4)) = 0$. But f_5 is injective so we have $d_A^4(a_4) = 0$ and thus $a_4 \in \operatorname{Ker} d_A^4 = \operatorname{Im} d_A^3$. This gives an $a_3 \in A_3$ with $a_4 = d_A^3(a_3)$ and $f_4(d_A^3(a_3)) = d_B^3(f_3(a_3)) = b_4$.

Next, let $f_3(a_3) = b_3' \in B_3$. Then we have $d_B^3(b_3') = d_B^3(b_3) = b_4$, or $d_B^3(b_3' - b_3) = 0$. Thus $b_3' - b_3 \in \operatorname{Ker} d_B^3 = \operatorname{Im} d_B^2$. So, there exists a $b_2 \in B_2$ with $d_B^2(b_2) = b_3' - b_3$. f_2 is surjective so there exists an $a_2 \in A_2$ with $f_2(a_2) = b_2$. Then we have $d_B^2(f_2(a_2)) = b_3' - b_3 = f_3(d_A^2(a_2))$. Thus we have

$$b_3 = b_3' - f_3(d_A^2(a_2))$$

$$
\begin{aligned}
&= f_3(a_3) - f_3(d_A^2(a_2)) \\
&= f_3(a_3 - d_A^2(a_2)),
\end{aligned}
$$

which proves surjectivity.

Problem 4.2 *Consider the following commutative diagram*

$$
\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0
\end{tikzcd}
$$

where the rows are exact. Show that there exists an exact sequence

$$
\begin{array}{ccccccccc}
0 & \to & \operatorname{Ker} f_1 & \to & \operatorname{Ker} f_2 & \to & \operatorname{Ker} f_3 & \to & \\
 & \to & \operatorname{Cok} f_1 & \to & \operatorname{Cok} f_2 & \to & \operatorname{Cok} f_3 & \to & 0.
\end{array}
$$

We can extend the columns by adding zeros before after, yielding the sequences

$$
\begin{gathered}
C^* : 0 \to A_1 \xrightarrow{f_1} B_1 \to 0 \\
D^* : 0 \to A_2 \xrightarrow{f_2} B_2 \to 0 \\
E^* : 0 \to A_3 \xrightarrow{f_3} B_3 \to 0.
\end{gathered}
$$

since $f_i \circ \iota = f_1(0) = 0$ and $0 \circ f_i = 0$ these sequences are actually chain complexes, from which we wish to define the short sequence

$$
0 \to C^* \xrightarrow{g} D^* \xrightarrow{h} E^* \to 0,
$$

where the chain maps g and h are the maps between columns of the original diagram. This sequence is then exact since each row of the original diagram is exact. Thus we can form the long exact cohomology sequence

$$
\begin{array}{ccccccccc}
0 & \to & H^0(C^*) & \xrightarrow{g^*} & H^0(D^*) & \xrightarrow{h^*} & H^0(E^*) & \xrightarrow{\partial^*} & \\
 & \to & H^1(C^*) & \xrightarrow{g^*} & H^1(D^*) & \xrightarrow{h^*} & H^1(E^*) & \xrightarrow{\partial^*} & 0.
\end{array}
$$

Now we have

$$
\begin{aligned}
H^0(C^*) &= \operatorname{Ker} f_1 \\
H^0(D^*) &= \operatorname{Ker} f_2 \\
H^0(E^*) &= \operatorname{Ker} f_3,
\end{aligned}
$$

and likewise

$$\begin{aligned}
H^1(C^*) &= B_1/\operatorname{Im} f_1 = \operatorname{Cok} f_1 \\
H^1(C^*) &= B_2/\operatorname{Im} f_2 = \operatorname{Cok} f_2 \\
H^1(C^*) &= B_3/\operatorname{Im} f_3 = \operatorname{Cok} f_3,
\end{aligned}$$

which establishes the result.

Problem 4.3 *In the commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^{0,0} \arrow[r] \arrow[d] & A^{1,0} \arrow[r] \arrow[d] & A^{2,0} \arrow[r] \arrow[d] & A^{3,0} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{0,1} \arrow[r] \arrow[d] & A^{1,1} \arrow[r] \arrow[d] & A^{2,1} \arrow[r] \arrow[d] & A^{3,1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{0,2} \arrow[r] \arrow[d] & A^{1,2} \arrow[r] \arrow[d] & A^{2,2} \arrow[r] \arrow[d] & A^{3,2} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{0,3} \arrow[r] \arrow[d] & A^{1,3} \arrow[r] \arrow[d] & A^{2,3} \arrow[r] \arrow[d] & A^{3,3} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

the horizontal $(A^{*,q})$ *and the vertical* $(A^{p,*})$ *are chain complexes where* $A^{p,q} = 0$ *if either* $p < 0$ *or* $q < 0$. *Suppose that*

$$\begin{aligned}
H^p(A^{*,q}) = 0 &\quad \text{for } q \neq 0 \text{ and all } p. \\
H^q(A^{p,*}) = 0 &\quad \text{for } p \neq 0 \text{ and all } q.
\end{aligned}$$

Construct isomorphisms $H^p(A^{*,0}) \to H^p(A^{0,*})$ *for all* p.

If we denote by d^{ij}_{lm} the map from $A^{i,j}$ to $A^{l,m}$, then commutativity of the diagram gives $d^{01}_{11} \circ d^{00}_{01} = d^{10}_{11} \circ d^{00}_{10}$. So these two maps have equal kernels but, since the second map of each composition is injective (H^0 of the first row and column are zero), the kernels of the first maps are isomorphic, and thus $H^0(A^{0,*}) = H^0(A^{*,0})$.

This is far as I've been able to get.

Problem 4.4 *Let* $0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \to 0$ *be a chain complex and assume that* $\dim A^i < \infty$. *The Euler characteristic is defined by*

$$\chi(A^*) = \sum_{i=0}^{n} (-1)^i \dim A^i.$$

Show that $\chi(A^*) = 0$ *if* A^* *is exact. Show that the sequence*

$$0 \to H^i(A^*) \to A^i/\operatorname{Im} d^{i-1} \xrightarrow{d^i} \operatorname{Im} d^i \to 0$$

is exact and conclude that

$$\dim A^i - \dim \operatorname{Im} d^{i-1} = \dim H^i(A^*) + \dim \operatorname{Im} d^i.$$

Show that

$$\chi(A^*) = \sum_{i=0}^{n} (-1)^i \dim H^i(A^*).$$

That $\chi(A^*) = 0$ when A^* is exact was proved in class - the alternating sum of dimensions of spaces in an exact sequence is zero (provided they are all finite).

Next we need to show that the sequence

$$0 \to \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}} \hookrightarrow \frac{A^i}{\operatorname{Im} d^{i-1}} \xrightarrow{d^i} \operatorname{Im} d^i \to 0,$$

is exact. We have $\operatorname{Ker} d^i \subset A^i$ so that the corresponding quotients are subsets and we can include the former in the latter. The inclusion is injective so the first map is exact.

Next, let $[x] \in A^i/\operatorname{Im} d^{i-1}$. We want to define $d^i([x]) = d^i(x)$. But we have

$$d^i(x) = d^i(y + d^{i-1}(z)) = d^i(y) + d^i(d^{i-1})(z) = d^i(y),$$

so that the map does not depend on representative. So, let $x \in \operatorname{Im} d^i$, then $x = d^i(y)$ for some $y \in A^i$. But then

$$d^i([y]) = d^i(y) = x,$$

so that the map d^i is surjective.

Finally we need to show exactness in the middle. Suppose $[x] \in \operatorname{Im} f$. $\operatorname{Im} d^{i-1} \subset \operatorname{Ker} d^i$ so that $x \in \operatorname{Ker} d^i$. Then

$$d^i([x]) = d^i(x) = 0,$$

so that $[x] \in \operatorname{Ker} d^i$. Conversely, suppose $[x] \in \operatorname{Ker} d^i$. Then $0 = d^i([x]) = d^i(x)$, so that $x \in \operatorname{Ker} d^i$. But then

$$[x] \in \operatorname{Ker} d^i/\operatorname{Im} d^{i-1} = \operatorname{Im} f.$$

Thus the sequence is exact.

Since the sequence above is short exact, we have immediately that

$$\begin{aligned}\dim(H^i(A^*)) + \dim(\operatorname{Im} d^i) &= \dim(A^i/\operatorname{Im} d^{i-1}) \\ &= \dim(A^i) - \dim(\operatorname{Im} d^{i-1}).\end{aligned}$$

Thus we can write

$$\sum_{i=0}^{n}(-1)^i \dim(A^i)$$

as

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^i \left(\dim(H^i(A^*)) + \dim(\operatorname{Im} d^i) + \dim(\operatorname{Im} d^{i-1})\right) \\ =\ &\sum_{i=0}^{n}(-1)^i \dim(H^i(A^*)) + \sum_{i=0}^{n}(-1)^i \dim(\operatorname{Im} d^i) + \sum_{i=0}^{n}(-1)^i \dim(\operatorname{Im} d^{i-1}).\end{aligned}$$

But, we can write the last sum as

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i \dim(\operatorname{Im} d^{i-1}) &= \sum_{i=1}^{n}(-1)^i \dim(\operatorname{Im} d^{i-1}) \\ &= \sum_{j=0}^{n-1}(-1)^j + 1 \dim(\operatorname{Im} d^j) \\ &= -\sum_{j=0}^{n-1}(-1)^j \dim(\operatorname{Im} d^j), \\ &= -\sum_{j=0}^{n}(-1)^j \dim(\operatorname{Im} d^j),\end{aligned}$$

where the last line follows since $\dim(\operatorname{Im} d^n) = 0$. Thus the final two sums above cancel and we are left with

$$\sum_{i=0}^{n}(-1)^i \dim(A^i) = \sum_{i=0}^{n}(-1)^i \dim(H^i(A^*)).$$

Problem 4.5 *Associate to two composable linear maps*

$$f : V_1 \to V_2, \quad g : V_2 \to V_3,$$

an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ker} f & \to & \operatorname{Ker}(g \circ f) & \to & \operatorname{Ker} g & \to & \\ & \to & \operatorname{Cok} f & \to & \operatorname{Cok}(g \circ f) & \to & \operatorname{Cok} g & \to & 0.\end{array}$$

More explicitly, we wish to construct the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}\,(f) & \hookrightarrow & \mathrm{Ker}\,(g\circ f) & \xrightarrow{f'} & \mathrm{Ker}\,(g) & \xrightarrow{\pi_1} & \\ & \to & V_2/\mathrm{Im}\,(f) & \xrightarrow{g'} & V_3/\mathrm{Im}\,(g\circ f) & \xrightarrow{\pi_2} & V_3/\mathrm{Im}\,(g) & \to & 0, \end{array}$$

where the first map is the inclusion which is injective. We note that for $x \in \mathrm{Ker}\,(f)$ that $(g \circ f)(x) = g(f(x)) = g(0) = 0$ so that $x \in \mathrm{Ker}\,(g\circ f)$ and the inclusion is defined. f' is the restriction of f to the indicated domain. If $x \in \mathrm{Ker}\,(g \circ f)$ then $f(x)$ is in $\mathrm{Ker}\,(g)$ so the map is well defined. g' is the map given by $g'([x]) = g(x)$, which is well defined since

$$[g(x)] = [g(y + f(z))] = [g(y) + g(f(z))] = [g(y)].$$

The map π_1 is the map sending x to its equivalence class in the quotient $\mathrm{Ker}\,(g)/\mathrm{Im}\,(f')$. This map is well defined since for $x \in \mathrm{Ker}\,(g\circ f)$ we have

$$0 = (g\circ f)(x) = g(f(x)) = g(f'(x)),$$

so that $f'(x) \in \mathrm{Ker}\,(g)$, that is, $\mathrm{Im}\,(f') \subset \mathrm{Ker}\,(g)$. Finally, since $\mathrm{Im}\,(g\circ f) \subset \mathrm{Im}\,(g)$ we have $V_3/\mathrm{Im}\,(g) \subset V_3/\mathrm{Im}\,(g\circ f)$ and π_2 is projection onto this subspace, which is surjective. It remains to prove exactness at the middle four steps.

First, since $\mathrm{Ker}\,(f) \subset \mathrm{Ker}\,(g\circ f)$, $\mathrm{Ker}\,(f') = \mathrm{Ker}\,(f)$ so $\mathrm{Im}\,\iota = \mathrm{Ker}\,(f')$.

Next, if $\pi_1(x) = 0$ then $x \in \mathrm{Im}\,(f')$ since $[0] = \mathrm{Im}\,(f')$. Conversely, if $x \in \mathrm{Im}\,f$ then $\pi_1(x) = [x]$, where $[x] = x + \mathrm{Im}\,(f) = \mathrm{Im}\,(f)$ so that $[x] = 0$.

Next, let $x \in \mathrm{Ker}\,(g)$, then $g'(\pi_1(x)) = g'([x]) = g(x) = 0$, so $[x] \in \mathrm{Ker}\,(g')$. Conversely, if $[x] \in \mathrm{Ker}\,(g')$ then $0 = g'([x]) = g(x)$, so $x \in \mathrm{Ker}\,(g)$, but then $[x] \in \mathrm{Im}\,(\pi_1)$.

Finally, let $[x] \in \mathrm{Ker}\,(\pi_2)$. Then $x \in \mathrm{Im}\,(g)$, that is, $x = g(y)$ for some y. But then $[g'[y]] = [g(y)] = [x]$, so that $[x] \in \mathrm{Im}\,(g')$. Conversely, if $[x] \in \mathrm{Im}\,(g')$ then $x \in \mathrm{Im}\,(g)$ and $\pi_2([x]) = 0$.

Cohomology Homework: Chapters 5 & 6

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Problem 5.3 *Can $\mathbb{R}^2$ be written as $\mathbb{R}^2 = U \cup V$ where U and V are open connected sets such that $U \cap V$ is disconnected?*

We have part of the Mayer-Vietoris sequence

$$0 \quad \to \quad H^0(U \cup V) \quad \to \quad H^0(U) \oplus H^0(V) \quad \to \quad H^0(U \cap V) \quad \to \quad H^1(U \cup V),$$

which becomes

$$0 \quad \to \quad \mathbb{R} \quad \to \quad \mathbb{R} \oplus \mathbb{R} \quad \xrightarrow{f} \quad H^0(U \cap V) \quad \to \quad 0,$$

since U, V, and $\mathbb{R}^2$ are all connected, and we know $H^1(\mathbb{R}^2) = 0$. This sequence is exact so the map f must be onto, so $\dim H^0(U \cap V) \leq 2$, in particular it is finite. Thus we have

$$\dim \mathbb{R} - \dim \mathbb{R} \oplus \mathbb{R} + \dim H^0(U \cap V) = 0,$$

or $\dim H^0(U \cap V) = 1$, so that $U \cap V$ must be connected. This result holds for each $\mathbb{R}^n$, since their respective cohomology groups are isomorphic.

Problem 5.4 *Suppose $p \neq q$ belong to $\mathbb{R}^n$. A closed set $A \subset \mathbb{R}^n$ is said to separate p from q when p and q belong to two different connected components of $\mathbb{R}^n - A$.*

Let A and B be two disjoint closed subsets of $\mathbb{R}^n$. Given two distinct points p and q in $\mathbb{R}^n - (A \cup B)$, show that if neither A or B separates p from q, then $A \cup B$ does not separate p from q.

Denote the open complements by $\tilde{A} = \mathbb{R}^n - A$ and $\tilde{B} = \mathbb{R}^n - B$. We have

$$\tilde{A} \cup \tilde{B} = (\mathbb{R}^n - A) \cup (\mathbb{R}^n - B) = \mathbb{R}^n - (A \cap B) = \mathbb{R}^n.$$

Suppose that both consist of a single connected component. Then we have part of the Mayer-Vietoris sequence

$$0 \quad \to \quad H^0(\tilde{A} \cup \tilde{B}) \quad \to \quad H^0(\tilde{A}) \oplus H^0(\tilde{B}) \quad \to \quad H^0(\tilde{A} \cap \tilde{B}) \quad \to \quad H^1(\tilde{A} \cup \tilde{B}),$$

which becomes

$$0 \quad \to \quad \mathbb{R} \quad \to \quad \mathbb{R} \oplus \mathbb{R} \quad \to \quad H^0(\tilde{A} \cap \tilde{B}) \quad \to \quad 0,$$

which shows that $H^0(\tilde{A} \cap \tilde{B})$ is 1-dimensional and thus consists of a single connected component. Since all the complements consists of a single connected component, none of them separate points which proves the theorem in this case. The full result will follow by showing that it can always be reduced to the present special case.

For more general sets A and B we can write them as a (possibly uncountable) sum over connected components

$$\begin{aligned} A &= \bigcup A_i \\ B &= \bigcup B_j, \end{aligned}$$

and we can write the complement as a (countable) sum over connected components

$$\begin{aligned} \tilde{A} &= \bigcup \tilde{A}_i \\ \tilde{B} &= \bigcup \tilde{B}_j. \end{aligned}$$

Now, since $A \cap B = \emptyset$ each connected component of one must be contained within a connected component of the complement of the other, that is, for each j there exists and i such that

$$\begin{aligned} B_j &\subset \tilde{A}_i \\ A_j &\subset \tilde{B}_i. \end{aligned}$$

Now, suppose that the points p and q are contained in the sets $\tilde{A}_{i_1}$ and $\tilde{B}_{j_1}$ (they are not separated). Then both points are contained in the intersection $\tilde{A}_{i_1} \cap \tilde{B}_{j_1}$. We will now removed the extraneous components of the complements of the sets A and B without changing the relevant intersection of the connected components containing the points.

For every $i \neq i_1$, add the component $\tilde{A}_i$ to A, creating a new closed set which we will continue to call A. This does not change A_{i_1}, so $\tilde{A}_{i_1} \cap \tilde{B}_{j_1}$ remains unchanged. Now, such a component of $\tilde{A}$ may contain a component B_j of B. If so, remove this component from the set B, obtaining a new closed set which will continue to call B. This may change the set $\tilde{B}_{i_1}$, but not its intersection with $\tilde{A}_{i_1}$, since the change is happening in an open set $\tilde{A}_i$, $i \neq i_1$. Repeat this process until $\tilde{A}_{i_1}$ is the only remaining component of the complement of A.

Now we do the same procedure with the components of the complement of B which also doesn't change the desired intersection of sets by the same argument as for A. We continue until the complement has the one remaining component $\tilde{B}_{i_1}$. Then each complement has exactly one component and the result follows from the special case.

Problem 6.1 *Show that "homotopy equivalence" is an equivalence relation in the class of topological spaces.*

First we need to first show that for any topological space X, $X \simeq X$. Let $f = g = \mathrm{id}_X$, then

$$f \circ g = g \circ f = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \simeq \mathrm{id}_X.$$

Next we need to show that if $X \simeq Y$, then $Y \simeq X$, but this is obvious from the definition.

Finally we need to show that if $X \simeq Y$ and $Y \simeq Z$ that $X \simeq Z$. We first need the following result. Suppose we have $f \simeq f'$, then $g \circ f \circ h \simeq g \circ f' \circ h$, where g and h are continuous maps. Let $F(x,t)$ be the homotopy with $F(x,0) = f(x)$ and $F(x,1) = f'(x)$, then define a new homotopy $G(x,t)$ by $G(x,t) = g(x) \circ F(x,t) \circ h(x)$. Then we have $G(x,0) = g(x) \circ F(x,0) \circ h(x) = g(x) \circ f(x) \circ h(x)$ and $G(x,1) = g(x) \circ F(x,1) \circ h(x) = g(x) \circ f'(x) \circ h(x)$. $G(x,t)$ is continuous since it is the composition of continuous maps.

Now, we have the following maps

$$\begin{array}{rclrcl} f &:& X \to Y & g \circ f &\simeq& \mathrm{id}_X \\ g &:& Y \to X & f \circ g &\simeq& \mathrm{id}_Y \\ f' &:& Y \to Z & g' \circ f' &\simeq& \mathrm{id}_Y \\ g' &:& Z \to Y & f' \circ g' &\simeq& \mathrm{id}_Z. \end{array}$$

We will define the maps $f'' : X \to Z$ and $g'' : Z \to X$ by

$$\begin{aligned} f'' &= f' \circ f \\ g'' &= g \circ g', \end{aligned}$$

then we have

$$\begin{aligned} g'' \circ f'' &= (g \circ g') \circ (f' \circ f) \\ &= g \circ (g' \circ f') \circ f \\ &\simeq g \circ \mathrm{id}_Y \circ f \\ &= g \circ f \\ &\simeq \mathrm{id}_X, \end{aligned}$$

$$\begin{aligned} f'' \circ g'' &= (f' \circ f) \circ (g \circ g') \\ &= f' \circ (f \circ g) \circ g' \\ &\simeq f' \circ \mathrm{id}_Y \circ g' \\ &= f' \circ g' \\ &\simeq \mathrm{id}_Z, \end{aligned}$$

where we have used our result above.

Problem 6.2 *Show that all continuous maps $f : U \to V$ that are homotopic to a constant map induce the 0-map $f^* : H^p(V) \to H^p(U)$ for $p > 0$.*

Since homotopic maps induce the same maps on cohomology groups (Thm. 6.8) we need only check the case when f is a constant map. So, let f be the constant map $f(x) = y_0$ for every x. The induced map on cohomology is given by

$$H^p(f) : [\omega] \to [f^*(\omega)],$$

where the induced map on (p-) forms is given by

$$(f^*\omega)_x(\xi_1, \ldots, \xi_p) = \omega_{f(x)}(D_x f(\xi_1), \ldots, D_x f(\xi_p)).$$

Now, if $p = 0$ this reduces to

$$(f^*\omega)_x = \omega_{f(x)},$$

the constant 0-form. However, if $p > 0$, then $D_x = 0$ since the map f is constant, and we have

$$(f^*\omega)_x(\xi_1, \ldots, \xi_p) = \omega_{f(x)}(0, \ldots, 0) = 0.$$

Thus $[f^*(\omega)] = [0] = 0$, so that $H^p(f)$ is the zero map for $p > 0$.

Problem 6.3 *Let $p_1, \ldots, p_k$ be k distinct points in $\mathbb{R}^n$, $n \geq 2$. Show that*

$$H^d(\mathbb{R}^n - \{p_1, \ldots, p_k\}) \cong \begin{cases} \mathbb{R}^k & \textit{for } d = n-1 \\ \mathbb{R} & \textit{for } d = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

We will first take the case $n = 1$ and suppose one point p is missing ($k = 1$). Then we set $U = (-\infty, p)$ and $V = (p, \infty)$, which gives $\mathbb{R} - \{p\} = U \cup V$. The Mayer-Vietoris sequences gives

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U \cap V) & \to & \\ & \to & H^1(U \cup V) & \to & H^1(U) \oplus H^1(V) & \to & H^1(U \cap V) & \to & 0, \end{array}$$

which becomes, since U and V are star-shaped with empty intersection,

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & \mathbb{R} \oplus \mathbb{R} & \to & 0 & \to & \\ & \to & H^1(U \cup V) & \to & 0 & \to & 0 & \to & 0, \end{array}$$

so that $\dim H^0(U \cup V) = 2$ and $\dim H^1(U \cup V) = 0$.

Now we proceed by induction on k to show that $H^1 = 0$ for $\mathbb{R}$ minus a finite number of points (we already know $\dim H^0 = k + 1$, the number of connected components, but we'll get this too).

Let $X = \mathbb{R} - \{p_1, \ldots, p_k\}$. We can write X as the union of $k + 1$ disjoint open intervals separated by the $\{p_i\}$. In particular, we write $X = U \cup V$, where

$$U = (-\infty, p_1) \cup (p_1, p_2) \cup \cdots \cup (p_{k-1}, p_k),$$

and

$$V = (p_k, \infty),$$

where we have assumed without loss of generality that $p_i < p_j$ for $i < j$. But then Mayer Vietoris gives

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U \cap V) & \to & \\ & \to & H^1(U \cup V) & \to & H^1(U) \oplus H^1(V) & \to & H^1(U \cap V) & \to & 0, \end{array}$$

which becomes

$$\begin{array}{ccccccccc} 0 & \to & H^0(U \cup V) & \to & \mathbb{R} \oplus \mathbb{R}^k & \to & 0 & \to & \\ & \to & H^1(U \cup V) & \to & 0 & \to & 0 & \to & 0, \end{array}$$

using the induction hypothesis on V which is a union of k disjoint intervals. Now we have the two exact sequences

$$0 \quad \to \quad H^0(U \cup V) \quad \to \quad \mathbb{R} \oplus \mathbb{R}^k \quad \to \quad 0,$$

and

$$0 \quad \to \quad H^1(U \cup V) \quad \to \quad 0.$$

Thus we get that $H^0(U \cup V) \cong \mathbb{R}^{k+1}$ and $H^1(U \cup V) \cong 0$.

Now that we know the cohomology groups of $\mathbb{R}$ minus a finite number of points, we will exploit Prop 6.11 to extend the result to $\mathbb{R}^n$. First, since the number of points is finite we can always find a diffeomorphism taking those points onto the subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, and we can calculate assuming this rearrangement since cohomology groups are diffeomorphism invariants. We will now assume our space to be replaced with this diffeomorphic image. Let A stand for the set up k points removed from the subspace $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$.

First we extend to $\mathbb{R}^2$. Prob 6.11 tells us that

$$\begin{array}{rcl} H^2(\mathbb{R}^2 - A) & \cong & H^1(\mathbb{R} - A) \cong 0 \\ H^1(\mathbb{R}^2 - A) & \cong & H^0(\mathbb{R} - A)/\mathbb{R} \cong \mathbb{R}^{k+1}/\mathbb{R} \cong \mathbb{R}^k \\ H^0(\mathbb{R}^2 - A) & \cong & \mathbb{R}, \end{array}$$

which is the intended result. Next we extend to $\mathbb{R}^3$

$$\begin{array}{rcl} H^3(\mathbb{R}^3 - A) & \cong & H^2(\mathbb{R}^2 - A) \cong 0 \\ H^2(\mathbb{R}^3 - A) & \cong & H^1(\mathbb{R}^2 - A) \cong \mathbb{R}^k \\ H^1(\mathbb{R}^3 - A) & \cong & H^0(\mathbb{R}^2 - A)/\mathbb{R} \cong \mathbb{R}/\mathbb{R} \cong 0 \\ H^0(\mathbb{R}^3 - A) & \cong & \mathbb{R}, \end{array}$$

and at this point the further induction to $\mathbb{R}^n$ is clear: H^0 stays $\mathbb{R}$, H^1 will be $\mathbb{R}/\mathbb{R} \cong 0$, $H^{n-2}(\mathbb{R}^{n-1} - A) \to H^{n-1}(\mathbb{R}^n - A) \cong \mathbb{R}^k$, and the rest stay zero, which establishes the result.

Problem 6.4 *Suppose that $f, g : X \to S^{n-1}$ are two continuous maps, such that $f(x)$ and $g(x)$ are never antipodal. Show that $f \simeq g$.*

Show that every non-surjective map $f : X \to S^{n-1}$ is homotopic to a constant map.

Regard the $n-1$ sphere as a subspace of Euclidean space:

$$S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}.$$

We intend to construct a homotopy $F : X \times [0,1] \to S^{n-1}$ between f and g by

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}},$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. We then have

$$F(x,0) = \frac{f(x)}{\sqrt{1 + 2(0)(1)(\langle f(x), g(x)\rangle - 1)}} = f(x),$$

and

$$F(x,1) = \frac{g(x)}{\sqrt{1 + 2(2)(0)(\langle f(x), g(x)\rangle - 1)}} = g(x),$$

so we need to show the map is well defined.

First, we must have the expression under the square root be non-negative. We have

$$\begin{array}{rcccl} -1 & \leq & \langle f(x), g(x)\rangle & \leq & 1 \\ -2 & \leq & \langle f(x), g(x)\rangle - 1 & \leq & 0 \\ -1/2 & \leq & t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 0 \\ -1 & \leq & 2t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 0 \\ 0 & \leq & 1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1) & \leq & 1, \end{array}$$

where the 3rd line follows since $0 \leq t(1-t) \leq 1/4$, since $0 \leq t \leq 1$. Thus we only have trouble when the expression is 0, which is when $\langle f(x), g(x)\rangle = -1$, that is, when $f(x)$ and $g(x)$ are antipodal, but by hypothesis this does not occur, so our map is continuous.

Second, we must have this map actually map into the sphere. We have for $|F(x,t)|$

$$\begin{aligned}
& \left| \frac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}} \right| \\
= & \left\langle \frac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}}, \frac{(1-t)f(x) + tg(x)}{\sqrt{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}} \right\rangle \\
= & \frac{(1-t)^2\langle f(x), f(x)\rangle + 2t(1-t)\langle f(x), g(x)\rangle + t^2\langle g(x), g(x)\rangle}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\
= & \frac{(1-t)^2 + 2t(1-t)f(x)\cdot g(x) + t^2}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\
= & \frac{1 - 2t(1-t) + 2t(1-t)\langle f(x), g(x)\rangle}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\
= & \frac{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)}{1 + 2t(1-t)(\langle f(x), g(x)\rangle - 1)} \\
= & 1,
\end{aligned}$$

where we used the fact that $\langle f(x), f(x)\rangle = \langle g(x), g(x)\rangle = 1$, since these maps are into the sphere. So our map does map into the sphere and is well defined.

As a corollary to this we prove the second part of the problem. If f is a non-surjective map into the sphere, then there exists a point $y_0 \in S^{n-1}$ such that $f^{-1}(y_0) = \emptyset$. Define the constant map $g : X \to S^{n-1}$ by $g(x) = y_1$ for every x, where y_1 is antipodal to y_0. Then $f(x)$ and $g(x)$ are never antipodal since y_0 isn't in the image of f, so we can construct a homotopy between them as outlined in the first half of the problem.

Problem 6.5 *Show that $S^{n-1} \simeq \mathbb{R}^n - \{0\}$. Show that two continuous maps*

$$f_0, f_1 : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\},$$

are homotopic iff their restrictions to S^{n-1} are.

That that these two spaces are homotopic was proved in class.

Now, suppose that the two maps f_0 and f_1 are homotopic. For clarity of notation let $X = \mathbb{R}^n - \{0\}$ and $Y = S^{n-1}$. We can regard the restrictions of these maps as being the the identity on X restricted to Y and the composed with the maps themselves:

$$f_0|_Y = f_0 \circ \mathrm{id}_X|_Y$$

and

$$f_1|_Y = f_1 \circ \mathrm{id}_X|_Y,$$

which are both compositions of continuous maps, so we have

$$f_0 \simeq f_1 \to f_0 \circ \mathrm{id}_X|_Y \simeq f_1 \circ \mathrm{id}_X|_Y,$$

or $f_0|_Y \simeq f_1|_Y$. There is nothing here particular to the case at hand - restrictions of homotopic maps are homotopic.

One the other hand, suppose the restricted maps are homotopic and let $g : Y \to X$ and $f : X \to Y$ be the maps defining the homotopy equivalence. Then we have

$$\begin{aligned} g \circ f &\simeq \mathrm{id}_X \\ f_i \circ (g \circ f) &\simeq f_i \end{aligned}$$

But, $g \circ f$ is in this case the map

$$x \mapsto \frac{x}{|x|},$$

mapping X onto Y. Thus we can regard $f_i \circ (g \circ f)$ as the restriction of f_i to Y. Thus $f_i|_Y \simeq f_i$, and we have

$$f_1 \simeq f_1|_Y \simeq f_2|_Y \simeq f_2,$$

since the restrictions are homotopic by hypothesis.

Problem 6.6 *Show that S^{n-1} is not contractible.*

A space is contractible if it has the same homotopy type as a point, or equivalently $\mathbb{R}^m$. Thus a space is contractible only if it has the same cohomology groups as $\mathbb{R}^m$. But we now know that $S^{n-1} \simeq \mathbb{R}^n - \{0\}$, so we have $(n \geq 2)$

$$H^p(\mathbb{R}^n - \{0\}) \cong H^p(S^{n-1}) \cong \begin{cases} \mathbb{R} & p = 0,\ n-1 \\ 0 & otherwise, \end{cases}$$

But, on the other hand, from Thm. 6.13 we have,

$$H^p(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & otherwise, \end{cases}$$

so these groups are not all isomorphic (there are two nontrivial groups for the first space, only one for the second).

Finally, for the case $n = 1$ we need to compute the groups for $\mathbb{R} - \{0\}$, which has two connected components, so $H^0(\mathbb{R} - \{0\}) = \mathbb{R} \oplus \mathbb{R}$, so the "0-sphere" is not contractible either.

Cohomology Homework: Chapter 7

Daniel J. Cross

December 13, 2006

Problem 7.1 *Show that $\mathbb{R}^n$ does not contain a subset homeomorphic to D^m when $m > n$.*

Let $\Delta \subseteq \mathbb{R}^n$ and $\phi : D^m \to \Delta$ a homeomorphism with $D^m \subset \mathbb{R}^m$. Then the restriction of ϕ to the interior of D^m is a homeomorphism onto the interior of Δ. But the interior of D^m is homeomorphic to $\mathbb{R}^m$, so we have $\mathring{\Delta} \cong \mathbb{R}^m$.

However, we can consider the inclusion $\iota : \mathring{\Delta} \hookrightarrow \mathbb{R}^m$, which maps $\mathring{\Delta}$ homeomorphically onto the proper subspace $\mathbb{R}^n \subset \mathbb{R}^m$. But this set is homeomorphic to $\mathbb{R}^m$ so must be open by invariance of domain, which is a contradiction since it is a subset of a proper subspace.

Problem 7.2 *Let $\Sigma \subseteq \mathbb{R}^n$ be homeomorphic to S^k $(1 \leq k \leq n-2)$. Show that*

$$H^p(\mathbb{R}^n - \Sigma) \cong \begin{cases} \mathbb{R} & \text{for } p = 0, n-k-1, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 7.8 we have the isomorphisms

$$H^p(\mathbb{R}^n - \Sigma) \cong H^p(\mathbb{R}^n - S^k),$$

so we will compute the latter groups. We will further consider $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$. First we have

$$H^p(\mathbb{R}^n - S^{n-1}) \cong H^p(\mathring{D}^n) \oplus H^p(\mathbb{R}^n - D^n).$$

The first set is star shaped, so we get $\mathbb{R}$ for $p = 0$ and 0 otherwise, while the second set is homeomorphic to $\mathbb{R}^n - \{0\}$, so we get $\mathbb{R}$ for $p = 0, n-1$ and 0 otherwise, so all together we get

$$H^p(\mathbb{R}^n - S^{n-1}) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & p = 0 \\ \mathbb{R} & p = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Next will iteratively apply Proposition 6.11 to extend this result. First we get

$$H^p(\mathbb{R}^{n+1} - S^{n-1}) \cong \begin{cases} \mathbb{R} & p = 0, 1, n \\ 0 & \text{otherwise,} \end{cases}$$

and more generally,

$$H^p(\mathbb{R}^{n+a} - S^{n-1}) \cong \begin{cases} \mathbb{R} & p = 0, a, n+a-1 \\ 0 & otherwise. \end{cases}$$

In the present case we set $n-1 = k$ and $m = n+a = n+k-1$. Thus $a = k-1$ and $n+a-1 = (m-a)+a-1 = m-1$, so we get

$$H^p(\mathbb{R}^m - S^k) \cong \begin{cases} \mathbb{R} & for\ p = 0, m-k-1, m-1 \\ 0 & otherwise. \end{cases}$$

Problem 7.3 *Show that there is no continuous map $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} \simeq id_{S^{n-1}}$.*

We follow the proof of Lemma 7.2 except that the function g satisfies $g|_{S^{n-1}} \simeq \mathrm{id}_{S^{n-1}}$. Then the function $g(tr(x))$ defines a homotopy between $g(r(x))$ and $g(0)$, a constant map. Thus we have

$$g(0) \simeq g(r(x)) \simeq \mathrm{id} \circ r(x) = r(x),$$

so that we still obtain a homotopy between $r(x)$ and a constant map, so the rest of the argument still holds.

Problem 7.4 *Let $f : D^n \to \mathbb{R}^n$ be a continuous map and let $r \in (0,1)$ be given. Suppose for all $x \in S^{n-1}$ that $||f(x) - x|| \leq 1 - r$. Show that $f(D^n)$ contains the closed disc with radius r and center 0.*

Suppose the conclusion false, then there exists a point x_0 with $||x_0|| \leq r$ and $x_0 \neq f(x)$ for any $x \in D^n$. As in the Brouwer Fixed Point Theorem, we we wish to define a function $g(x)$ to be the intersection of the half line from x_0 to $f(x)$ with S^{n-1}, which is well-defined as x_0 and $f(x)$ are always distinct. The function $g(x)$ is defined as

$$g(x) = x_0 + t\frac{x_0 - f(x)}{||x_0 - f(x)||},$$

with t given so that $||g(x)|| = 1$, that is

$$t = -x_0 \cdot u + \sqrt{1 - ||x_0||^2 + (x_0 \cdot u)^2},$$

where

$$u = \frac{x_0 - f(x)}{||x_0 - f(x)||},$$

and thus $g(x)$ is continuous.

If we can show that the restriction of $g(x)$ to S^{n-1} is homotopic to the identity then by problem 7.3 we have a contradiction, so that $g(x)$ cannot exist and neither the point x_0. We prove this next.

Let $x \in S^{n-1}$ and let D_x be the solid ball of radius $1-r$ centered on x, and let D_0 be the solid ball of radius r centered on 0. Then D_0 and D_x intersect in an unique point p, and moreover this intersection defines a $n-1$ dimensional tangent hyperplane in $\mathbb{R}^n \simeq \mathbb{R}^{n-1} \times \mathbb{R}$, and let z be the coordinate function in the direction orthogonal to the hyperplane oriented so that $z(p) < 0$ ($z(p) \neq 0$ since D_0 has radius $r > 0$). Then for every $y \in D_x$, $y \neq p$, $z(y) < z(p) < 0$, in particular $z(x) < 0$ and $z(f(x)) < 0$.

Now, if $x_0 = p$ then $f(x) \neq p$ and $z(x_0) = z(p) > z(f(x))$. Likewise if $x_0 \neq p$ then $z(x_0) > z(p) \geq z(f(x))$. Thus in either case we will have $z(x_0) > z(f(x))$, but $z(f(x)) < 0$, so that $z(g(x)) < 0$ as well. Thus x and $g(x)$ are always on the same side of the hyperplane, so, in particular, they are never antipodal points. Thus by problem 6.4 $g(x)$ is homotopic to the identity on S^{n-1}.

Problem 7.5 *Assume given two injective continuous maps $\alpha, \beta : [0,1] \to D^2$ such that*

$$\begin{array}{rclcrcl} \alpha(0) & = & (-1,0), & & \alpha(1) & = & (1,0), \\ \beta(0) & = & (0,-1), & & \beta(1) & = & (0,1). \end{array}$$

Prove that the two curves α and β intersect.

We will use α and β to denote the maps and their images in D^2. α may be in the boundary of D^2 at places but we can assume that $\alpha \cap \beta(1) = \emptyset$. Thus there must exist a neighborhood N of $\beta(1)$ with $B \cap \alpha = \emptyset$. Moreover there are points $t_1, t_2 \in [0,1]$ with the property that $\alpha(t_1), \alpha(t_2) \in S^1$, $\alpha(t) \notin S^1$ for $t_1 < t < t_2$, and $\beta(1)$ is between $\alpha(t_1)$ and $\alpha(t_2)$. We have, at the least, that $t_1 = 0$ and $t_2 = 1$.

We consider S^1 to be parametrized by angle in the usual way and let θ_1 and θ_2 be angles corresponding to the points $\alpha(t_1)$ and $\alpha(t_2)$ respectively. We note that $\theta_1 > \theta_2$.

Now we wish to define a map $\phi : S^1 \to D^2$ by

$$\phi(\theta) = \begin{cases} \alpha\left(T_1 t_2 + (1-T_1) t_1\right) & 0 \leq \theta \leq \theta_2 \\ (\cos(\theta), \sin(\theta)) & \theta_2 \leq \theta \leq \theta_1 \\ \alpha\left(T_2 t_2 + (1-T_2) t_1\right) & \theta_1 \leq \theta \leq 2\pi, \end{cases}$$

where we have

$$\begin{aligned} T_1 &= \frac{\theta - \theta_1 + 2\pi}{\theta_2 - \theta_1 + 2\pi} \\ T_2 &= \frac{\theta - \theta_1}{\theta_2 - \theta_1 + 2\pi}. \end{aligned}$$

This function is certainly piecewise continuous, but it's easy to see that the pieces agree at their points of overlap (including $\phi(0) = \phi(2\pi)$, so ϕ is continuous. Moreover, it is injective, so the image, Σ, of ϕ is homeomorphic to S^1 (I suppose this needs further justification - such as showing that ϕ is an open map, sending open sets to (relatively) open sets in the image).

Thus, by the Jordan-Brouwer Separation Theorem, $\mathbb{R}^n - \Sigma$ has two connected components U_1 and U_2, the former bounded, the latter unbounded, and Σ is their common boundary. Moreover we have that $\Sigma \cap \partial D^2 = \phi(\theta)$ for $\theta_2 \leq \theta \leq \theta_1$.

We have $\beta(1) \in \Sigma$ and $\beta(0) \in U_2$. There is a $t_0 \in (0,1)$ such that for every $t < t_0$, $\beta(t) \cap \Sigma = \emptyset$. We can suppose that $\beta(t_0) \neq \alpha(t_1), \alpha(t_2)$, so that for some $t' < t_0$ we have $\beta(t') \in U_1$. Thus we have a curve from U_1 to U_2 which must intersect the boundary Σ along $\phi(\theta)$ for $0 \leq \theta \leq \theta_1$ or $\theta_1 \leq \theta \leq 2\pi$, that is, it must intersect along α.

Cohomology Homework: Various Problems

Daniel J. Cross

January 6, 2010

Problem 8.4 *Set $T^n = \mathbb{R}^n/\mathbb{Z}^n$, i.e. the set of cosets for the subgroup $\mathbb{Z}^n$ of $\mathbb{R}^n$ with respect to vector addition. Let $\pi : \mathbb{R}^n \to T^n$ be the canonical map and equip T^n with the quotient topology. Show that T^n is a compact topological manifold of dimension n. Construct a differentiable structure on T^n, such that π becomes smooth and every $p \in \mathbb{R}^n$ has an open neighborhood that is mapped diffeomorphically onto an open set in T^n by π. Prove that T^1 is diffeomorphic to S^1.*

To show T^n is a topological manifold we must show that it is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$. We first show π is a local homeomorphism. In the quotient $a \sim b$ iff $a - b$ is a vector with integer coefficients. It follows that if $||a - b|| < 1$ then $a \sim b$ iff $a = b$. Thus on any open ball of radius 1/2 of any point $p \in \mathbb{R}^n$, π is injective and a local inverse exists. To show π is a local homeomorphism we need only show that the local inverses π^{-1} are continuous, i.e. that π is an open map. Let $U \in \mathbb{R}^n$ be open. We need to show $V = \pi(U)$ is open. But in a domain of injectivity, $U = \pi^{-1}(V)$ is open, thus V is open by definition of quotient topology. Thus π is open and therefore a local homeomorphism.

Now we show T^n is Hausdorff. Actually, the maximal domain on which π is a local homeomorphism are hypercubes of side length one. Let $p \neq q \in T^n$. Choose pre-images of p and q within distance one of each other and consider a domain of injectivity containing those pre-images. Since $\mathbb{R}^n$ is Hausdorff there are disjoint open sets U and V separating those pre-images. They can be assumed small enough to be contained in the domain of injectivity. Since π is a local homeomorphism $\pi(U)$ and $\pi(V)$ separate p and q.

Finally, we show T^n is second countable. Let $U \in T^n$ be open. Then $V = \pi^{-1}(U)$ is open in $\mathbb{R}^n$. But $\mathbb{R}^n$ is second countable so V can be expressed in this countable basis. The image under π of these basis elements gives U. Thus the image under π of the basis for $\mathbb{R}^n$ yields a countable basis for T^n. Thus T^n is a topological manifold.

Next we show that T^n is compact. Suppose that $\cup U_\alpha$ is an open cover of T^n. The pre-image under π of that cover covers $\mathbb{R}^n$. Now any closed hypercube of $\mathbb{R}^n$ of side length one is a compact subset that maps onto T^n under π. Choose a finite subcover of the hypercube. Its image under π provides a finite subcover for the original cover of T^n.

An atlas for T^n can be constructed using domains of injectivity for π. The overlap maps are of the form $\pi^{-1} \circ \pi$ and can be seen to correspond to translations in $\mathbb{R}^n$, which are smooth. The map π is smooth if the composition $\pi^{-1} \circ \pi$ with a chart map is smooth, which we have already showed.

Finally we show $T^1 \cong S^1$. Consider S^1 as the unit complex numbers, $S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$, so that $z = e^{i\theta}$, $\theta \in \mathbb{R}$. Define a map $\phi : T^1 \to S^1$ by $\phi(x) = \exp 2\pi i \pi^{-1}(x)$. This is well defined for if y_1 and y_2 are both pre-images of x under π then $y_2 = y_1 + a$, $a \in \mathbb{Z}$ and

$$\begin{aligned}
\phi(x) &= \exp 2\pi i y_2 \\
&= \exp 2\pi i (y_1 + a) \\
&= \exp 2\pi i y_1 + \exp 2\pi i a \\
&= \exp 2\pi i y_1.
\end{aligned}$$

Similarly it follows that ϕ is injective and it is easy to show ϕ is surjective. It is tedious but straightforward to show that ϕ is both continuous and smooth, from which the result follows since T^1 is compact.

Problem 9.9 *In the vector space $M = M_n(\mathbb{R})$ of real-valued $n \times n$ matrices we have the subspace of symmetric matrices S_n. Define a smooth map $\phi : M \to S_n$ by*

$$\phi(A) = A^t A,$$

where A^t is the transpose of A. Note that the pre-image $\phi^{-1}(I)$ of the identity matrix is exactly the set of orthogonal matrices $O(n)$. Show that for $A \in M$ and $B \in M$ we have

$$D_A \phi(B) = B^t A + A^t B.$$

Apply Ex. 9.6 to show that $O(n)$ is a differentiable submanifold of $M_n(\mathbb{R})$.

Interpreting tangent vectors as velocity vectors to curves we have

$$D_A \phi(B) = \left. \frac{d}{dt} \right|_0 \phi(\gamma(t)),$$

where γ is any curve that satisfies $\gamma(0) = A$ and $\dot{\gamma}(0) = B$. Since M is the set of all matrices the curve $\gamma(t) = A + tB$ will suffice. We then have

$$D_a \phi(B) = \left. \frac{d}{dt} \right|_0 (A + tB)^t (A + tB) = A^t B + B^t A.$$

Now $O(n) = \phi^{-1}(I)$ is a submanifold if ϕ is a submersion. Now $D\phi : T_A M \to T_{\phi(A)} S_n$. Since $M \simeq \mathbb{R}^{2n}$, $T_A M \cong M$. Similarly S_n is a linear subspace of M (since for any symmetric S_1, S_2, $aS_1 + bS_2$ is symmetric for real a, b) we have $T_B S_n \cong S_n$. So we must show that for any symmetric matrix S there is a solution B to the equation $A^t B + B^t A = S$. If we write $B = AC$ then we have

$$A^t B + B^t A = A^t (AC) + (C^t A^t) A = 2C,$$

so we are done if we take $C = S/2$ since then $C^t = C$.

Problem 9.10 *A* Lie Group G *is a smooth manifold, which is also a group, such that both*

$$\mu : G \times G \to G;\ \mu(g_1, g_2) = g_1 g_2,$$

and

$$i : G \to G;\ i(g) = g^{-1},$$

are smooth. Show that the group $O(n)$ of orthogonal $n \times n$ matrices is a Lie group.

The previous exercise showed $O(n)$ is a differentiable manifold. It is a group under matrix multiplication since the determinant condition is preserved. It remains to show that the group operations are smooth. But these are rational analytic functions in the entries of the matrix (Cramer's rule for the inverse), and so smooth.

Problem 10.1 *Let $\pi : \mathbb{R}^2 \to T^2$ be as in Ex. 8.4, and let*

$$U_1 = \pi\left(\mathbb{R} \times (0,1)\right),\ U_2 = \pi\left(\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right).$$

Show that U_1 and U_2 are diffeomorphic to $S^1 \times \mathbb{R}$, and that $U_1 \cap U_2$ has two connected components, which are both diffeomorphic to $S_1 \times \mathbb{R}$. Note that $U_1 \cup U_2 = T^2$. Use the Mayer-Vietoris sequence and Cor. 10.14 to show that

$$H^0(T^2) \simeq H^2(T^2) \simeq \mathbb{R} \text{ and } H^1(T^2) \simeq \mathbb{R}^2.$$

It is apparent from 8.4 that $T^2 \cong S^1 \times S^1$ and that π commutes with the Cartesian product. Hence we can write $\pi(A \times B) = \pi_1(A) \times \pi_1(B)$, in an obvious notation. Now $\pi_1(\mathbb{R}) = S^1$. On the other hand $(0,1) \cong \mathbb{R}$, but is equal to a maximal domain of injectivity of π_2, hence the image of $\pi_2((0,1))$ is diffeomorphic to $\mathbb{R}$, and the product is therefore diffeomorphic to $S^1 \times \mathbb{R}$, which has the homotopy type of S^1.

If we take $(0,1)$ as fundamental domain for the second factor then $(0,1)$ and $\pi_2^{-1}\pi_2(-1/2, 1/2)$ have two overlapping regions, $(0, 1/2)$ and $(1/2, 1)$, both intervals diffeomorphic to $\mathbb{R}$. Therefore the images under π_2 are two disjoint open sets diffeomorphic to $\mathbb{R}$ in S^1. The intersection of U_1 and U_2 are therefore the Cartesian product of these intervals with S^1.

The Mayer-Vietoris sequence for U_1, U_2, $U_1 \cap U_2$, and $U_1 \cup U_2 \cong T^2$ is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(T^2) & \longrightarrow & H^0(U_1) \oplus H^0(U_2) & \longrightarrow & H^0(U_1 \cap U_2) & \longrightarrow & \\
 & \longrightarrow & H^1(T^2) & \longrightarrow & H^1(U_1) \oplus H^1(U_2) & \longrightarrow & H^1(U_1 \cap U_2) & \longrightarrow & \\
 & \longrightarrow & H^2(T^2) & \longrightarrow & H^2(U_1) \oplus H^2(U_2) & \longrightarrow & H^2(U_1 \cap U_2) & \longrightarrow & 0.
\end{array}$$

Since U_1 and U_2 have the homotopy type of S^1 their cohomology groups are given by $H^0(U_i) \cong H^1(U_i) \cong \mathbb{R}$ and $H^2(U_i) \cong 0$. Since $U_1 \cap U_2$ is the disjoint union of two sets, each diffeomorphic to either U_1 or U_2, its cohomology groups is the direct sum of the groups of each U_i. The sequence becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(T^2) & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & H^1(T^2) \\ & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & H^2(T^2) & \longrightarrow & 0, \end{array}$$

and there is not yet enough information to determine the relevant groups. However, since the alternating sum of dimensions must vanish we have $\dim H^0 + \dim H^2 = \dim H^1$.

Now by Cor. 10.14 if a space M is a compact and connected manifold then $H^n(M) \cong \mathbb{R}$. A further corollary is that for M compact $H^n(M) \cong H^0(M)$, since the latter counts connected components. This can be considered a special case of Poincaré duality. It follows that $\dim H^0 = \dim H^2$. Since T^2 is connected we have $H^0(T^2) \cong \mathbb{R}$, and the result follows immediately.

Problem 10.2 *In the notation of Ex. 10.1 we have smooth submanifolds*

$$C_1 = \pi(\mathbb{R} \times \{a\}),\ C_2 = \pi(\{b\} \times \mathbb{R}),\ (a, b \in \mathbb{R}),$$

of T^2 which are diffeomorphic to S^1. They are given the orientations induced by $\mathbb{R}$. Show that the map

$$\Omega^1(T^2) \to \mathbb{R}^2;\ \omega \mapsto \left(\int_{C_1} \omega, \int_{C_2} \omega \right),$$

induces an isomorphism $H^1(T^2) \to \mathbb{R}^2$. Show that this isomorphism is independent of a and b.

It follows by Stoke's theorem that this map is well defined on cohomology since the integration domains are boundaryless. It is manifestly linear, thus a homomorphism. It remains to show it is injective. The kernel consist of those closed one forms ω such that $\int_{C_1} \omega = \int_{C_2} \omega = 0$. Now

$$\begin{aligned} \int_C \omega &= \int_{\pi^{-1}(C)} \pi^* \omega &(1)\\ &= \int_{\pi^{-1}(C)} f dx + g dy &(2)\\ &= \int_0^1 f(\gamma(t)) \frac{dx}{dt} dt + g(\gamma(t)) \frac{dy}{dt} dt, &(3) \end{aligned}$$

where the functions f and g must both be periodic in (x, y) with period one.

For C_1 we take $\gamma = (t, b)$ and for C_2 $\gamma = (a, t)$ with $t \in [0, 1]$. In the first case the integral reduces to

$$\int_0^1 f(t, b) dt,$$

and in the second to

$$\int_0^1 g(a, t) dt.$$

Now, since ω is closed and exterior differentiation commutes with pullbacks we have $d\pi^*\omega = 0$, which gives

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = 0,$$

or

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

Now consider the integral over C_1. We have

$$\begin{aligned}
\left.\frac{\partial}{\partial y}\right|_{y=b} \int_0^1 f(t,y)dt &= \int_0^1 \left.\frac{\partial f(t,y)}{\partial y}\right|_{y=b} dt \\
&= \int_0^1 \frac{\partial g(t,b)}{\partial x} dt \\
&= g(1,b) - g(0,b) \\
&= 0,
\end{aligned}$$

where we used the Fundamental Theorem of Calculus and that g is periodic. A similar results holds for the integral over C_2. In both cases the result is that the values of the integrals are independent of the choices of b and a.

Now every closed form on $\mathbb{R}^2$ is exact, so there exists a function F with $dF = \pi^*(\omega) = fdx + gdy$. Hence $f = F_x$ and $g = F_y$. If F is periodic is defines a function on T^2 whose exterior derivative is ω. Now the integral over C_1 can be written

$$\begin{aligned}
0 &= \int_0^1 f(t,y)dt \\
&= \int_0^1 \frac{\partial F(t,b)}{\partial x} dt \\
&= F(1,b) - F(0,b),
\end{aligned}$$

and from the C_2 integral we have $F(a,1) = F(a,0)$, and both of these are independent of a and b. It follows that F is periodic, from which the result follows.