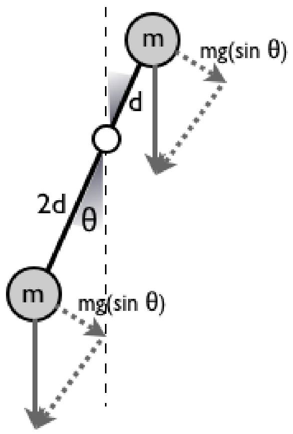
PHYS 201 WINTER 2010

HOMEWORK 1

Solutions

1. A Lopsided Pedulum



The only force acting on this system is gravity. The above picture shows the weight mg of each sphere broken into components parallel and perpendicular to the rod. (The component of weight parallel to the rod is not needed; it simply stretches or squishes the rod, which in this case is assumed to be nearly rigid.)

1.1. **Write an equation of motion for this object.**

The torque on each sphere is $mg(\sin\theta)$ times the leverage. Choosing clockwise as the positive direction for θ, the total torque is $\tau = \tau_{top} - \tau_{bottom}$.

$$\tau = -2dmg(\sin\theta) + dmg(\sin\theta) = -dmg(\sin\theta) = -50(\sin\theta)\ N \cdot m$$

The moment of inertia for a point mass is $I = mr^2$. For the two masses here,

$$I = md^2 + m(2d)^2 = 5md^2 = 25\ Kg \cdot m^2$$

Substituting these into $\tau = I\ddot{\theta}$, the equation of motion is

$$-dmg(\sin\theta) = 5md^2(\ddot{\theta}) \quad \Leftrightarrow \quad -g(\sin\theta) = 5d(\ddot{\theta}) \quad \Leftrightarrow \quad -2(\sin\theta) = \ddot{\theta}$$

1.2. **Solve the equation of motion for small oscillations and zero friction.**

For small angles, $\sin\theta \approx \theta$. The resulting equation of motion is $-g\theta = 5d(\ddot{\theta})$, which is the (undamped) spring equation with g and $5d$ in place of of k and m.

$$\theta(t) = A\cos(\omega t + \phi) \qquad \omega = \sqrt{\frac{g}{5d}} = \sqrt{2}$$

The initial conditions given were $\theta(0) = 0.87$ radians and $\dot{\theta}(0) = 0$.

$$0.87 = \theta(0) = A\cos(\phi) \qquad 0 = \dot{\theta}(0) = -\sqrt{2}A\sin(\phi)$$

Since A is not zero, the second condition requires $\phi = 0$ (or $\pm\pi, \pm 2\pi, \pm 3\pi$, etc). Plugging that back into the first condition requires $A = 0.87$. The answer is:

$$\theta(t) = 0.87\cos(\sqrt{2}t)$$

1.3. **How does the oscillation frequency change if friction is included?** Including friction, the small-angle equation of motion is

$$-dmg(\theta) - \beta(\dot{\theta}) = 5md^2(\ddot{\theta}) \quad \Leftrightarrow \quad -50(\theta) - 10(\dot{\theta}) = 25(\ddot{\theta})$$

This is the same equation as a damped spring oscillator, but with dmg, β, and $5md^2$ in place of k, b, and m. The frequency of a damped spring oscillator is:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \rightarrow \sqrt{\frac{dmg}{5md^2} - \frac{\beta^2}{4(5md^2)^2}}$$

Plugging in the numbers given in the problem,

$$\omega' = \sqrt{\frac{50}{25} - \frac{100}{4(625)}} = \sqrt{2 - \frac{1}{25}} = \sqrt{\frac{49}{25}} = \frac{7}{5}\ rad/s$$

2. Energy Dissipation in a Shock Absorber

The solution to $m\ddot{x} + b\dot{x} + kx = 0$ given in the textbook is:

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi) \qquad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

2.1. **At what time are the oscillations half as large as they were at $t = 0$?**

The solution can be split into (amplitude function) $\times$ (oscillating function). The amplitude function is $Ae^{-\frac{b}{2m}t}$. At $t = 0$, this function is simply A. Let t_{half} denote the time at which the amplitude is $\frac{1}{2}A$.

$$Ae^{-\frac{b}{2m}t_{half}} = \frac{1}{2}A \quad \Rightarrow \quad -\frac{b}{2m} = \ln\frac{1}{2} = -\ln 2 \quad \Rightarrow \quad t_{half} = 2(\ln 2)\frac{m}{b}$$

2.2. **Show that this system conserves energy if $b = 0$. If b is positive, show that the system loses energy. Where does the energy go?**

With no friction, the equation is $m\ddot{x} + kx = 0$. The solution from the book is:

$$x(t) = A\cos(\omega t + \phi) \qquad \omega = \sqrt{\frac{k}{m}}$$

The energy of the system at any time t is $E = \text{potential} + \text{kinetic} = \frac{1}{2}kx^2 + \frac{1}{2}m(\dot{x})^2$, which is not too hard to calculate directly:

$$x^2 = A^2\cos^2(\omega t + \phi) \qquad (\dot{x})^2 = \big[-\omega A\sin(\omega t + \phi)\big]^2 = \omega^2 A^2 \sin^2(\omega t + \phi)$$

Plugging these into the equation for energy and using $\omega = \sqrt{\frac{k}{m}}$, we find

$$E = \frac{1}{2}\big[kA^2\cos^2(\omega t + \phi) + m\omega^2 A^2 \sin^2(\omega t + \phi)\big]$$

$$= \frac{1}{2}kA^2\big[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\big] = \frac{1}{2}kA^2$$

using the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$. Notice that we have found an explicit expression for the energy in any undamped simple harmonic oscillator! It depends only on the amplitude and spring constant and does not change in time.

If $b > 0$, then consider what happens after a very long time $t \to \infty$.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi) = 0$$

The oscillating function $\cos(\omega t + \phi)$ does not approach any limit as $t \to \infty$; it just oscillates between 0 and 1. The amplitude function $Ae^{-\frac{b}{2m}t}$ approaches zero as $t \to \infty$, so the oscillations "die out" and eventually the shock absorber is approximately at equilibrium with no potential or kinetic energy.

Conservation of energy requires that the energy go *somehwere*. In this case, it is spent stirring the fluid. Viscous fluids dissipate energy as heat when stirred.

3. Bonus Problem

Find the first and second derivatives of the complex-number solution to $x(t)$:

$$\dot{x}(t) = rAe^{rt} + sBe^{st} \qquad \ddot{x}(t) = r^2Ae^{rt} + s^2Be^{st}$$

Plug them back into the equation $m\ddot{x} + b\dot{x} + kx = 0$ to find

$$(mr^2 + br + k)Ae^{rt} + (ms^2 + bs + k)Be^{st}$$

In order for this to be true for *all* times t and any choice of initial conditions A, B, the terms in parentheses must be zero. The complex exponents r and s must be the two roots of the quadratic equation $mr^2 + br + k = 0$.

$$r, s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

For a critically-damped system, $b = 2\sqrt{km} \Rightarrow b^2 - 4mk = 0$. Thus a spring is critically damped if and only if its complex exponents are double roots.

For an underdamped system, $b < 2\sqrt{km}$. This means $b^2 - 4mk$ is negative, so

$$\sqrt{b^2 - 4mk} = \sqrt{-(4mk - b^2)} = \imath\sqrt{4mk - b^2}$$

The complex exponents r and s can now be written

$$r, s = \frac{-b \pm \imath\sqrt{4mk - b^2}}{2m} = -\frac{b}{2m} \pm \imath\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The imaginary part is the damped-oscillator frequency ω' and the real part is the exponent for the amplitude function $e^{-\frac{b}{2m}t}$. This is the advantage of using complex exponents to solve linear differential equations: they turn hard problems (solve a damped-oscillator equation) into easy problems (use the quadratic formula).