PHYS 201 WINTER 2010

HOMEWORK 6

Solutions

1. Space Oddity

Since we'll be using Einstein's γ repeatedly for this problem, here it is:

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{2.80\cdot 10^8}{3.00\cdot 10^8}\right)^2}} \approx \frac{1}{\sqrt{1-\left(\frac{14}{15}\right)^2}} \approx 2.785$$

1.1. By definition, electromagnetic radiation travels at 1 ly/year. The spaceship travels at $\frac{14}{15}$ times the speed of light, which is $\frac{14}{15}$ ly/year. Spaceships obey distance = rate × time just like any other object; the tricky part in relativity is that "distance" and "time" can be different to different observers. From Earth's point of view, the distance is 20.3 ly and the ship's speed is $\frac{14}{15}$ ly/year, so

$$20 \text{ ly} = \tfrac{14}{15} \text{ ly/year} \times t \quad \Rightarrow \quad t \approx 21 \text{ years}$$

The ship arrives about a year after Earth receives the radio message. (If you did this problem using meters and seconds instead of light-years and years, you should get the same answer: 21 years $\approx 6.6 \cdot 10^8$ seconds.)

1.2. The distance to Gliese 581 is 20 ly from the point of view of anyone who is *not* moving relative to either planet. That's the "proper length" ΔL_0 of the journey. The navigators onboard the ship see a contracted length ΔL:

$$\Delta L = \frac{1}{\gamma}\Delta L_0 \approx \frac{1}{2.785}(20) \approx 7.2 \text{ ly}$$

1.3. From the navigators' point of view, the distance to Earth is $\frac{1}{\gamma}20 \approx \frac{20}{2.785} \approx 7.2$ light-years and Earth is moving towards them at $\frac{14}{15}$ ly/year:

$$d = rt \quad \Rightarrow \quad \frac{20}{2.785} \text{ ly} = \frac{14}{15} \text{ ly/year} \times t \quad \Rightarrow \quad t \approx 7.7 \text{ years}$$

From Earth's point of view, the ship takes $20 \cdot \frac{15}{14} \approx 21$ years to arrive. But to Earthlings, the ship's clocks appear to run slow due to time dilation:

$$\Delta t = \gamma(\Delta t_0) \quad \Rightarrow \quad 20 \cdot \frac{15}{14} = 2.785(\Delta t_0) \quad \Rightarrow \quad t_0 \approx 7.7 \text{ years}$$

Earthlings and navigators agree that the ship's "proper" time - the time shown on its own clock - increases about 7.7 years during the journey. Either method works, but be careful not to use d from one point of view and t from another!

1.4. To an Earth observer, the ship's speed is $2.80 \cdot 10^8$ m/s. Using $\gamma = 2.79$,

$$K = (\gamma - 1)mc^2 = (1.79)(250,000 \text{ kg})(3.00 \cdot 10^8 \text{ m/s})^2 \approx 4.0 \cdot 10^{22} \text{ joules}$$

This is more than the total energy used by all humans during the 20th century.

2. X-Ray Specs

The peak intensities are at the original wavelength of $\lambda_0 = 50.0$ picometers and another wavelength λ' given by the formula for Compton scattering:

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta)$$

The angle between the beam direction and the doctor is 135°. Plugging in the usual values for h and c and the electron mass $m_e \approx 9.11 \cdot 10^{31}$ gives

$$\lambda' = \lambda_0 + \frac{h}{m_e c}\left(1 - \frac{-1}{\sqrt{2}}\right) = 5.41 \cdot 10^{-11} = 54.1 \text{ picometers}$$

3. Goalie de Broglie

The momentum of a hockey puck at 150 km/h is

$$mv = 0.150 \text{ kg} \cdot 150 \text{ km/h} \cdot \frac{1000 \text{ m}}{\text{km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} \approx 6.25 \text{ kg} \cdot \text{m/s}$$

Planck's constant in MKS units is $\approx 6.626 \cdot 10^{-34}$ J·s, so

$$\lambda = \frac{6.626 \cdot 10^{-34}}{6.25} \approx 1.06 \cdot 10^{-34} \text{ meters}$$

This is absurdly small compared to the size a typical hockey puck. A useful rule-of-thumb is that wave behavior of physical objects is negligible when an object's de Broglie wavelength is much smaller than the object itself. This is why we don't notice diffraction and interference for common objects like hockey pucks.

4. What's In The Box?

The wavenumber k for this function is just the number that goes in front of x: $k = \frac{7\pi}{L}$. Since $k = \frac{2\pi}{\lambda}$, the wavelength is $\lambda = \frac{2}{7}L$. (To get a better visual idea of what this means, pick a number for L and use a calculator or computer to plot $\psi_7(x)$. The distance between peaks will be $\frac{2}{7}$ times whatever number you chose.)

We can rewrite $E = \frac{p^2}{2m}$ in terms of de Broglie wavelength λ:

$$E = \frac{p^2}{2m} = \frac{(\frac{h}{\lambda})^2}{2m} = \frac{h^2}{2m\lambda^2}$$

Plugging in our result $\lambda = \frac{2}{7}L$, we find

$$E = \frac{h^2}{2m(\frac{4}{49})L^2} = \frac{49h^2}{8mL^2}$$

Notice that this answer matches the formula in the book $E_n = \frac{n^2h^2}{8mL^2}$.

5. Uncertainty Principle

Δx is given, so we can find the minimum Δp_x easily:

$$(1 \cdot 10^{-10}\ \mathrm{m})\Delta p_x \geq \frac{h}{2\pi} \approx 1.055 \cdot 10^{-34}\ \mathrm{J \cdot s}$$
$$\Rightarrow\ \Delta p_x \geq 1.055 \cdot 10^{-24}\ \mathrm{kg \cdot m/s}$$

Notice the direction of the inequality: this is the *minimum* uncertainty for p_x.

6. Bonus Problem

This problem simplifies greatly because E and U_0 are given:

$$G = 16\left(\frac{2}{5}\right)\left(1 - \frac{2}{5}\right) = \frac{96}{25} = 19.2$$
$$\kappa = \frac{\sqrt{2(9.11 \cdot 10^{-31}\ \mathrm{kg})(3 \cdot 10^3\ \mathrm{eV})(1.602 \cdot 10^{-19}\mathrm{J/eV})}}{1.055 \cdot 10^{-34}\ \mathrm{J \cdot s}}$$
$$\kappa = \frac{2.959 \cdot 10^{-23}\ \mathrm{kg \cdot m/s}}{1.055 \cdot 10^{-34}\ \mathrm{kg \cdot m^2/s}} \approx 2.80 \cdot 10^{11}\ \mathrm{m}^{-1}$$

Note that we have to convert $(U_0 - E)$ from keV to joules or else κ will have the incredibly confusing unit kg·keV·J^{-1}s^{-1} instead of m^{-1}. (I didn't convert to joules before finding G because it is dimensionless: $\frac{E}{U_0}$ is the same in any units.)

We know the tunneling probability T as a function of L, so we can solve for L:

$$T = Ge^{-2\kappa L}\ \Rightarrow\ \ln\left(\frac{T}{G}\right) = -2\kappa L\ \Rightarrow\ L = \frac{1}{2\kappa}\ln\left(\frac{G}{T}\right)$$

There are *two* barriers, so the probability of the electron being detected outside the barriers is *twice* the value of T. Setting $2T = 0.01$ and solving for L, we find

$$L = \frac{1}{2(2.80 \cdot 10^{-11}\ \mathrm{m}^{-1})}\ln\left(\frac{19.2}{.005}\right) \approx 1.47 \cdot 10^{-11}\ \text{meters} = 14.7\ \text{picometers}$$

This distance is on the order of the Bohr radius, which is about 53 picometers. Tunneling may be negligible for hockey-puck-sized objects, but in chemistry and nuclear physics it is extremely important.