Quantum Picture of the Josephson Junction

------A promising candidate for qubits in quantum computing

In this project, I introduce the idea of Josephson Junction, which has drawn great attention of researchers with its potential to server as a quantum bit in a quantum computer. I use the approaches we discussed in class to study the quantum picture of a current-biased Josephson Junction. The lowest two states of such a Junction are singled out as two states of a quantum bit. I calculated the tunneling rate for each state and explain how we determine which state a qubit is in, with the information about the tunneling rates of each state. Because of the existence of thermal noise, ultra-low temperature is required (usually around 10 mk) if we want to observe these quantum properties experimentally. There are many other issues that should be considered when we study such a system, such as electromagnetic isolation and decoherence and dissipation of the Junction. All these are not taken into account when I sketch the quantum picture of the Junction here.

Quantum Picture of the Josephson Junction

------A promising candidate for quantum bits in quantum computing

1. Physical Properties of a Josephson Junction

Electrons in a superconductor form **Cooper pairs*** for temperatures below the critical temperature of the superconductor[1]. These pairs are in a collective motion corresponding to the ground state of the system. The state can be described with a wave function:

$$\Psi(\vec{r},t) = A(\vec{r},t)e^{i\theta(\vec{r},t)} \tag{1.1}$$

The phase of the wave function is coherent throughout the superconductor.

If two superconductors are separated by a very thin layer of insulator, a Josephson Junction is formed (Fig.1). When the insulator thickness is small enough, the electronic wave functions from the two sides can overlap. The cooper pairs in these two superconductors can tunnel to the other and the phase of the wave functions in two superconductors are correlated.

If the phase difference is $\gamma = \theta_1 - \theta_2$, the relation between the tunneling current I flowing through the junction and the voltage V across the junction is given by **Josephson Relations**[2]:

$$I = I_0 \sin\gamma \tag{1.2}$$

$$V = \frac{\Phi_0}{2\pi}\frac{d\gamma}{dt} \tag{1.3}$$

Where $\Phi_0 = \frac{h}{2e} = 2.07\times10^{-15} T\cdot m^2$ is the **flux quantum****, and I_0 is the critical current of the junction.

---

* Animations of the formation of Copper Pairs and how they contribute to superconductivity can be found at:
http://www.chemsoc.org/exemplarchem/entries/igrant/bcstheory_noflash.html

** Flux quantum: The magnetic flux in a superconducting loop is quantized. The allowed flux is: $n\cdot\Phi_0$,where $n = 0,1,2...$

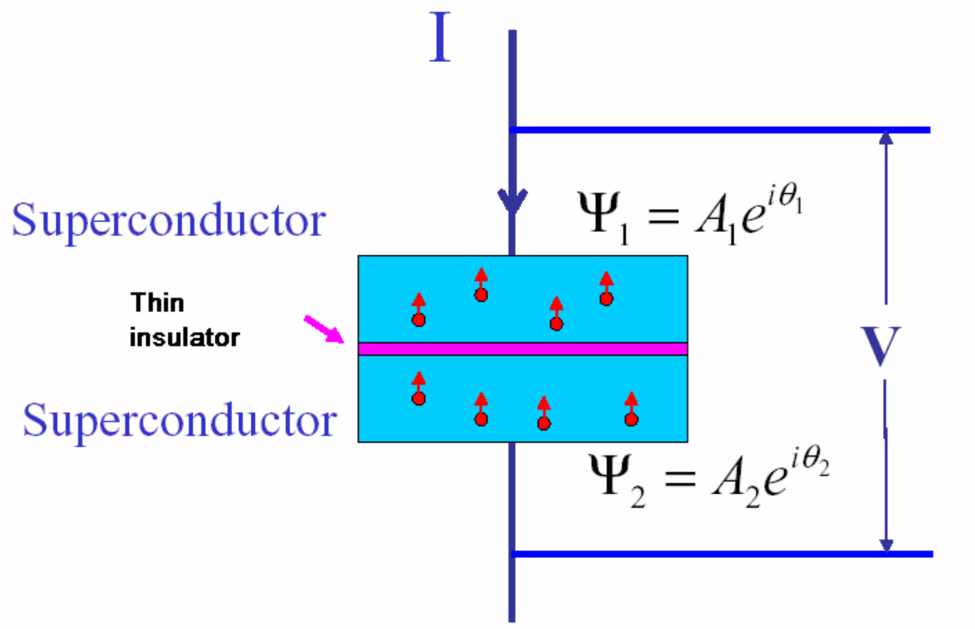


Fig.1 Schemetic of a Josephson Junction

In an actual Josephson Junction, there is a capacitance between the two superconducting plates, and a shunt resistance parallel to the Junction. In this model, which is called RCSJ Model, a Josephson Junction can be indicated as Fig.2. The cross is the bare Junction; I_b is the total current flowing through, which is called **Bias Current**.

Thus, the equation for the total current is:

$$I_b = I_j + C\dot{V} + \frac{V}{R} = I_0 \sin(\gamma) + C\frac{\Phi_0}{2\pi}\ddot{\gamma} + \frac{\Phi_0}{2\pi R}\dot{\gamma} \tag{1.4}$$

in which the Josephson Relations are plugged.

We study Josephson Junction as a **Phase Qubit**, which means the states of the qubit (0 and 1 in classical computer) are indicated by different states of the phase (γ)of the Junction. I am going to make an analogy between the problem of a Josephson Junction and a dynamic problem using equation 1.4.

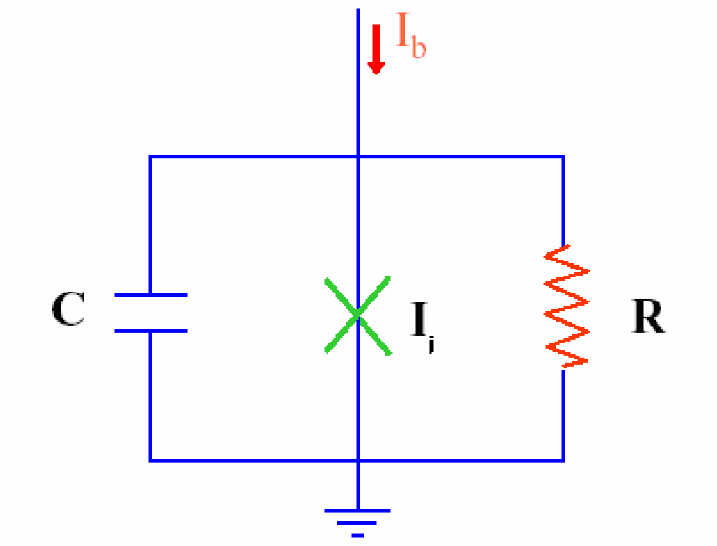


Fig.2 RCSJ Model

2. Dynamic analogy for a Josephson Junction

Consider a particle with mass $m = C\left(\frac{\Phi_0}{2\pi}\right)^2$ moving in a potential:

$$U(\gamma) = -\frac{\Phi_0}{2\pi}\left(I_0 \cos(\gamma) + I_b\gamma\right) \quad (2.1)$$

subjected to a damping force $\left(\frac{\Phi_0}{2\pi\sqrt{R}}\right)^2 \dot{\gamma}$. The equation of motion is:

$$m\ddot{\gamma} = -\frac{dU(\gamma)}{d\gamma} - \left(\frac{\Phi_0}{2\pi\sqrt{R}}\right)^2 \dot{\gamma} \quad (2.2)$$

which is exactly equation 1.4. So, the problem of phase state of a Josephson Junction is analogized to problem of motion of a particle with displacement γ in potential $U(\gamma)$, subjected to a damping force.

We study this potential quantum mechanically. Note that the potential depends on the bias current I_b, The sketch of the potential for $I_b = 0.1I_0$, $I_b = 0.3I_0$ and $I_b = 0.9I_0$ is shown in Fig.3.

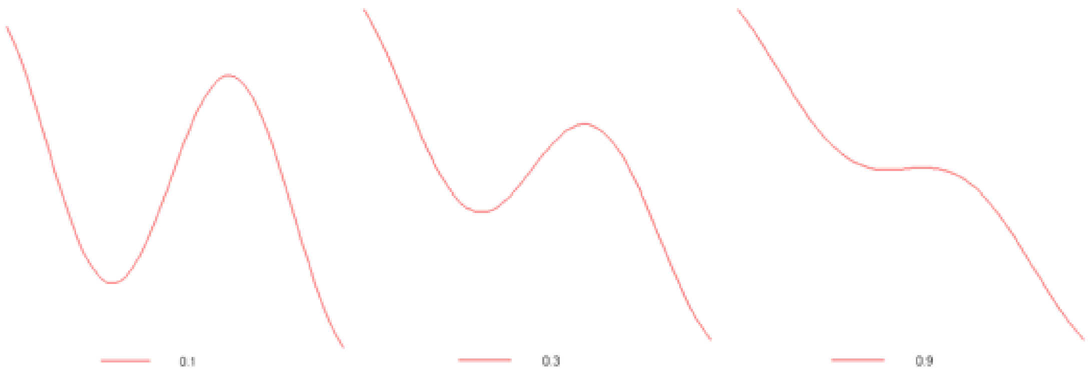


Fig.3

Through adjusting the bias current I_b, we can control the deepness of the well, thus the number of bound states.

3. Eigen states of the tilted wash board potential

When a Josephson Junction is used as a qubit, one need to find the bias current I_b such that there are only several eigen states existing in the well. As we discussed in class, we can discretize the potential and diagonalize the Hamiltonian matrix to get the eigen values and eigen functions. The Hamiltonian of this system is simply:

$$H = -\frac{\hbar}{2m}\frac{d^2}{d\gamma^2} + U(\gamma) \tag{3.1}$$

where $U(\gamma)$ depends on the bias current I_b.

With the typical parameter: $I_0 = 10\mu A; C = 1pF; L = 0.15nH$, my calculation showed that when $I_b = 0.97 I_0$, there are only 4 bound states in the well, which is a good condition where we can regard the Josephson Junction as a qubit. The eigen energies of the bounded states are (I chose $U(r = 0)$ as the zero potential):

$$E_0 = -5.0314563659704 \times 10^{-21};$$

$$E_1 = -5.02280861941517 \times 10^{-21};$$

$$E_2 = -5.01468472553116 \times 10^{-21};$$

$$E_3 = \text{-}5.00738407077634 \times 10^{\text{-}21}$$

Let us label the energy space in frequency between i-th state and j-th state as: ω_{ij}, the energy spaces are:

$$\omega_{10} = 82GHz\ ;$$

$$\omega_{21} = 77GHz\ ;$$

$$\omega_{32} = 69GHz$$

Unlike the Harmonic Oscillator Potential, the energy spaces of the wash board potential are not equal. This is a significant quality of this system that makes it a candidate for qubit, as discussed later.

The potential in my calculation and the eigenfunctions for the 4 lowest states are shown in Fig. 4. The eigenfunctions look similar with eigenfunctions of Harmonic Oscillator Potential. However, one can see that on the right edge of the functions for state 2 and state 3 the functions are no longer zero. In fact, since right barrier of the well is not infinitely high (actually very low in this case), there must be a transmission rate (or quantum tunneling rate) for each state. From the sketch of the functions, we can roughly tell that the tunneling rate of state 2 and 3 are much larger than that of state 0 and 1. Actually, this difference in tunneling rate is another basis for our effort to design a qubit with a Josephson Junction. In next section, I am going to calculate the tunneling rate of each state and explain how to measure the state of such a qubit through the quantum tunneling of it.

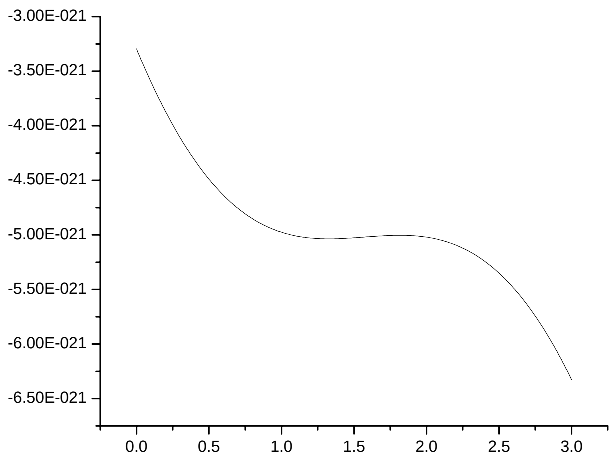


Fig.4a Potential (Ib=0.97*Io)

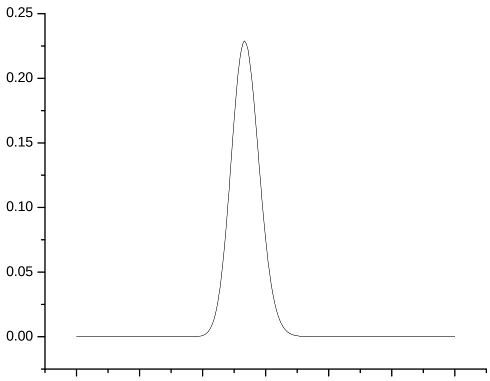


Fig.4b Eigen function of State 0

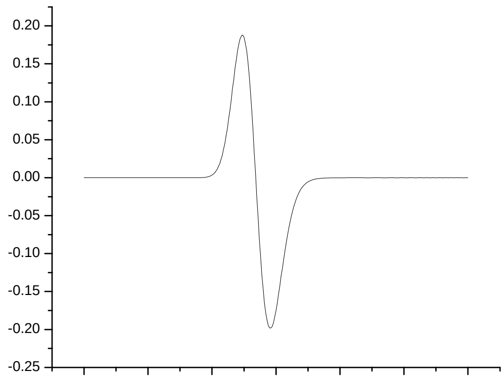


Fig.4c Eigenfunction of State 1

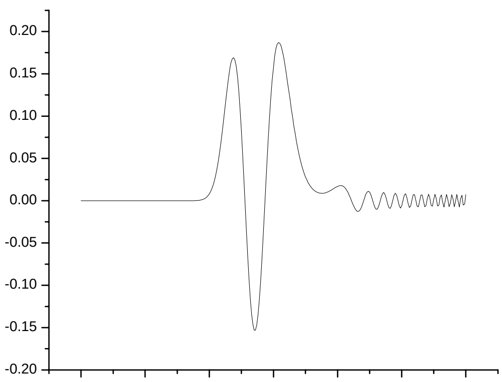


Fig.4d Eigen function of state 2

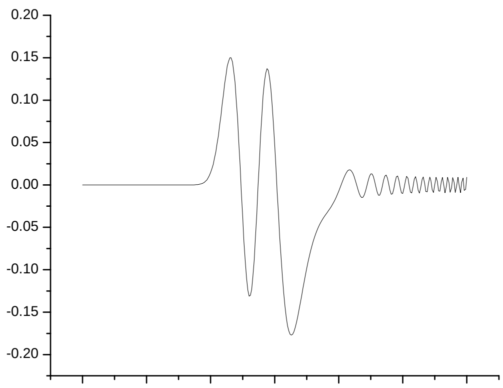


Fig.4d Eigen function of state 3

4. Tunneling rate of the states

As we learned in class, for an arbitrary potential $V(x) > E$, the tunneling rate is:

$$T \sim e^{-2\int_a^b K(x)dx} \tag{4.1}$$

where $K(x) = \sqrt{\frac{2m}{\hbar^2}\left(V(x)-E\right)}$; $V(a) = V(b) = E$, $a < b$. This relation can be derived from WKB approximation. I am going to calculate the tunneling rate with the approach of transfer matrix. As reference, the ratio of tunneling rates for higher stats to ground state with equation 4.1 are:

$$T_1 \sim 10^3 T_0$$

$$T_2 \sim 10^6 T_0$$

$$T_3 \sim 10^8 T_0$$

We can see that the tunneling rate for higher states are much higher than those of lower states.

I discretize the potential curve to a sequence of constant potential V_i with step length δ, as Fig.5 shows. Note that in the total region II, we have $E < V(\gamma)$. And I set the lowest point of the well to be zero energy. Then I calculated the M matrix as we defined in class[3]:

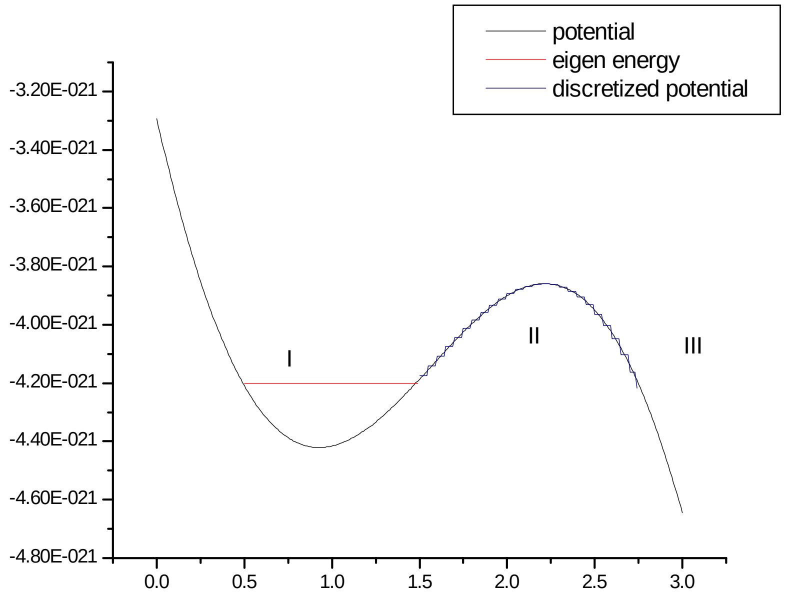
$$M_i = \begin{pmatrix} \cosh(\delta K_i) & -\frac{1}{K_i}\sinh(\delta K_i) \\ -K_i \sinh(\delta k_i) & \cosh(\delta K_i) \end{pmatrix} \tag{4.2}$$

where K_i represents the momentum in i-th rectangle potential: $K_i = \sqrt{2m(V_i - E)/\hbar^2}$. Since my goal is to compare the tunneling rates for each state, instead of calculating the exact tunneling rates, let us assume that the momentum in region I is the same with that in region III: $K_0 = \sqrt{2mE/\hbar^2}$. The relation between the amplitude in region I and region III then becomes:

$$\begin{pmatrix} A_I \\ B_I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{iK_0} \\ 1 & -\frac{1}{iK_0} \end{pmatrix} \cdot M_1 \cdot M_2 \cdots M_i \cdot \begin{pmatrix} 1 & 1 \\ iK_0 & -iK_0 \end{pmatrix} \cdot \begin{pmatrix} A_{III} \\ B_{III} \end{pmatrix} \tag{4.3}$$

where $B_{III} = 0$ and the phase matrices are ignored. The tunneling rate is $\left(\frac{A_{III}}{A_I}\right)^2 = \left(\frac{1}{t_{11}}\right)^2$, where t_{11} is the first element of the transfer matrix:

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{iK_0} \\ 1 & -\frac{1}{iK_0} \end{pmatrix} \cdot M_1 \cdot M_2 \cdots M_i \cdot \begin{pmatrix} 1 & 1 \\ iK_0 & -iK_0 \end{pmatrix} \tag{4.4}$$



Discretization of the potential

With the typical parameters listed above and $\delta = 0.05$, I calculated the tunneling rates for each state.

$$T_0 = 1.0 \times 10^{-10}$$

$$T_1 = 2.4 \times 10^{-8}$$

$$T_2 = 2.0 \times 10^{-5}$$

$$T_3 = 4.9 \times 10^{-3}$$

The results match the results from WKB approximation with $T_2 / T_1 \sim 10^3$ and $T_3 / T_2 \sim 10^2$, but $T_1 / T_0 \sim 10^2$ in this calculation is slightly different from $T_1 / T_0 \sim 10^3$ as in WKB approximation. Actually, there is a factor of ~ 500 if we calculate with the exact expressions in WKB approximation for the tunneling rate of this two state. Anyway, we can get the idea that the tunneling rates of higher states are much lager than those of lower states.

The measurement of the state of such a qubit is based on the ideas that: (1) the energy spaces of different pairs of states are not equal; (2) the tunneling rate of the higher states are much larger than that of the lower states.

Tunneling means the system goes to the continuous running state, ie γ keeps changing.

According to the Josephson Relation, the system goes to a finite voltage stage from zero voltage stage. Therefore, people measure the voltage across the Junction to evaluate the tunneling. Under low temperature, where the qubit is in either state 0 or state 1, one can apply a microwave with frequency ω_{10} to the system to determine the state of a qubit. Because the energy space between each two states are not equal, this microwave can only be absorbed when the system is in state 0. If this happens, an enhancement in tunneling rate will be observed. That tell people the qubit is in state 0. Another similar way to do this is applying a microwave with frequency ω_{31} to the system. If the qubit is in state 1, the microwave will be absorbed, thus a significant tunneling will be observed.

References:

1. M. Tinkham, Introduction to Superconductivity, 2nd ed. (McGraw-Hill, New York, 1996)
2. B.D. Josephson, Phys. Lett. **1**, 251 (1962)
3. R. Gilmore, Elementary Quantum Mechanics in One Dimension, Baltimore, MD: Johns Hopkins University Press, 2004