Dynamics of a Superconducting Qubit Coupled to an LC Resonator

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Abstract: We investigate the dynamics of a current-biased Josephson junction quantum bit or "qubit" coupled to an LC resonator. The Hamiltonian of this superconductor-based system is developed and the energies and eigen-functions of the entangled states are studied by Harmonic approximation. Since such a superconducting junction behaves as a two-level artificial atom coupled to a harmonic oscillator, this system can be treated as a solid-state analog of an atom in a cavity – which is the fundamental system in the well-developed field of Cavity Quantum Electrodynamics.

1 .Josephson Junction: A superconducting qubit

The structure of a Josephson Junction is shown in Fig.1. As we already know, the relation between the tunneling current flowing through the junction and the voltage across the junction is given by Josephson Relations[1]:

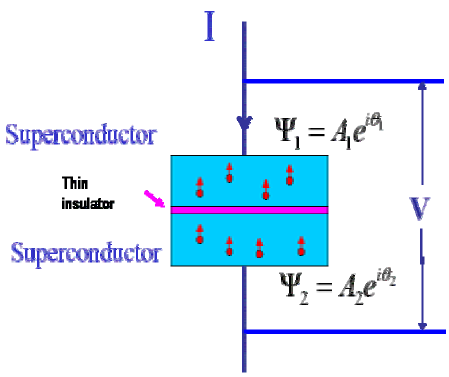


Fig.1 Schematic of a Josephson Junction

$$I = I_0 \sin\gamma$$

$$V = \frac{\Phi_0}{2\pi}\frac{d\gamma}{dt}$$

where, γ is the phase difference between the two superconductors; I_0 is the critical current of the junction and $\Phi_0 = \frac{h}{2e} = 2.07\times10^{-15} T\cdot m^2$ is the flux quantum.

2. A Josephson Junction coupled to a LC resonator

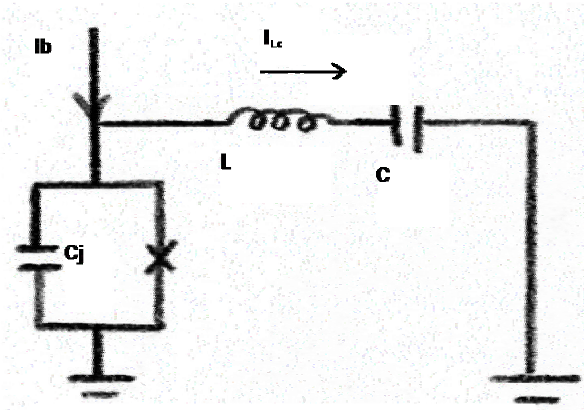


Fig.2 A Josephson Junction coupled to a LC resonator

The system to be considered is shown in Fig.2, a Josephson Junction coupling to a LC resonator in series. The relation between current passing through the resonator and current through the junction can be obtained by Kirchhoff's Law:

$$\begin{cases} I_b = I_j + C_j \dot{V}_j + C\dot{V}_c \\ V_c = V_j - L\dot{I}_{LC} \end{cases}$$

Plug Josephson Relations into these equations and make a substitution

$$\gamma_2 = \frac{2\pi L}{\Phi_0} \cdot I_{LC}$$

We have

$$\begin{cases} (C_j + C)(\frac{\Phi_0}{2\pi})\ddot{\gamma}_1 - C \cdot \frac{\Phi_0}{2\pi}\ddot{\gamma}_2 + I_c \sin(\gamma_1) - I_b = 0 \\ C \cdot \frac{\Phi_0}{2\pi}(\ddot{\gamma}_1 - \ddot{\gamma}_1) - \frac{\Phi_0}{2\pi L}\gamma_2 = 0 \end{cases}$$

The Lagrangian of the system is easily obtained from these two equations and thus the Hamiltonian:

$$H = \overbrace{\frac{P_1^2}{2m_1} + U(\gamma)}^{H_{qubit}} + \overbrace{\frac{P_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2\gamma_2^2}^{H_{LC}} - \overbrace{\xi\frac{P_1P_2}{\sqrt{m_1m_2}}}^{H_{coupling}}$$

with conjugate momenta:

$$P_1 = C_J(\frac{\Phi_0}{2\pi})^2 \dot{\gamma}_1 + C(\frac{\Phi_0}{2\pi})^2(\dot{\gamma}_1 - \dot{\gamma}_2) \qquad P_2 = C(\frac{\Phi_0}{2\pi})^2(\dot{\gamma}_1 - \dot{\gamma}_2)$$

and

$$m_2 = (\frac{\Phi_0}{2\pi})^2 C_J \qquad m_2 = (\frac{\Phi_0}{2\pi})^2 (\frac{C_J C}{C_J + C}) = m_1 \xi^2$$

$$\omega_2 = \sqrt{\frac{C_J + C}{L \cdot C \cdot C_J}} \qquad \xi = \sqrt{\frac{C}{C + C_J}}$$

$$U(\gamma_1) = -\frac{\Phi_0}{2\pi} I_c(\cos(\gamma_1) + J\gamma_1) \qquad J = \frac{I_b}{I_c}$$

3. Decoupled momentum and coupled potential

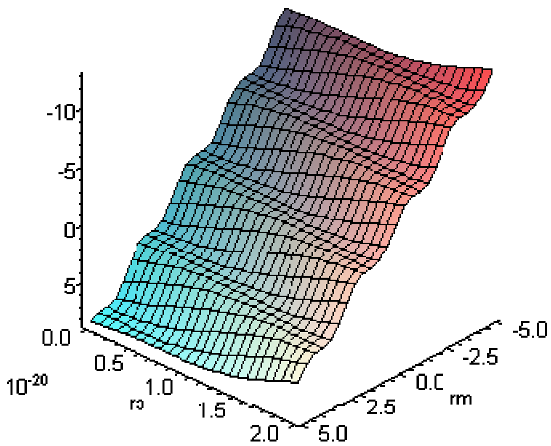
Let consider a transformation defined by:

$$P_\pm = \sqrt{\frac{1}{2} \mp \frac{\xi}{2}} \cdot \left(P_1 \pm \frac{P_2}{\xi}\right) \qquad \gamma_\pm = \left(\gamma_1 \pm \frac{\gamma_2}{\xi}\right) \Big/ \sqrt{\frac{1}{2} \mp \frac{\xi}{2}}$$

Under this transformation, the Hamiltonian becomes

$$H = \frac{1}{2m_1}\left(P_+^2 + P_-^2\right) + U'(\gamma_+, \gamma_-)$$

This potential is shown below at $J = J_0$, where the lowest two states of the Junction have an energy space that equals the energy space of the LC resonator.



4. Eigen energy and states of the system

We can start directly from the Hamiltonian

$$H = \overbrace{\frac{P_1^2}{2m_1} + U(\gamma)}^{H_{qubit}} + \overbrace{\frac{P_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2\gamma_2^2}^{H_{LC}} - \overbrace{\xi\frac{P_1P_2}{\sqrt{m_1m_2}}}^{H_{coupling}}$$

to calculate the eigen-value and states of the system using numerical method. This involves a large matrix that can not be solved by general algorithms.

Here I will use harmonic oscillator approximation to solve this problem.

Near the lowest point of the washboard potential, we can expand the potential and drop all the constants to get a new Hamiltonian:

$$H = \frac{1}{2}\omega_1^2 P_\alpha^2 + \frac{1}{2}x_1^2 + \frac{1}{2}\omega_2^2 P_\beta^2 + \frac{1}{2}x_2^2 - \xi\omega_1\omega_2 P_\alpha P_\beta$$

Where,

$$x_1 = m_1^{\frac{1}{2}} \cdot \omega_1\left(\gamma_1 - \arcsin\left(J\right)\right) \qquad x_2 = m_2^{\frac{1}{2}} \cdot \omega_2 \cdot \gamma_2$$

$$P_\alpha = m_1^{-\frac{1}{2}} \cdot \omega_1^{-1} P \qquad P_\beta = m_2^{-\frac{1}{2}} \cdot \omega_2^{-1} \cdot P_2$$

$$\omega_1 = \left(\frac{2\pi I_c}{\Phi_0 C_J}\right)^{\frac{1}{2}}\left(1 - J^2\right)^{\frac{1}{4}}$$

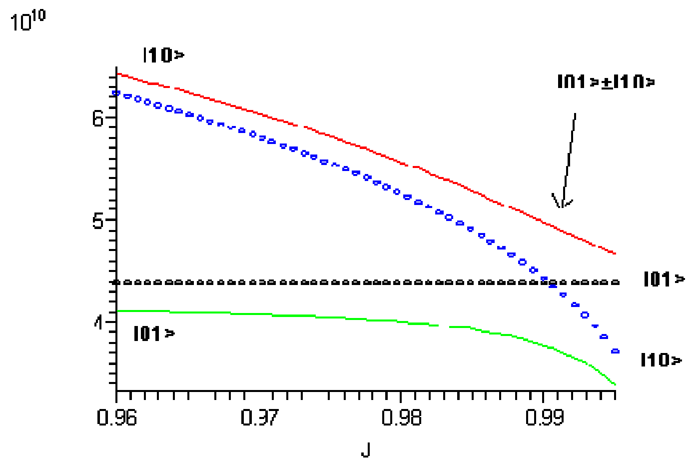
This Hamiltonian can be written in matrix form:

$$H = \frac{1}{2}P^T\begin{pmatrix} \omega_1^2 & -\xi\omega_1\omega_2 \\ -\xi\omega_1\omega_2 & \omega_2^2 \end{pmatrix}P + \frac{1}{2}x^T x$$

Diagonalizing gives us the energy:

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\xi^2\omega_1^2\omega_2^2}\right)$$

When we tune the bias current of the Josephson Junction (change the frequency of the qubit), we can observe the entangled states between the LC resonator and the qubit, denoted by an avoided crossing in the plot of Energy VS bias current.



- Energy of the system as a function of J, the normalized bias current in the Junction. The state notation is /Junction, LC>. Circles denote the uncoupled /0> to /1> level spaces for Junction (blue) and LC resonator (black).
- At the resonance point where $J \approx 0.9902$, an avoided crossing occurs with a split of $2\xi\omega_2$

In experiment, the system is first cooled down to ground state /00>.
For low and high bias, the energy level transitions are from the ground state /00> to excited states /01> or /10>, depending on the frequency of microwave we apply to the system. At the resonance point, the first two excited states become:

$$\frac{|01> \pm |10>}{\sqrt{2}}$$

At the resonant point, the energy of the system is: $\omega_{\pm}^2 = (1 \pm \xi)\omega_2^2$ and the normal modes are: $P_{\pm} = \frac{1}{\sqrt{2}}\left(P_\alpha \pm P_\beta\right)$

We can construct the first few states of the system by using the creation operators:

$$a_{\pm}^{\dagger} = \frac{1}{\sqrt{2}}\left(a_\alpha^{\dagger} \pm a_\beta^{\dagger}\right)$$

and the definition:

$$\left|\Psi_{j,k}\right\rangle = (j!k!)^{-\frac{1}{2}} a_{+}^{\dagger j} a_{-}^{\dagger k} \left|00\right\rangle$$

The first few states are:

$$\left|\Psi_{0,0}\right\rangle = \left|00\right\rangle$$

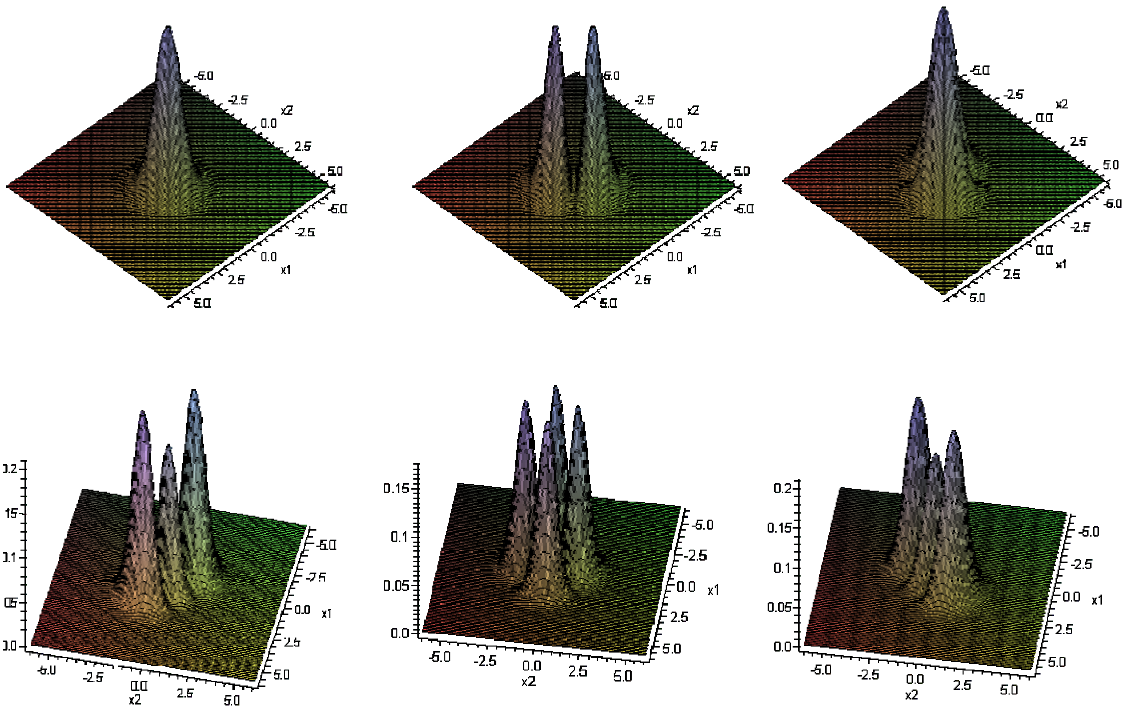
$$\left|\Psi_{0,1}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|10\right\rangle - \left|01\right\rangle\right)$$

$$\left|\Psi_{1,0}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|10\right\rangle + \left|01\right\rangle\right)$$

$$\left|\Psi_{0,2}\right\rangle = \frac{1}{2}\left(\left|02\right\rangle + \left|20\right\rangle\right) - \frac{1}{\sqrt{2}}\left|11\right\rangle$$

$$\left|\Psi_{1,1}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|20\right\rangle - \left|02\right\rangle\right)$$

$$\left|\Psi_{2,0}\right\rangle = \frac{1}{2}\left(\left|02\right\rangle + \left|20\right\rangle\right) - \frac{1}{\sqrt{2}}\left|11\right\rangle$$



These are the eigen-functions of the system under Harmonic Oscillator Approximation with transformed coordinates.

5. Conclusion and future work

We derived the Hamiltonian for the system of a Josephson phase qubit coupled to a LC resonator. Harmonic approximation is utilized to calculate the energy of the system and the entangled states are verified by an avoided crossing in the energy spectrum. First six states of the system are obtained.

Although harmonic approximation is sufficient for us to understand the energy and entanglement of the system, it is too rough for describing the energy and state explicitly. Plus, it becomes less valid when the well of the washboard potential goes shallow, which is in fact the right case for a qubit. Therefore a more powerful technique is required in future work to solve for the energies and states of the system.

Considering this system as a solid-state analog of an atom in a cavity[2], we shall explore the possibility of observing in such a system, quantum-mechanical effects such as the AC Stark shift, which has been observed in an atom-cavity system[3] and more recently, in a Cooper pair box-resonator system[4]. This might be useful in developing non-demolition types of measurements for determining the state of a qubit.

References:

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