Order parameter and segregated phases in a sandpile model with two particle sizes

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We study the behavior of two one-dimensional sandpile models with two different particle sizes in dependence on the size dispersion and on the difference in the surface creep between the two types of particles. We investigate the phase space and find several types of particle segregation that occur: two oppositely totally segregated states, a striped state, and two oppositely partially segregated states. By defining an order parameter for the size segregation we investigate the effect of the size dispersion and of the creep difference between the two types of particles. At very small size dispersions the creep difference induces the size segregation; if the sign of the creep difference is reversed, the size segregation in the sandpile is reversed too and the order parameter changes its sign. In one of our models, in the absence of the creep difference the order parameter grows continuously with the size dispersion from zero to its maximal value, a behavior that is reminiscent of the behavior of the order parameter in the vicinity of a continuous phase transition in the absence of external fields. [S1063-651X(97)07108-0]

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I. INTRODUCTION

Recently, much work has been done on trying to understand the unusual properties of granular matter [1]. In particular, a size segregation of a large number of granular particles of different sizes under external vibrations has been investigated [2]. A similar phenomenon occurs in sand ripples or dunes where heavy grains accumulate on the crest and light grains on the trough of a ripple [3]. Recently, a different phenomenon, a self-stratification on top of the size segregation, has been observed in the absence of any external perturbation [4].

There are several theoretical approaches that are used to study the phenomena described above. In 1987 Bak *et al.* [5] introduced a cellular automaton model as a paradigm of the concept of self-organized criticality (SOC) [6,7]. Regardless of an initial configuration, the SOC system ''self-organizes'' into a critical state with power-law distributions of avalanche amplitudes, provided that certain conditions, such as slow external driving [8], are met. However, most experiments on granular materials disagree with power-law predictions; see, e.g., Ref. [9] and references therein. Rather, the sandpile behavior is reminiscent of the properties of an equilibrium system close to a first-order phase transition with hysteresis [10]. Along these lines a continuum description of the dynamics of sandpile surfaces has been developed that takes into account two populations of grains, immobile and rolling [11], and as such yields better agreement with experimental findings.

In order to explain the self-stratification phenomenon [4], the continuum description has been extended to a pair of species [12]. Further studies have shown that the key requirement for self-stratification in granular mixtures is a difference in the repose angles of the two pure species, whereas for a size segregation of the species to take place the size dispersion is needed [13]. In this specific problem of explaining the self-stratification, the cellular automata models proved to predict the behavior very well. Also, in another case of the segregation in sand ripples under the external wind a cellular automaton model has been successfully applied recently [14] and the difference in creep dynamics between the two types of particles has been shown to be crucial to explain the particle size segregation.

Although most of the sandpile models do not quite mimic a real pile of sand, they may shed light on different aspects of the behavior of the granular matter. Here we investigate two critical-slope sandpile models that differ from the ''classical'' sandpile models by two properties: (i) the size dispersion (two different sizes of particles) and (ii) the difference in the creep dynamics between the two particle species, in order to study systematically different types of particle segregation during the evolution of the pile. Our aim is not to explain the mechanisms of size segregation or self-stratification, but rather to explore the phase space defined by two parameters: (i) size dispersion and (ii) difference in creep dynamics. In order to classify different sandpile configurations we define an order parameter that differentiates between random and phase-separated (size-segregated) configurations.

We consider two different sandpile models in the limit of slow driving rather than having a constant flux of incoming particles. The two models under consideration differ in the definition in the local slope. In the first model, A, the local slope depends on the size of the rolling particle, whereas in the second model, B, the size of the rolling particle does not affect the local slope. In the limit of a very small size dispersion both models coincide.

In Sec. II a brief description of the models under consideration is given. A comparison with other existing models is made. In Sec. III the order parameter is defined and its importance stated. In Sec. IV we present the results of both models. In addition to the expected size segregation in the sandpile (large particles on the bottom and small on the top) we find the opposite segregation (small particles on the bottom and large particles on the top). Although we are not able to change the difference in the repose angles of pure species,

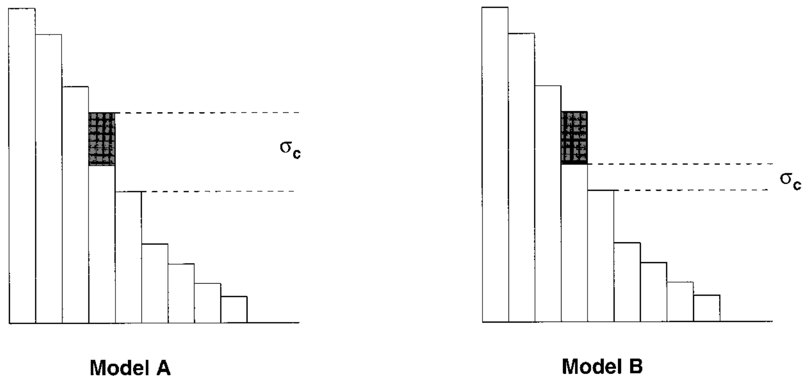

FIG. 1. Schematic diagrams for the two models considered. The reflecting and absorbing walls are at the left and right of the pile, respectively. In both cases A and B the shaded particle is the rolling particle at the active site. Note that in model B the criterion for toppling is independent of the size of the rolling particle.

the difference in the two repose angles arises, however, for a certain range of parameters due to the finite size of particles and thus we also find a self-stratified, i.e., striped configuration, such as observed and theoretically explained [4,12,13]. In addition to the three configurations described above we find within both models two partially segregated configurations that exist only at small size dispersions. We pay attention specifically to the behavior of the order parameter in two limiting cases. First, at zero creep difference within model B the order parameter is zero for all values of the size dispersion, whereas in model A the order parameter increases continuously from zero with size dispersion. Second, in the limit of a very small size dispersion both models coincide and the corresponding order parameter shows a discontiniuous behavior as the sign of the creep difference between the two particle species is changed. The conclusions are drawn in Sec. V.

II. MODELS

The models considered in this article are simplified versions of the local-limited (LL) model introduced by Kadanoff *et al.* [15]. The models presented are one-dimensional critical-slope sandpile models with one reflecting and one absorbing wall. The starting condition for the simulation is an empty lattice in which particles are added to the first site, adjacent to the reflecting wall. Because the addition of a particle is followed by a relaxation of the system, the pile builds up with the site heights decreasing from the reflecting to the absorbing wall. Similarly to other critical-slope models, the relaxation step takes place only on active sites, defined as a site i with local slope $h_i - h_{i+1}$ that exceeds a critical value σ_c. An active site relaxes so that the topmost particle ''topples'' from site i to site $i+1$. This toppling may yield the neighboring site to be active in the next step causing the relaxation rule to be repeated until all the local slopes are smaller or equal to σ_c. The iterative relaxing process readily produces slides, or avalanches, to occur in the system. In our case this relaxation means that the added particle rolls downhill until it reaches a stable site. As in the Bak-Tang-Wiesenfeld (BTW) model [5], particles are added to the system (in our case to site 1) only when all the sites are found to be stable and no more topplings occur. Under this condition, the system is said to be driven very slowly because the response of the system is much faster than the external perturbation [16]. Typical results from the BTW simulations result in power-law scaling of quantities such as the distribution function of avalanche sizes [5]. Under certain conditions the distribution functions in one-dimensional sandpile models have been found to scale in a multifractal way [15].

Four differences between the LL model and the models presented here are that (i) our models topple at most one particle from an active site, (ii) particles are added exclusively to the first site at the reflecting wall, (iii) the added particles are a random mixture of two species, i.e., two sizes, and (iv) the critical slope depends on the type of the rolling particle and the particle below it, i.e., the critical slope is a local variable. Differences (i) and (ii) are actually simplifications of the LL model, which in the case of only one type of particle yield a trivial and thus uninteresting behavior. As for (iii), we choose two different heights of particles, whereas their widths are the same. Other possibilities for achieving (iii) not considered in this paper are to choose two different species that are of equal sizes but differ by some other quantity, either by the mass [14] or by the angles of repose of pure species. By (iv) we introduce into our models different creep dynamics for different combinations of the rolling particle and the particle below it. This difference comes from a difference in friction (which in real sandpiles could come from different shapes of grains) between different particles and is an intrinsic property that distinguishes granular matter from a fluid.

The pile is initially built by choosing at random one of the two types of particles to be added to an empty lattice. For convenience, we choose to have equal ''volumes'' in the pile at any given moment of the simulation for each of the two size populations. This means that if, for example, we have sizes $s_1 = 1$ and $s_2 = 2$, then particles with s_1 will be chosen with 67% probability while particles with s_2 will be chosen with 33% probability. This condition is introduced so that spatial ordering can be easily identified visually.

We consider two different relaxation rules that define the two models A and B. In model A, a site i becomes active when a new particle is added to it at time t and the local slope defined between this site i and site $i+1$ is bigger than σ_c (see Fig. 1). If site i is active, the following relaxing rule applies:

$$H(i,t+1) = H(i,t) - s,$$

$$H(i+1,t+1) = H(i+1,t) + s, \tag{1}$$

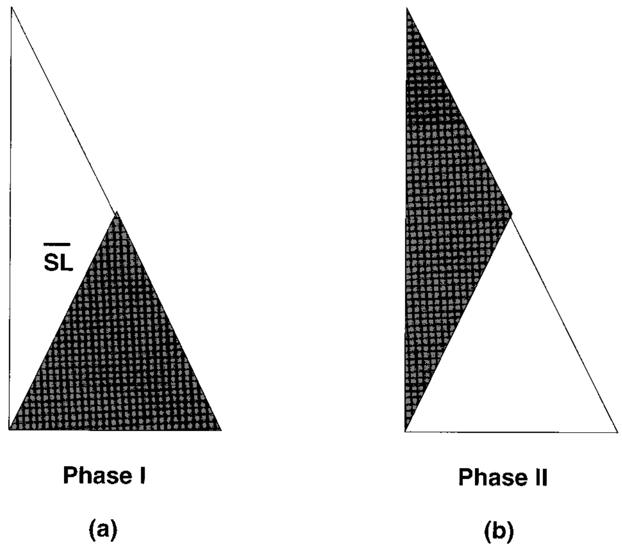

FIG. 2. Schematic diagrams for the ordered phases exhibited by the size segregation. The line $\overline{SL}$ indicates the boundary between the two different particles with the small, phase I ($\eta=1$), and large, phase II ($\eta=-1$), particles on the top.

provided that $H(i,t)-H(i+1,t)>\sigma_c$ [see Fig. 2(a)]. Here s is the height of the rolling particle and $H(i,t)$ is the total height at site i at time step t. In model B, a site becomes active only when the relative height of its site minus the size of the rolling particle $H(i,t)-s$ and the height of the next site $H(i+1,t)$ exceeds σ_c [see Fig. 1(b)]. As a consequence of this rule, in model B the condition for a site to be active is independent of the size of the rolling particle.

Rule (i) is similar to that of the LL model, but the presence of the size dispersion in the system introduces an additional noise that may affect the configuration of the system. This additional noise is not present in the BTW and LL models because the particles are of the same size and once they are ''buried'' inside the pile they cannot affect any of the dynamics that happen on the surface where the slides occurs. In the presence of the size dispersion, since the total height at a given site is made of a sequence of buried particles of different individual sizes, the order in which particles arrive and the amount of each type will affect the dynamics as well as the landscape of the surface. There is an implicit dependence of the behavior of the system on the way that the pile is built, or a history dependence.

Surface creep difference is introduced in each of the two models by modifying the value of σ_c according to the type of a rolling particle and the particle below. In particular, we choose the equation

$$\sigma_c=\sigma_0\left[1+\gamma\ \frac{s_T-s_B}{s_{\max}}\right], \tag{2}$$

where s_T and s_B are the sizes of the rolling particle (on top) and the particle below, respectively. σ_0 is the critical slope of the rolling particle on the top of the particle of the same size ($s_T=s_B$), $s_{\max}$ is the value of the size of the largest particle in the system, and γ is an adjustable parameter that affects the strength of the creep difference. The functional form of Eq. (2) allows for the modification to σ_c when the rolling particle and the particle below it are different. In particular, the above equation differentiates between the two situations: One is achieved when a small rolling particle is on the top of a large one and the other is the reverse. In fact, it has been pointed out that large grains may roll more easily on top of small grains than small grains roll on top of large grains [13]. This situation is realized in our models for $\gamma<0$. If the rolling particle is of the same type as the one below it, then $\sigma_c=\sigma_0$. This means that if the pile was formed by a single species, no matter which of the two types of particles we would take, the final repose angle would be the same. This is of course a simplification. However, in the general case where the size s of the particles is not commensurate with the critical slope σ_0, the angle of repose within model A will be ns, where n is the largest integer such that $ns<\sigma_0$ and $(n+1)s>\sigma_0$. Within model B under the same conditions, the angle of repose is equal to $(n+1)s$ (since the size of the rolling particle does not enter the rules in model B). This means that by an appropriate choice of the two sizes and the slope parameter σ_0 we can achieve a difference in the repose angles of the two types of particles.

III. SIZE SEGREGATION AND ORDER PARAMETER

Because the sandpile models under consideration are composed of particles of two different sizes, the outcome of a simulation exhibits different phenomena related to this alternative degree of freedom. In order to charaterize and quantify different realizations, we define an order parameter that takes into account the spatial distribution of the particles in the pile.

Under certain choices of parameters, both of the two models exhibit one of two ordered phases in which the particles segregate according to sizes (see Fig. 2). The boundary of the segregated system can be approximated by a diagonal line $\overline{SL}$ that cuts the pile into two regions. The two ordered phases are distinguished from each other; in phase I the small particles accumulate on the top of the pile and in phase II the big particles are on the top. We introduce an order parameter η as defined in the order-disorder transition of AB-type alloys [17]. We choose phase I to be the ordered state with the maximal order parameter $\eta=1$. Our order parameter then measures the fraction of particles correctly placed relative to the phase I. To be more specific, we define two sublattices a and b. One covers the positions of particles above the diagonal line $\overline{SL}$, the other one the positions below that line, respectively. In the completely ordered phase I, the sublattice a accommodates small and the sublattice b large particles. The order parameter η is then defined by

$$\frac{N}{4}\ (1+\eta)=N_a^A, \tag{3}$$

$$\frac{N}{4}\ (1-\eta)=N_b^A,$$

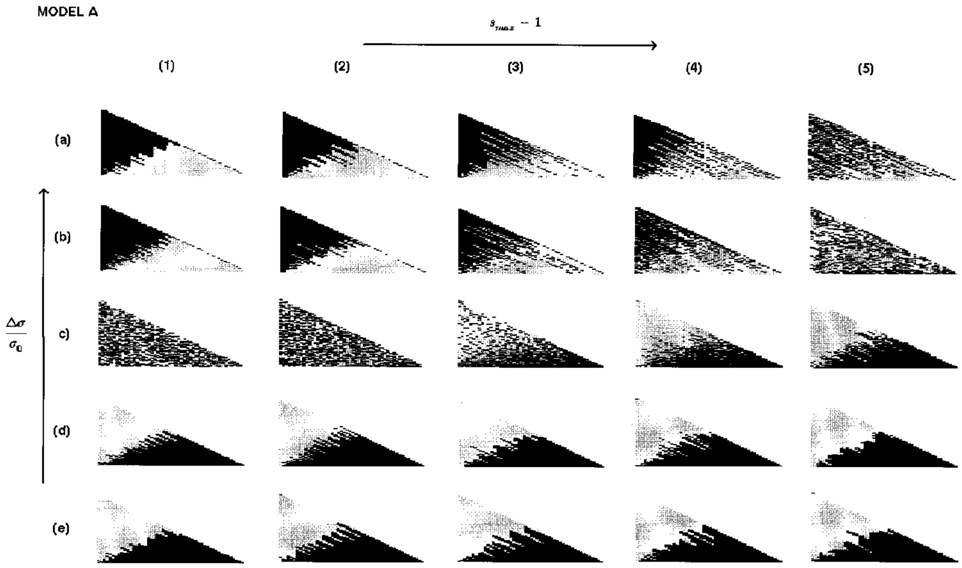

FIG. 3. Configurations obtained within model A at several relative creep differences $\Delta\sigma/\sigma_0 = 1(1)$, 0.5(2), 0(3), $-0.5(4)$, $-1(5)$ and at several dispersions $s_{\max}-1=0.001$ (a), 0.02 (b), 0.37 (c), 1.06 (d), and 3.12 (e). The system size is $L=50$ and $\sigma_0=8.28$. The values of $s_{\max}-1$ are chosen in such a way that the angle of repose of small particles σ_{SS} is smaller than the angle of repose of large particles σ_{LL}.

$$\frac{N}{4}\,(1-\eta)=N_a^B,$$

$$\frac{N}{4}\,(1+\eta)=N_b^B,$$

where N_y^X is the number of X particles in the y sublattice. From the above definition we choose $\eta=1$ for phase I and therefore obtain $\eta=-1$ for phase II (all particles are opposite to the ordering in phase I). Defined in this way, the range of values of η is from -1 to 1. In a disordered phase the order parameter vanishes since there are many wrongly positioned small and large particles in each respective sublattice.

IV. RESULTS

In all our simulations we set the size of the small particles to 1 and vary the size of the large particles $s_{\max}$. The size dispersion is then defined as $s_{\max}-1$. The other relevant parameter in our study is the relative creep difference $\Delta\sigma/\sigma_0$, where $\Delta\sigma=\gamma(s_{\max}-1)/s_{\max}$. The critical slope for a small particle on the top of a large one is then $\sigma_{SL}=\sigma_0-\Delta\sigma$ and the critical slope for a large particle on the top of a small one is $\sigma_{LS}=\sigma_0+\Delta\sigma$. In this section we study the configurations and order parameter in dependence on the above two relevant parameters.

A. Model *A*

Figure 3 shows typical configurations of the sandpile at several surface creep differences $\Delta\sigma/\sigma_0$ and several size dispersions $s_{\max}-1$. The dark and light colors correspond to large and small particles, respectively.

As might be expected, at $\Delta\sigma/\sigma_0=0$ and small $s_{\max}-1$ the arrangement of the particles is random, as a consequence of the random sequence of particle drops. As a function of increasing $s_{\max}$ (for all $\Delta\sigma/\sigma_0$), the system tends to assemble with large particles at the bottom of the pile. This tendency can be particularly observed for all $s_{\max}$ at $\Delta\sigma/\sigma_0=0$. It is a result of large particles satisfying the criterion to roll easily to the bottom of the pile as $s_{\max}$ increases.

The creep difference has an effect of switching the segregation of large and small particles that can be noticed particularly at the smallest dispersion $s_{\max}-1=0.001$. At this small dispersion, even small positive and negative values of $\Delta\sigma/\sigma_0$ can cause the opposite segregation of large and small particles. Namely, at positive values of the relative creep difference, large particles tend to be on top of small particles. This occurs due to the reduced critical slope $\sigma_{SL}=\sigma_0-|\Delta\sigma|$ for small on top of large particles as compared to large on top of small particles $\sigma_{LS}=\sigma_0+|\Delta\sigma|$. This means that small particles slide easily over large particles. On the other hand, negative creep differences correspond to smaller critical slopes for large on top of small particles $\sigma_{LS}<\sigma_{SL}$, meaning that large particles slide easily over small particles. For negative creep difference there is only one phase because the negative creep difference adds up to the overall tendency of large particles to pile at the bottom of the sandpile.

At positive creep differences and large dispersions $s_{\max}-1\gg1$ a competition effect is more apparent. The competition is between the tendency of small particles to roll easily and the large particles to have a high friction over small and the tendency of large particles to violate the critical condition for rolling easier than small particles. As one can notice in Fig. 3, the corresponding configurations vary from almost random (e.g., $s_{\max}=4.12$ and $\Delta\sigma/\sigma_0=0.5$) to a more ordered, almost stripelike configuration (e.g., $s_{\max}=2.06$ and $\Delta\sigma/\sigma_0=1.0$).

Figures 4(a) and 4(b) show the order parameter η as defined in Eq. (3), as a function of the size dispersion $s_{\max}-1$ and the relative creep $\Delta\sigma/\sigma_0$, respectively. There are two interesting limiting cases that are worth considering: (i) the zero relative creep difference $\Delta\sigma/\sigma_0=0$ and (ii) the limit of small size dispersion $s_{\max}-1\ll1$.

In case (i), Fig. 4(a), filled diamonds, the order parameter η can be studied in dependence on the size dispersion $s_{\max}-1$. As presented in Fig. 4(a), the order parameter vanishes at zero dispersion. An interesting dependence of η on $s_{\max}-1$ at zero creep difference is shown in Fig. 4(c), where η is presented together with a fit $\eta(\Delta s)=\eta_0\arctan(\alpha\Delta s)$,

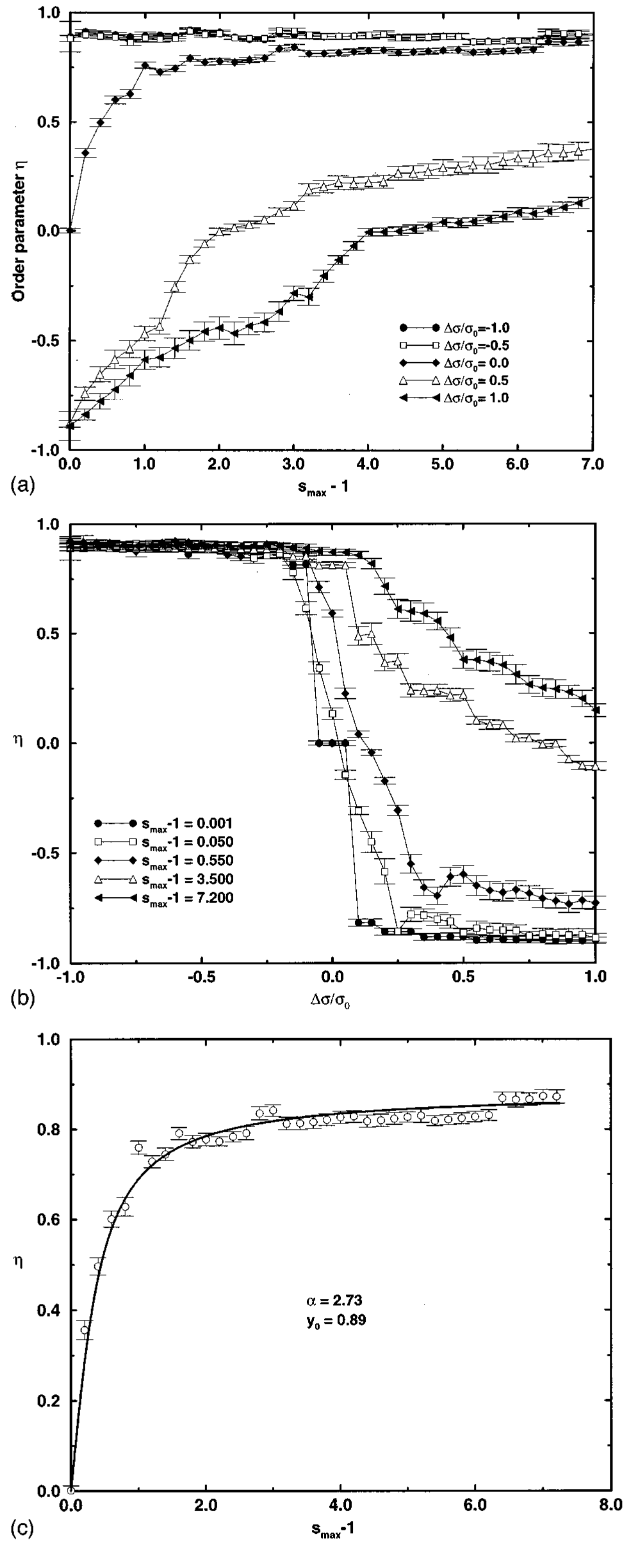

FIG. 4. (a) Order parameter η in dependence on the size dispersion $s_{\max}-1$ at several fixed values of the relative creep difference $\Delta\sigma/\sigma_0=-1,-0.5,0,0.5,1$ and (b) η in dependence on the relative creep difference $\Delta\sigma/\sigma_0$ at several fixed values of the size dispersion $s_{\max}-1=0.001,0.05,0.55,3.5,7.2$ within model A. (c) The order parameter η at a zero creep difference in dependence on $s_{\max}-1$ behaves as $\eta_0\arctan[\alpha(s_{\max}-1)]$ (solid line) with $\alpha=2.73$ and $\eta_0=0.89$. Each individual point in the three graphs is a result of averaging over 100 different configurations. The size of the system is $L=50$ and $\sigma_0=8.28$.

where $\Delta s=s_{\max}-1$. We find $\alpha=2.73$ and $\eta_0=0.89$. We confirmed that the value of η_0 depends on the system size, i.e., the larger the system, the closer η_0 will be to the maximal value of 1. This behavior is reminiscent of the behavior of the order parameter in the vicinity of the continuous phase transition, i.e., the order parameter goes to zero continuously with the exponent $\beta=1$ [found by expanding arctan(x) at small x]. The dispersion parameter $s_{\max}-1$ can be considered as the difference in concentration between the two types of particles. Because the two partial volumes of the two types of particles are kept equal, the relative number of large particles N_L versus the number of small particles N_S decreases as the size dispersion increases $N_L/N_S=s_{\max}^{-1}$. The difference in relative concentrations may be defined as $(N_S-N_L)/(N_S+N_L)=(s_{\max}-1)/(s_{\max}+1)$.

In case (ii), Fig. 4(b), filled circles, we consider the order parameter η in dependence on the relative creep difference $\Delta\sigma/\sigma_0$. In analogy to phase transitions, we can associate the creep difference with the role of an external ordering field, which by inverting the sign inverts the sign of the order parameter. As shown in Fig. 4(b), there are two sharp jumps in the order parameter around $\Delta\sigma/\sigma_0=0$. We conclude that the relative creep difference (external field) induces a discontinuous transformation from a disordered ($\eta=0$) to an ordered ($\eta\neq 0$) configuration at small size dispersions. At larger size dispersions the transformation becomes continuous and the system goes through a continuum of configurations, from a disordered to one of the two possible segregated states.

B. Model *B*

In model B, the overall tendency of large particles to assemble at the bottom of the sandpile is absent because the dynamical rules do not take into account the size of the rolling particle when changing the local slope. This means that there is no mechanism to induce size segregation in the absence of the creep difference, as is the case in model A. This is particularly apparent in Fig. 5 ($\Delta\sigma/\sigma_0=0$) and Fig. 6(a), where as a function of $s_{\max}$ the system remains in a disordered state, in contrast to Fig. 3 at the same parameters.

On the other hand, the creep difference preserves the ability to induce size segregation. As in model A for $s_{\max}-1\ll 1$, negative creep differences allow large particles to slide easily on the top of small particles, so that large particles tend to go to the bottom of the pile ($\eta>0$), while positive creep differences cause large particles to experience a large friction for rolling over small ones, so that large particles tend to remain at the top of the pile ($\eta<0$). At larger $s_{\max}$ and positive $\Delta\sigma/\sigma_0$, the part of the pile occupied by the large particles becomes ''infected'' by small particles. A striped configuration is seen at large $s_{\max}$ and negative $\Delta\sigma/\sigma_0$.

Figures 6(a) and 6(b) show the behavior of the order parameter η in dependence of the size dispersion $s_{\max}-1$ at a few fixed values of the relative creep difference $\Delta\sigma/\sigma_0$, and η in dependence on the relative creep difference $\Delta\sigma/\sigma_0$ at several fixed values of the size dispersion $s_{\max}-1$, respectively.

The limiting case of η versus $\Delta\sigma/\sigma_0$ for small size dispersions has the same behavior as in model A, case (ii) [see

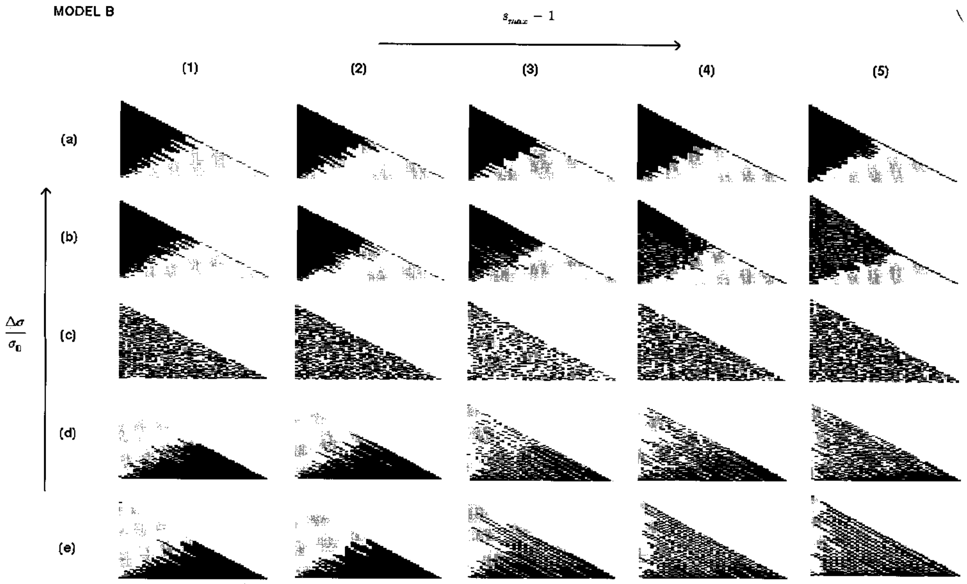

FIG. 5. Configurations obtained within model B at several relative creep differences $\Delta\sigma/\sigma_0 = 1(1)$, $0.5(2)$, $0(3)$, $-0.5(4)$, $-1(5)$ and at several dispersions $s_{\max}-1=0.001$ (a), 0.15 (b), 1.55 (c), 2.6 (d), and 3.55 (e). The system size is $L=50$ and $\sigma_0=8.28$. The values of $s_{\max}-1$ are chosen in such a way that the angle of repose of small particles σ_{SS} is smaller than the angle of repose of large particles σ_{LL}.

Fig. 6(b)]. At $s_{\max}-1\ll 1$, there is a jump in the order parameter η at the point where $\Delta\sigma/\sigma_0$ changes sign, meaning that the external field induces a transition into the two ordered states ($\eta=\pm 1$). The fact that in model B the random configuration ($\eta=0$) occurs at $\Delta\sigma/\sigma_0=0$ for all size dispersions ''pins'' the curves in Fig. 6(b) to go through the same point $\eta(\Delta\sigma/\sigma_0=0)=0$, whereas in Fig. 4(b) the curves cross the $\eta=0$ line at different $\Delta\sigma/\sigma_0$, if at all.

C. Striped configuration

In addition to the two totally segregated configurations that always appear at small dispersions in both models for opposite signs of the surface creep difference, there is another interesting configuration that is found. This configuration is the striped configuration, created by layer segregation (self-stratification) composed of a layer of small followed by a layer of large particles. This configuration has been observed in a sandpile experiment [4] and has been studied theoretically [12,13]. It has been shown that two requirements have to be fulfilled in order for self-stratification to occur: $\sigma_{SS}<\sigma_{LL}$ (the repose angle of small particles is smaller that the repose angle of large particles) and $\sigma_{LS}<\sigma_{SL}$ (large grains roll more easily on the top of small grains than small grains roll on the top of large grains).

In Figs. 3 and 5 we have chosen such values of $s_{\max}$ that the first requirement $\sigma_{SS}<\sigma_{LL}$ is met, whereas the second requirement is met for negative values of the creep difference. Although within model A the striped configuration is not very pronounced, a slight tendency towards such an ordering can be observed at positive creep differences, where the tendency of large particles to assemble at the bottom of the pile competes with the tendency of the external field to push the small particles to the bottom. This stripy configuration is accompanied by the opposite size segregation (small particles at the bottom and large at the top of the pile) as opposed to the original observation [4]. In model B, on the other hand, we find the striped configuration accompanied by size segregation (as found in original work [4]) at negative creep differences, which promote large particles to assemble at the bottom, under the same conditions that were recently established [13].

By observing the formation of the striped configuration in our simulation, we detect a kink that grows from the bottom to the top of the pile and creates a double layer of small particles below and large particles above. The kink is formed by a wall started by a large particle at the bottom of the pile. This wall acts as a barrier that particles cannot cross. Since large particles slide easily on top of small ones, they are stopped only by this barrier at the bottom. The formation of a double layer ends when the wall moves to the top of the pile. Stripes, observed in model B, are thus formed in double layers, as seen in experiment as well as modeled and treated theoretically [4,13].

D. Order parameter at small size dispersions

Here we consider the behavior of the order parameter η in dependence on the slope parameter σ_0, i.e., the critical slope of a particle on the top of another particle of the same type. We are interested in the region of very small size dispersions $s_{\max}-1\ll 1$ where both models display similar behavior. In both models at a small dispersion $s_{\max}-1=0.001$ the order parameter η shows a discontinuous behavior. For example, in Figs. 4(b) and 6(b), at negative creep differences the order parameter is close to the maximal value 1, whereas at positive creep differences it is close to its minimal value -1.

There is a plateau in order parameter η at small creep differences around $\eta=0$. In other words, at very small dispersions there is only a discrete set of possible configurations that differ by the value of the order parameter. The pile may be totally segregated (with either η close to 1 or η close to -1) or totally disordered with $\eta=0$. However, the exact number of possible configurations still depends on the slope parameter σ_0. Figure 7 shows different plateaus for different σ_0 at a fixed size dispersion $s_{\max}=1.001$. At $\sigma_0=1.1$, for example, there are three plateaus: The two outer ones correspond to $\eta\approx -1$ and $\eta\approx 1$ and the middle plateau corresponds to $\eta=0$. At certain values of σ_0, for example, $\sigma_0=2.8$, there are additional two plateaus, i.e., configura-

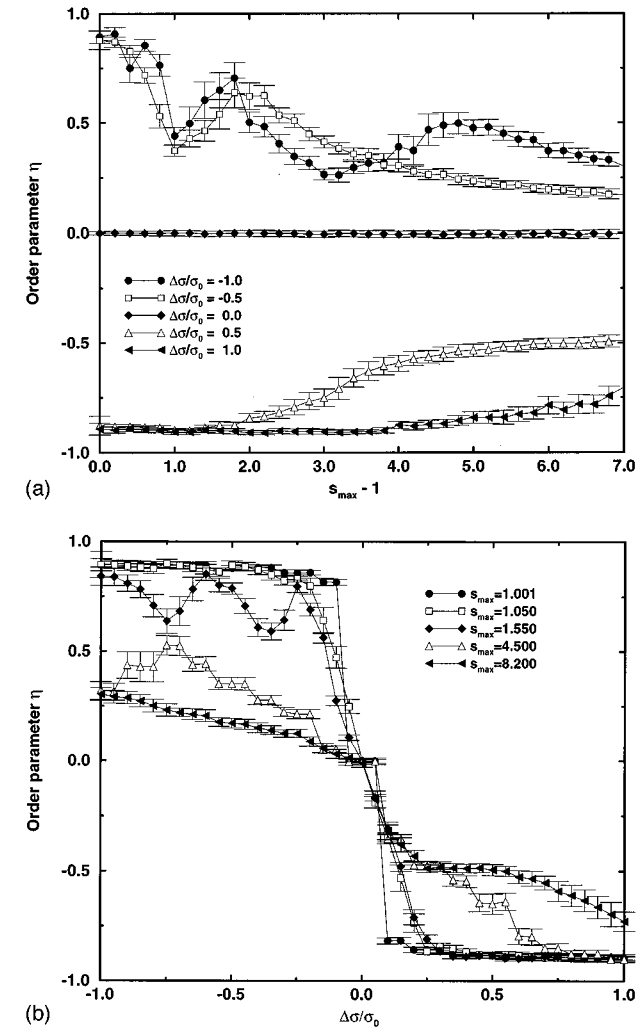

FIG. 6. (a) Order parameter η in dependence on the size dispersion $s_{\mathrm{max}}-1$ at several fixed values of the relative creep difference $\Delta\sigma/\sigma_0=-1,-0.5,0,0.5,1$ and (b) η in dependence on the relative creep difference $\Delta\sigma/\sigma_0$ at several fixed values of the size dispersion $s_{\mathrm{max}}-1=0.001,0.05,0.55,3.5,7.2$ within model B. Each calculated point on the graphs is a result of averaging over 100 different configurations. The size of the system is $L=50$ and $\sigma_0=8.28$.

tions with $\eta=\pm 0.33\pm 0.01$, between the totally disordered state with $\eta=0$ and the two outer totally segregated states. They correspond to partially segregated configurations. The absolute value of the inner limits of the two outer plateaus at integer values σ_0 decrease with σ_0; we find the decrease to be a power law with the power -0.97.

Next, we study all possible configurations within the diagram $\Delta\sigma/\sigma_0$ versus $s_{\mathrm{max}}-1$. The results for model A are presented in Fig. 8(a) and for model B in Fig. 9(a). The slope parameter σ_0 is chosen such that there are five different possible configurations at small dispersions $s_{\mathrm{max}}-1\ll 1$. The corresponding configurations are plotted in Fig. 8(b) for model A and Fig. 9(b) for model B. The partially segregated configurations consist of small (large) particles on the bottom and a random mixture of small and large particles on the top. Only at very small size dispersions are the transformations between different configurations discontinuous. At larger dispersions the transformations between configurations become smooth and continuous and all the plateaus disappear.

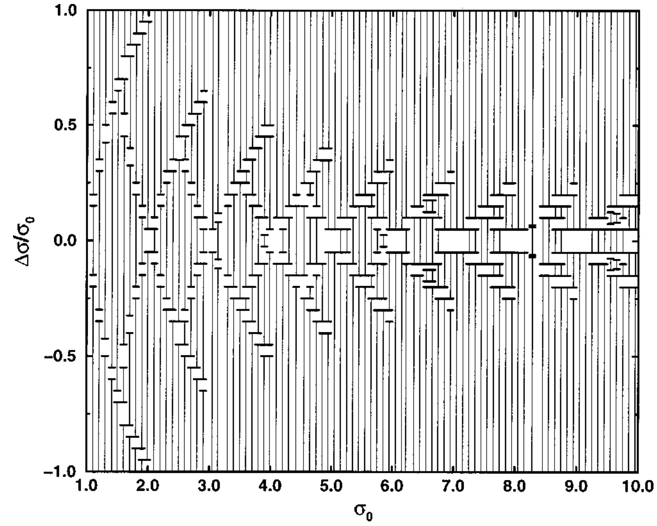

FIG. 7. Plateau regions in the order parameter in dependence on the slope parameter σ_0. The outer regions correspond to two totally segregated configurations with $\eta\approx 1$ and $\eta\approx -1$, while the central plateau around $\Delta\sigma/\sigma_0=0$ corresponds to a disordered configuration with $\eta=0$. Note that the two additional, partially segregated configurations appear between $\sigma_0=n+0.5$ and $\sigma_0=n+1$, where n is an integer.

By comparing the diagrams of both models in Figs. 8(a) and 9(a) one can notice that all the curves are symmetric around $\Delta\sigma/\sigma_0=0$ in model B and are slightly asymmetric at larger dispersions in model A. This is due to the fact that the random configuration persists in model B in the absence of the creep difference, while it is shifted to positive creep differences in model A.

V. CONCLUSION

In this paper we study size segregation in two critical-slope sandpile models that differ in the definition of the local slope. In order to describe the system, we introduce an order parameter that is zero for a disordered system and varies from -1 to 1 as a function of external parameters of the models. In addition to the size dispersion we introduce a surface creep difference between the two types of particles. We investigate a two-dimensional phase space defined by the two relevant parameters, i.e., the size dispersion and the surface creep difference. We find several different possible segregated states: two totally segregated states with order parameters $\eta\approx\pm 1$, a striped configuration, and two partially segregated configurations with $\eta\approx\pm 0.33$ that appear only at very small size dispersions.

We show that in one of the models (model A) size dispersion induces the size segregation even in the absence of the creep difference, with an order parameter that grows continuously from zero to its maximal value. In both models the surface creep difference acts as an external field. At very small dispersions, by changing the sign of the surface creep, a discontinuous transformation from a segregated to an oppositely segregated state is obtained.

In this contribution we restrict our study to different sizes of particles. This leaves untouched several questions. One of them is what would happen if we keep the sizes of the two species equal and change only the two repose angles of pure

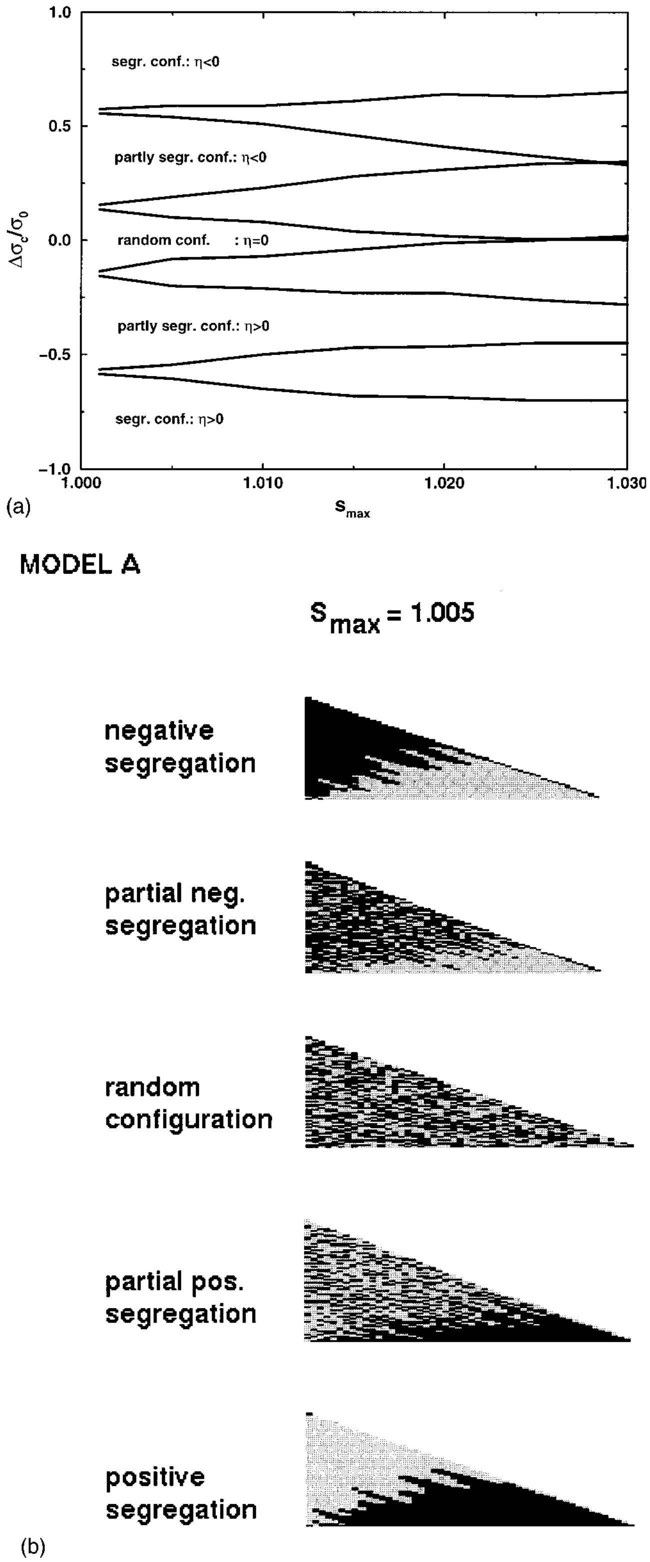

FIG. 8. Diagram of (a) segregated configurations within model *A* and (b) the corresponding sandpile configurations.

species. It would be interesting to explore the phase space by including the difference in the two repose angles into the models. Also, our models can be extended easily to account for more than two different sizes of particles. Work is under way to see the outcome of considering a specific distribution of particle sizes.

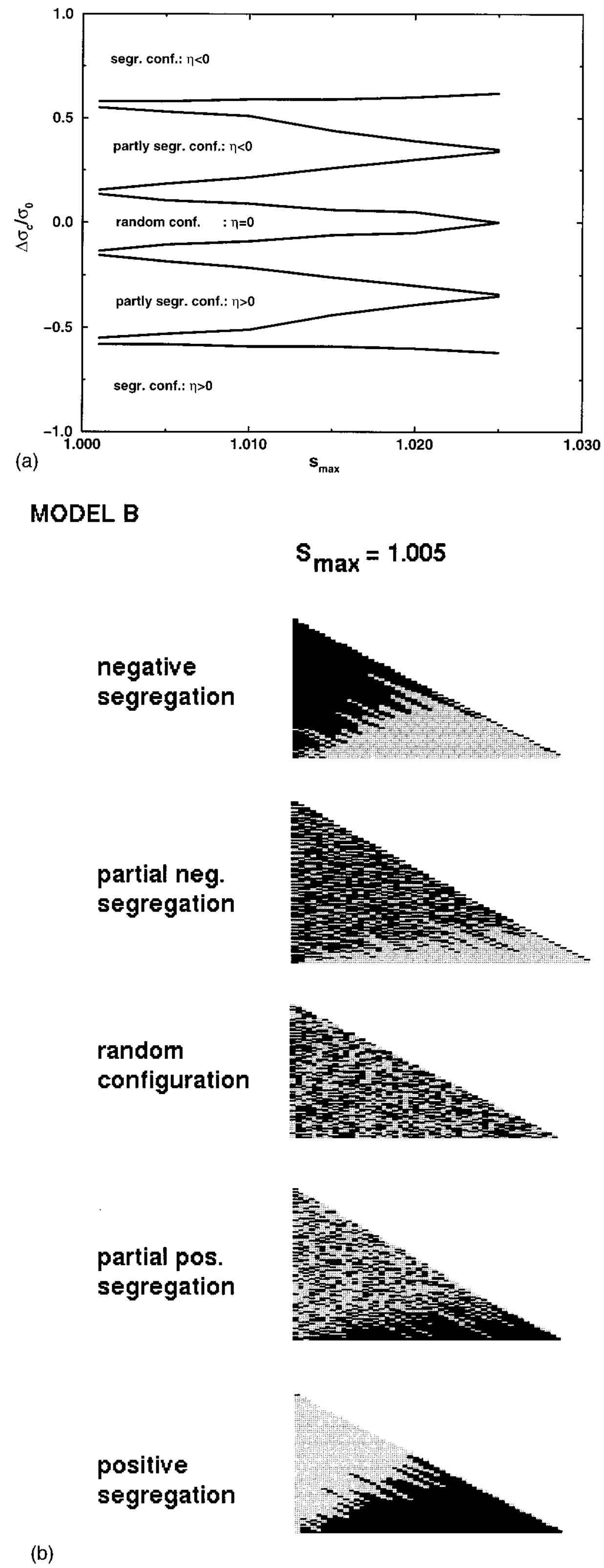

FIG. 9. Diagram of (a) segregated configurations within model *B* and (b) the corresponding sandpile configurations.

In conclusion, our models display a rich variety of different segregated phases of which only a few have been observed experimentally [4], namely, the size segregation where the large particles are found on the bottom and the small particles on the top of the sandpile and the self-stratified configuration. In the theoretical description [12,13] of the observed configurations the study was restricted to negative surface creep differences under the assumption that large grains roll more easily on top of small grains than the reverse. The question arises if the opposite segregation (with large particles on the top and small particles on the bottom of the sandpile) is possible to be achieved experimentally by an appropriate choice of grains.

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