The physics of Superconducting Quantum Interference Devices (SQUIDs)

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Josephson junctions are interesting devices that provide a great deal of physics to research, both experimental and theoretical. The device was first proposed by Brian Josephson in short concise paper in 1962 [1], which predicts a resistance-less supercurrent that passes through the very thin insulating layer in between two macroscopic size superconductors. Depending on the circuit configuration and biasing methods, many useful devices can be made for use in research and industry, the most widely used being the Superconducting Quantum Interference Device (SQUID). In this report, I present the underlying physics of these devices.

What is a SQUID?

Making a SQUID is fairly simple: it is a superconducting ring interrupted by one or more Josephson junctions. An example is shown in figure 1.

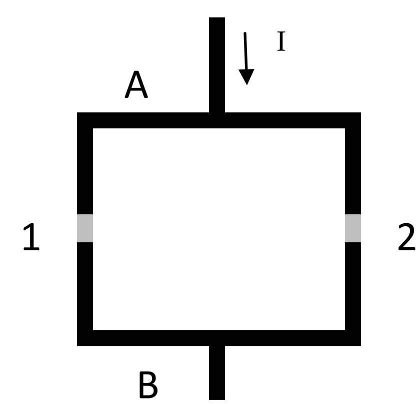

Figure 1 – Schematic of a 2-Junction SQUID. The black lines forming the box are made of superconducting material and grey interruptions are made of insulating material. These two interruptions are the two Josephson Junctions.

Based on this description, we can foresee some of the physics governing this device. First, we have the Josephson effect. Due to the ring geometry and the possibility of multiple junctions, we should encounter interference effects. Second, there is a loop containing electric supercurrent which will result in effects due magnetic flux. We will start simple and look at the flux due to currents in a superconducting ring. We will follow this with the basic concepts of a Josephson junction alone, and then add them to the superconducting ring.

Magnetic flux in a superconducting ring

To start we need some equations describing the superconducting state. As shown by the BCS theory of superconductivity, the electrons in the superconducting state form correlated pairs of electrons with opposite spin and momentum called cooper pairs [2]. The coherence length of these pairs is sufficiently large that they overlap and the relative phase factors of the wavefuntions of each pair match. This results in a coherent phase throughout the superconductor. We will then assume, a la Feynman, that we can take the ensemble average of the cooper pair wavefunctions and desecibe all cooper pairs with a macroscopic wavefunction of the form [3][4]

$$\Psi = |\Psi(\boldsymbol{r})|e^{i\boldsymbol{K}\cdot\boldsymbol{r}} \tag{1}$$

Where $\boldsymbol{K}$ is the net wave vector of all cooper pairs and $|\Psi(\boldsymbol{r})|$ is the ensemble-average function when $K = 0$. The quantity $\boldsymbol{K}\cdot\boldsymbol{r}$ is the position dependent phase of our macroscopic wavefunction, so we will write this as $\boldsymbol{K}\cdot\boldsymbol{r} = \theta(\boldsymbol{r})$. We can normalize this wavefuction so that

$\int_V \Psi^*\Psi d\mathbf{r} = N$, the total number of electron pairs in the superconductor. This means we can write our wavefunction as

$$\Psi = \sqrt{n(\boldsymbol{r})}e^{i\theta(\boldsymbol{r})} \tag{2}$$

Where $n(\boldsymbol{r})$ is the local cooper pair density. From here on we will take the density to be spatially invariant, which is approximately true in weak fields.

The classical canonical momentum of a particle of charge q and mass m in a magnetic field of vector potentional **A** is $\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}$. For a pair of mass m* and charge e*, we have

$$\boldsymbol{p} = m^*\boldsymbol{v} + q^*\boldsymbol{A} \tag{3}$$

We can find the momentum density by multiplying equation (3) by the local density $n(\boldsymbol{r}) = n$. Quantum mechanically, $n\boldsymbol{p}$ is the expectation value of the canonical-momentum operator $-i\hbar\boldsymbol{\nabla}$, so we have

$$n\boldsymbol{p} = \langle\Psi|-i\hbar\boldsymbol{\nabla}|\Psi\rangle = n\hbar(\boldsymbol{\nabla}\theta) \qquad (4).$$

The pair current density is given by $\boldsymbol{J} = ne^*\boldsymbol{v}$, so we can write momentum as

$$\boldsymbol{p} = \hbar\boldsymbol{\nabla}\theta = e^*\Lambda\boldsymbol{J} + e^*\boldsymbol{A} \tag{5}$$

where $\Lambda = m^*/ne^{*2}$ [3].

Now we can look at a superconducting ring. We will consider a closed contour within a superconductor that surrounds a hole. Then integrate equation (5) around this contour:

$$\hbar\oint\boldsymbol{\nabla}\theta\cdot d\boldsymbol{l} = e^*\oint(\Lambda\boldsymbol{J} + \boldsymbol{A})\cdot d\boldsymbol{l} \tag{6}$$

The phase must be coherent around the contour, so the right-hand-side integral must come in integer multiples of 2π. If the contour is chosen deep inside the superconductor, $\boldsymbol{J} \approx 0$, and using Stokes' theorem, the left-hand-side integral becomes

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) dS = \Phi_\mathrm{s} \tag{7}$$

Where Φ_s is the flux penetrating our loop. The integral on the left we found to be quantized, so the flux must be quantized as

$$\Phi_\mathrm{s} = \frac{lh}{e^*} \qquad l = 0, 1, 2, 3, \ldots \tag{8}$$

Since each pair consists of two electron charges, we can define the flux quantum as $\Phi_0 = \frac{\mathrm{h}}{2\mathrm{e}}$ [3].

The quantization of magnetic flux was actually first discovered experimentally. The experiment was done in 1961 by two groups: Doll and Näbauer in Munich and Deaver and Fairbank in Stanford [6] In summary, because the phase in a superconductor is coherent, the flux in a ring must be quantized in units of the flux quantum.

The Josephson Junction

We model a Josephson junction as a thin insulator sandwiched between two

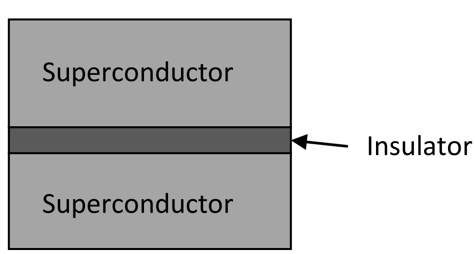

Figure 2 - Simple schematic of a Josephson Junction.

superconductors, as shown in figure 2.

Under much skepticism, Brian Josephson was able to predict in a short paper in 1962 that under zero applied voltage cooper pairs between the two superconductors should tunnel through the insulation barrier resulting in a resistance-less supercurrent [1]. His paper is very often cited, but his derivations are very rarely used possibly in part because it has errors to be corrected and gaps to be filled. Instead the Feynman approach has been used widely [4], as we will here.

We can represent each superconductor of the junction with their own macroscopic wavefunctions, equation (2). One way to interpret the Josephson effect in superconductors is these two wavefunctions, which describe cooper pairs and not individual electrons, penetrate into the insulator. If the insulator is sufficiently thin, these wavefunctions will have an overlap which means there is a probability that cooper pairs will tunnel through the barrier. Representing this wavefunction overlap as a coupling constant, one can derive this cooper pair current [4]. The result is the current depends on the difference in the phases of the two superconductors at the insulator boundary, which is written as

$$I_s = I_c \sin(\Delta\varphi) \tag{9}$$

In the presence of a magnetic field the current will be altered, so the phase is not gauge invariant. In terms of the vector potential $\boldsymbol{A}$, the gauge invariant phase difference is [5]

$$\gamma \equiv \Delta\varphi - (\frac{2\pi}{\Phi_0}) \int \mathbf{A} \cdot d\boldsymbol{l} \tag{10}$$

where the integration is taken across the insulating layer. If we replace the phase difference in equation (9) with equation (10), we get the true Josephson current relation in a magnetic field.

The dc-SQUID

Now we will put two Josephson junction in a superconducting loop, such as in figure 1, which is called a dc-SQUID. To compute the line integral of the vector potential around the loop to calculate the magnetic flux, we have to separate the integral for the superconducting electrodes and the Josephson junctions, which will be referred as links [5]. If the electrodes are thicker than the London penetration depth, then we can choose a contour deep inside the electrodes where the supercurrent velocity, $\boldsymbol{v}$, introduced in equation (3), is zero. Therefore, from equation (5), the vector potential along the contour inside the electrodes is $\boldsymbol{A} = (\Phi_0/2\pi)\boldsymbol{\nabla}\varphi$. The flux looping the dc-SQUID is [5],

$$\Phi = \oint \boldsymbol{A} \cdot d\boldsymbol{l} = \left(\frac{\Phi_0}{2\pi}\right)\int_{electrodes} \boldsymbol{\nabla}\varphi \cdot d\boldsymbol{l} + \int_{links} \boldsymbol{A} \cdot d\boldsymbol{l} \tag{11}$$

The phase around the loop must be single valued, so the integral in the first term in the right side of equation (11) plus the phase differences across the links must be an integer multiple of 2π. Combining equation (11) with equation (10) for both links, difference in the gauge-invariant phase differences is [5]

$$\gamma_1 - \gamma_2 = 2\pi\Phi/\Phi_0 (\mathrm{mod}\ 2\pi). \tag{12}$$

For a single Josephson junction, the maximum current is when the gauge-invariant phase difference is $\gamma = \pi/2$. When two are coupled in a loop, there is an interference effect which restricts the possible values for the phases. Unless the total magnetic flux is an integral multiple of the flux quantum, the max current passing through the SQUID must be less than the sum of the critical currents of each of the junctions. If each junction has identical crictical currents, I_c, it can be shown that the maximum supercurrent is [5]

$$I_m = 2 * I_c |\cos(\pi\Phi/\Phi_0| \quad (13)$$

The plot of the max current against the flux looks an awful lot like the interference pattern from Young's Double slit experiment. So SQUIDs exhibit the analogous superconducting experiment. This derivation treated the junctions as point contacts, so variation along the boundary was not considered. If we extend the size of a rectangular junction, we find that the current varies sinusiodally with position. The maximum current as a function of the flux becomes [5]

$$\frac{I_m(H)}{I_m(0)} = \left| \frac{\sin\left(\frac{\pi\Phi}{\Phi_0}\right)}{\left(\frac{\pi\Phi}{\Phi_0}\right)} \right| \quad (14)$$

This is the Fraunhofer diffraction pattern, just like when light passes through a narrow rectangular slit. If the junction is circular, we get an airy diffraction pattern [5].

Other typers of SQUID devices can be made by using different numbers of Josephson junctions. A superconducting loop with one Junction is called an rf-SQUID. These are fairly similar in function to dc-SQUIDs, but dc-SQUIDS are used more often because they were the first to be built [3]. BY using three junctions, one can make a flux qubit or persistent current qubit. These qubits use the clockwise and counterclockwise directed currents as basis states which can create superposition states.

Applications of the SQUIDs

SQUIDs have found their way into many facets of research and industry. They are widely used at highly sensitive magnetometers. Looking at equations (13) and (14), we can see that we can measure changes in current due to changes in flux on the order of a flux quantum, which is $\Phi_0 = 2.07 \times 10^{-7}\ \mathrm{G} \cdot \mathrm{cm}^2$; compare this to the earth's magnetic field which $\sim 1\ G$. In current research, they are used to read out the states of superconducting qubits. With a SQUID located in close proximity to either a phase or flux qubit. The flux induced by changing currents can be detected. They are also used to measure the magnetic susceptibility of materials and measure faint signals of biological systems suchas the human hear t and brain [3].

References

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