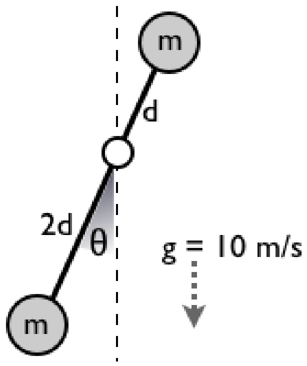
PHYS 201 WINTER 2010

HOMEWORK 1

Write down all steps towards the solution to obtain maximum credit. Don't forget to specify units!

1. A Lopsided Pedulum



Consider the lopsided pendulum pictured above. Assume it is free to rotate in the plane of the page but otherwise fixed. Each sphere has uniform density and mass m. Neglect the mass of the rod and assume $g = 10\ m/s^2, m = 5\ Kg$, and $d = 1\ m$.

Define θ to be the angle between the rod and a vertical line passing through the ball bearing. (You can define clockwise to be positive or negative; either is OK.)

1.1. **Write an equation of motion for θ (assuming zero friction).**

(1) The equation of motion for a rigid, rotating object is $\tau = I\alpha$, where τ is total torque on the pendulum, I is its moment of inertia, and $\alpha = \ddot{\theta}$ is the angular acceleration of the pendulum. (Dots mean time derivatives here. $\tau = I\alpha$ is just the rotational version of $F = ma$.)

(2) The torque on each sphere is (component of force in the direction of rotation) $\times$ (distance between sphere and axis of rotation). Draw vectors to indicate forces, add the torques to find total τ, and watch your $\pm$ signs!

(3) To find I for a complicated object, break it into pieces, find I for each piece, then add those I's together. The spheres can be treated as point masses at distances d and $2d$ from the axis of rotation.

1.2. **The pendulum is held at an angle $\theta_0 = 5^\circ \approx 0.87$ radians and then released. Solve the equation of motion for small oscillations.**

Hopefully, you found that I is some positive number and τ is some negative number (call it $-\kappa$) times $\sin\theta$. (If not, look up "physical pendulum" in your textbook and try again!) Since $\sin\theta \approx \theta$ for small values of θ, the equation of motion is:

$$-\kappa\theta = I\ddot{\theta}$$

This is a second-order differential equation, but you don't have to solve it from scratch. Instead, look up the general solution for "simple harmonic motion" and adjust any undetermined constants to fit this particular problem.

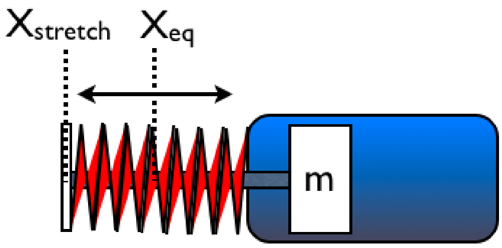
1.3. **How does the oscillation frequency change if friction is included?**

In the real world, ball bearings are not frictionless. Assuming that friction produces a torque proportional to $\dot{\theta}$, a more accurate equation of motion is:

$$\tau_{gravity} + \tau_{friction} = I\alpha \qquad \Rightarrow \qquad -\kappa\theta - \beta\dot{\theta} = I\ddot{\theta}$$

where β is some positive constant. (The minus sign ensures that friction always opposes motion - check it for yourself!) Find the frequency of oscillations if $\beta = 10\ N \cdot m \cdot s$ and compare it to the frequency from your answer to part 1.2.

2. Energy Dissipation in a Shock Absorber



A piston can move through a container of fluid, and a spring is wound around a shaft connecting an endplate to the piston. The equilibrium position for the endplate is x_{eq}. Something stretches the spring to the position $x_{stretch}$ and then releases it. The piston, shaft, and endplate together have mass m. When the piston moves, fluid viscosity produces a force $F_v = -b\dot{x}$. The spring force is $F_s = -kx$.

Neglecting ordinary friction (but not fluid viscosity!) and the mass of the spring itself, the equation of motion for x is $F_s + F_v = m\ddot{x}$, or equivalently, $-kx - b\dot{x} = m\ddot{x}$.

2.1. **At what time are the oscillations half as large as they were at $t = 0$?**

Look up the solution $x(t)$ for this equation of motion. Ignore the oscillating part of the solution and set the amplitude equal to half of what it was at $t = 0$.

2.2. **Show that this system conserves energy if $b = 0$. If b is positive, show that the system loses energy. Where does the energy go?**

At any time t, the energy $E(t)$ of the system is potential + kinetic $= \frac{1}{2}kx^2 + \frac{1}{2}m(\dot{x})^2$.

(1) For the case $b = 0$, calculate $E(t)$ explicitly and show that it is constant.
(2) For the case $b > 0$, show that $E(t) < E(0)$ for very large values of t. (To be precise, find the limit of $E(t)$ as $t \to \infty$.)

3. Bonus Problem

Young and Freedman p.441 says it is "straightforward but tedious" to verify that

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega' t + \phi) \qquad \omega' \equiv \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

is a solution to the damped-spring equation $m\ddot{x} = -kx - b\dot{x}$, where dots over the x's mean time derivatives. (They're right. I've done it, and it's really boring.)

Verifying this solution is less tedious using complex numbers. The infamous *De Moivre formula* $e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta)$ lets us rewrite the solution like this:

$$x(t) = Ae^{rt} + Be^{st}$$

where r and s are two distinct complex numbers. **Plug this solution into the damped-spring equation and use the quadratic equation to find r and s.** (Taking the derivative of an exponential is easy: $\frac{d}{dt}e^{rt} = re^{rt}$ for any complex r.)

Quadratic equations have two real roots, one "double" real root, or two complex conjugates. **Show that s and r are double roots if and only if the spring is critically damped: $b = 2\sqrt{km}$.** If the spring is underdamped ($0 < b < 2\sqrt{km}$), **show that the imaginary parts of s and r are $\pm\, \omega'$.**

The imaginary part of s tells us how rapidly x oscillates and the real part tells us how quickly the oscillations are dissipated. (Notice that a critically-damped or overdamped spring has a purely real s and doesn't oscillate at all.)