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Phys 517

## *Basic Principles of Quantum Field Theory*

In our description of nature, isolating our view to individual particles does give us enough descriptive power to describe our reality. An accurate description requires the introduction of a quantum field concept. In order to better understand the idea of fields in quantum mechanics, I decided to research some of the basic formulations of quantum field theory (QFT). The issues and formalism I'm going to talk about in this paper are all referenced from a very user friendly text in QFT by Robert Klauber. Throughout this paper, I am going to mention quantum mechanics (QM) versus QFT in several instances and I should clarify now what I mean by the two terms QM and QFT. When I refer to QM, I will be talking about quantum mechanics prior to the introduction of quantum field theory into the modern physics mix; QFT is the quantum mechanical framework with the new field concept. This paper will focus first on the necessity of the field introduction to quantum mechanics, then discuss first and second quantization of classical entities, and finally the paper will end after a more detailed look at the Klein-Gordan field equation.

To understand why QFT is necessary, we first have to look at its strengths and shortcomings. Quantum mechanics has been immensely successful in describing how nature treats matter. Our understanding of reality was thrown a massive curveball when it was discovered that particles were no longer well defined specimens or even particular. Despite the initial shock, this interpretation of our universe led to a very rich mathematical framework in which we could accurately detail the evolution of various particle states in specified potentials. Yet, as mentioned

above, QM has its shortcomings. One of these failings is that the theory could not describe an actual transmutation of particles such as what happens in matter-antimatter annihilation. A particle state bound in the potential of another particle state was certainly doable within the framework of the theory, the hydrogen atom wave equation being one of the biggest successes of QM, but the actual creation of a photon from an electron-positron interaction and then the recreation of an electron-positron pair from said photon was problematic for the early quantum theory. It is precisely that kind of interaction problem which QFT aims to solve, and the theory does so by looking to quantize classical fields much as the young quantum mechanics quantized classical particles.

The method by which QFT goes about quantizing the fields from classical theories is a procedure called second quantization. The natural question to ask here is, "If this is second quantization, then what was first quantization?" First quantization is what took place when QM quantized classical *particles*. From a mathematical point of view, the whole first versus second quantization terminology is an exercise in semantics; the procedure for both is largely the same. The difference between the two quantizations lies in the physical meaning of the terms that we are dealing with rather than in the mathematical structure. First quantization deals with physical parameters that I like to call *isolated properties*. I call them isolated properties because these are values that we can directly measure when we look at a single, "isolated" quantum state. So this would include such things as momentum, mass, spin, position, etc. Now to follow through conceptually to second quantization: particles are, in a sense, isolated entities and so could be quantized by looking at isolated properties. Fields, on the other hand, are distributed throughout a region and so it would seem to be a good idea to look at *distributed properties* if we want to quantize a classical field. Distributed properties is the term that I use to describe physical

entities which arise only when you consider several particles or states at once. The best examples of these kinds of properties are densities. This definitely makes sense considering that we know that fields can be described with the densities from a few well know classical arguments, such as Gauss's Law:

$$\bullet E = \rho \epsilon 0 .$$

By using distributed properties instead of isolated properties, we can create a whole set of field equation versions of very familiar quantities, such as Lagrangians and Hamiltonians. From this point, using the Poisson Bracket  commutator relationship that we have talked about in class, the quantum interpretation of a field manifests quite naturally. Take, as an example, the field equivalent of the position and momentum relationship. Let $\phi(x)$ represent the field strength at a particular position x and let $\pi(x)$ represent the field's canonical momentum density at the point x. These two values are canonical values, where $\phi$ plays the role of the generalized coordinate and $\pi$ of the generalized momentum. With these two variables so defined, I will now proceed to demonstrate how the Poisson brackets and commutators behave and from there it will be fairly clear that the typical quantum relationship between the Poisson brackets and commutators is upheld.

Derivation 1: Possion Brackets and Commutators

$$\phi, \pi = d\phi x d\phi x 0 d\pi x d\pi x 0 - d\phi x d\pi x 0 d\pi x d\phi x 0 = \delta x - x 0 \quad \text{(Eqn. 1)}$$

For the quantum commutator, the momentum will assume an operator form, just as in the particle arguments:

$$\phi, \pi = \phi\pi - \pi\phi = \phi\pi - \pi\phi f\phi$$

$f(\phi)$ is a test function that I introduce so that I can go through the operator arithmetic. Once all of the work is done, I will drop the test function term.

Noting that $\pi = \hbar i \partial\partial\phi$, I can say that

$$\phi, \pi = \phi\hbar i\partial\partial\phi(x0)f(\phi) - \hbar i\partial\partial\phi(x0)\phi f(\phi)$$

$$= \phi\hbar i\partial f\partial\phi - \hbar i f\phi\delta x - x0 - \phi\hbar i\partial f\partial\phi = -\hbar i f\phi\delta x - x0 = i\hbar f\phi\delta x - x0$$

Now that all of the arithmetic is done, I drop any test functions still remaining and find that

$$\phi, \pi = i\hbar\delta x - x0 \quad \text{(Eqn. 2)}$$

On comparing equations (1) and (2) we can see that they differ from each other by a coefficient of $\hbar$, and so the commutator and Poisson brackets are related in the usual way. In fact, these expressions are not all that different from the particle notation, with the exception that there is a delta function present in the solution. This delta function highlights the fact that we are considering a field that has a functional behavior defined by position. In order to use the field as a canonical variable, we have to isolate our considerations to only one point in space: i.e. one element of the field. In summary, quantizing a field follows very much the same formula as many other quantizations familiar from particle quantum mechanics. All of the background is

now out of the way, so I will turn to the heart of this paper: the behavior of free fields as modeled by the Klein-Gordan equation.

What is a free field? A free field is a field that has no potential acting on it, much as a free particle is a particle that isn't in any potential. Once the general equation of a free force is worked out, we will have a blank canvas with which to add potential effects and gradually be able to explain more interesting physical phenomena. Despite the usefulness of a free field equation in deriving more physically relevant situations, it is a constructive exercise to pause and consider what a physical example of a free field would be. We can get to a very interesting place if we consider that particles like electrons and quarks are excitations of "electron fields" and "quark fields." For brevity, I'm going to coin the term *material fields* to refer to these types fields collectively. In this worldview, one could view the excitation of these material fields as leading to the creation of particles sharing the field's namesake. Should a potential be acting on the material field, then any particles created from the material field will be governed by the effective potential as well. I should mention that this point of view is entirely my own. An example of a free field is remarkably hard to find so I put together my own ideas for an interpretation.

There is a lot to say about the foundational form of the free field equations. One of the biggest differences between ordinary QM and QFT is that there isn't one equation that satisfies all cases. Rather, there are three equations that fill the role of Schrödinger's equation in QFT: the Klein-Gordan equation (for spin-0 fields), the Dirac equation (for spin-1/2 fields) and the Proça equation (for spin-1 fields). Covering all of these equations in one small term paper would be unrealistic, so I will focus on examining the Klein-Gordan equation as I stated at the beginning

of this paper. One note about these equations is that all of them are relativistic, which makes QFT an inherently relativistic theory.

The derivation of the of the Klein-Gordan equation starts by using the relativistic energy equation to derive the Hamiltonian:

$$E2=p2c2+ m2c4$$

Letting E→H and quantizing p p→ -iℏ∂∂xi2=-iℏ∂i2, I get the Hamiltonian:

$$H=-iℏc2∂i2+m2c4 \quad \text{(Eqn. 3)}$$

The ∂i2 term is written in Einstein notation so it is a sum of derivatives in all available spatial dimensions. There is, however, an issue with the Hamiltonian expression in that there is an operator under the square root. This issue is circumvented by squaring the operator portions of the time-dependent Schrodinger Equation:

$$iℏddt2ϕ=H2ϕ$$

This goes to:

$$iℏddt2ϕ=-iℏc2∂i2+m2c4ϕ$$

Once we get to this step, all that remains to do is rearrange the equation. After some simple arithmetic, the expression for the Klein-Gordan equation is found:

$$∂μ∂μ+μ2ϕ=0 \quad \text{(Eqn. 4)}$$

where μ2=m2c2ℏ2=m2 in natural units

and ∂μ∂μ=∂2c2∂t2-

The

2 comes from the ∂i2 term in the Hamiltonian.

There is a very crucial distinction to be made here regarding the meaning of ϕ in the QFT version of the Klein-Gordan equation as opposed to the QM version. In QFT, ϕ is *not* a state; rather ϕ is an operator and solving the Klein-Gordan equation in this format will yield quantum operator fields as its solutions rather than quantum states. This also means that QFT assumes the Heisenberg interpretation of quantum mechanics, wherein the quantum wave state is stationary and the quantum operator that acts on the state is what evolves in time. The nature of the solutions isn't all that changes in QFT: actually calculating those solutions proceeds in a remarkably different fashion from the standard nonrelativistic quantum mechanics: the probability density is no longer a multiplication complex conjugates. The proper way to evaluate the probability density for the Klein-Gordan equation requires the use of the continuum equation:

$$\partial\rho\partial t+$$

$$\bullet j=0$$

In the continuum equation, j is known as the current density and ρ is the volumetric density. The idea behind deriving the probability density for the Klein-Gordan equation is to manipulate the time-dependent equation and get into the form listed above. The terms that take the place of ρ in the continuum equation comprise the probability density we need to solve problems. The probability density of the Klein-Gordan equation ends up being:

$$i(\partial\phi\partial t\phi\dagger-\partial\phi\dagger\partial t\phi)\, d3x=1$$

In order to discuss results that are more unique to QFT and not simply relativistic QM, I will forgo a proof of this normalization condition. Suffice to say that this difference in computation is not something one can simply overlook. In fact, the normalization condition being present in this form ensures that the probability of a wave function is a relativistic invariant.

A plane wave solution of the Klein-Gordan equation is:

$$\phi x= k12V\omega k(Ake-ikx+Bk\dagger eikx) \quad \text{(Eqn. 5)}$$

Again, as long as the $\phi$ terms here are taken to be operators and not states, then this is the Klein-Gordan solution that will be used in QFT. The form of Eqn. 5 is specifically for discrete wave solutions. While there do exist integral versions of this solution to the KG Equation that denote continuous wave solutions, I will not deal with them in this paper. The operator solutions to the KG equation are obtained when Hamiltonian is the energy density of a system. Fortunately, due to the similarity in the mathematics of both the isolated and distributed Hamiltonian cases, the forms of the KG solutions are identical; only the meanings of the terms change.

The only thing left to do is to use the plane wave solution of the Klein-Gordan field equation to find the form of the quantum field Hamiltonian. The Hamiltonian is obtained by solving the following equation. For brevity, I will simply state the initial equation and then give the result for the Hamiltonian. The math really just involves dealing with a lot of differentials but it can get pretty messy looking:

$$H00= \partial\phi\partial t\partial\phi\dagger\partial t+$$

$$\phi\dagger\bullet$$

$$\phi+\mu 2\phi\phi\dagger d3x$$

$$=k\omega kNak+12+Nbk+12 \quad \text{(Eqn. 6)}$$

$$\text{where } Na=a\dagger kak \text{ and } Nb=b\dagger kb(k) \quad \text{(Eqn. 7)}$$

Eqn. 7 details $N_a$ and $N_b$ . The N's are number operators that explain how many "a" particles and how many "b" particles exist with a certain 3-momentum k. With this definition in mind, the meaning of the Hamiltonian in Eqn. 6 takes on a rather intuitive meaning. The energy density of the field is essentially expressed using the energy of the various particles present within the field. Yet, there remains one other interesting quality about Eqn. 6, the ½ constants. Should no particles be present in the field, the Hamiltonian density will still have a nonzero value because of these constants. The ½ terms thus get the name *zero-point energy*. This the same energy that lead many to point to a quantum foam of particles popping into and out of existence at small distance scales.

The equations described above are only free fields and, even more pointedly, for spin-0 particles. QFT is obviously far more expansive then what I was able to capture in this paper. The largest implication from the ideas discussed so far is that, despite how foreign and threatening the concept of fields may initially seem to be, the equations are very relatable to "ordinary" QM by focusing on the similarity of the foundational mathematics and by making use of the Poisson bracket commutator relationship that shows up ubiquitously in many quantum calculations. Though the results of QFT equations are operators, the use of these operators adds a richness and predictability to physics that would otherwise be lacking.

**Reference**

Klauber. R. “Quantum Field Theory: Basic Principles and Quantum Electrodynamics” First 3 chapters: http://www.quantumfieldtheory.info/#Chap01