Scattering Matrices of Josephson Junction Networks

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I. Introduction

In the ultra-cold microcosms of the mesoscopic scale, matter can behave in very bizarre ways. In metals, such as aluminum, electrons seem to live in a world of absolute freedom; they can move anywhere unobstructed, and can be anywhere and everywhere at the same time, well… as long as they have a buddy to go along with them. What I am talking about is the superconducting state of matter. It is a state that some metals, such as aluminum and lead (but not copper), undergo when cooled below a certain threshold temperature. These temperatures are usually on the order of 1 K. There are high temperature superconducting materials such as the famous cuprates $YBa_2Cu_3O_7$ or “YBCO” and $Bi_2Sr_2CaCu_2O_8$ or “BSCCO” that have critical temperatures of 93 K and 13 K, respectively[1]. The superconducting state is characterized by zero resistance and dissipation of magnetic fields[1]. As humans in the macroscopic universe, we have absolutely no intuition about matter at this scale, but over the past century, quantum mechanics has helped us at least describe the superconducting phenomena in specific, mathematical language. Whether we actually *understand* the phenomena through quantum mechanics is a whole other topic.

What I am concerned with here is the more practical side of the superconductors. Many useful devices have been made such as ultrasensitive magnetic field detectors call

S.Q.U.I.Ds. These devices use various arrangements of seemingly simple devices called Josephson Junctions (JJ). They are simple in the sense that they are made by sandwiching a very thin insulting oxide layer between two superconducting metals, as shown in fig. 1.

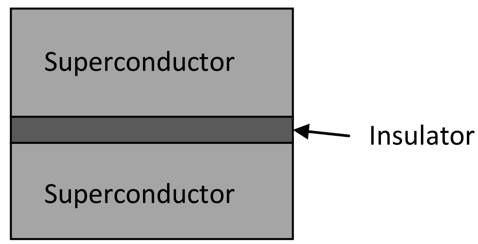

Figure 1 - Simple schematic of a Josephson Junction.

My goal here is to present a potentially useful description of a Josephson Junction with scattering or S-matrices. I will then show how we can use these S-matrices to calculate the properties of complex networks of JJs connected to each other with superconducting wires.

II. Basic Theory of Superconductivity and Josephson Junctions

Here, we will not concern ourselves with an in-depth description of superconductivity. We simply want enough of a foundation to achieve our goal of an S-matrix. Richard Feynman's pedagogical description is sufficient for our purposes [2]. In the superconducting state, electrons condense down into a single macroscopic ground state. If you are wondering how fermionic electrons can occupy the same quantum mechanical state, refer to references on the BCS theory of superconductivity [1]. To ease our concerns, the basic idea is that at super-low temperatures, the phonon interaction cause electrons to attract slightly and form into pairs with opposite spin due to the Pauli Exclusion Principle. They form a single entity (whatever that is) called a cooper pair which behaves as a boson because their total spin is zero. Bosons can occupy a single quantum state, so we can proceed with our macroscopic ground state.

If we represent the wavefunction of a single electron as $|\psi\rangle$, then the normalization condition is $\langle\psi|\psi\rangle = 1$. If we have a macroscopic wavefunction $|\boldsymbol{\psi}\rangle$ describing all electrons in a superconductor, we would expect the normalization condition to be $\langle\boldsymbol{\psi}|\boldsymbol{\psi}\rangle = N$ where N is the total number of electrons. At a position $\boldsymbol{r}$ and time t in the superconductor we would expect $\langle\boldsymbol{\psi}(\boldsymbol{r},t)|\boldsymbol{\psi}(\boldsymbol{r},t)\rangle = n(\boldsymbol{r},t)$ where $n(\boldsymbol{r},t)$ is the local density. We can then write the macroscopic wavefunction as,

$$\boldsymbol{\psi}(\boldsymbol{r},t) = \sqrt{n(\boldsymbol{r},t)}e^{i\phi(\boldsymbol{r},t)} \tag{1}$$

In a Josephson Junction, there are two superconductors, so we have two macroscopic wavefunction,

$$\boldsymbol{\psi_1}(\boldsymbol{r},t) = \sqrt{n_1(\boldsymbol{r},t)}e^{i\phi_1(\boldsymbol{r},t)}$$

$$\boldsymbol{\psi_2}(\boldsymbol{r},t) = \sqrt{n_2(\boldsymbol{r},t)}e^{i\phi_2(\boldsymbol{r},t)} \tag{2}$$

With only a thin barrier between the superconductors, the wavefunction have a small overlap, which means there is a probability that the super electrons will tunnel through the barrier. We now ask, what is the electric current associated with this tunneling? Feynman's approach is to couple the wavefunction via coupled Schrödinger equations [2]

$$\frac{\partial \boldsymbol{\Psi}_1}{\partial \mathrm{t}} = \frac{\text{-i}}{\hbar}(\mathrm{E}_1\boldsymbol{\Psi}_1 + \mathrm{K}\boldsymbol{\Psi}_2)$$

$$\frac{\partial \boldsymbol{\psi}_2}{\partial t} = \frac{-i}{\hbar}(E_2\boldsymbol{\psi}_2 + K\boldsymbol{\psi}_1) \tag{3}$$

where E_1 and E_2 and the energies of the two wavefunction and K is a general unknown coupling energy. If we plug equations (2) into (3) and separate the real and imaginary parts, we obtain

$$\frac{1}{2}\frac{\dot{n}_1}{\sqrt{n_1}} = \frac{K}{\hbar}\sqrt{n_2}\sin(\phi_2 - \phi_1)$$

$$\frac{1}{2}\frac{\dot{n}_2}{\sqrt{n_2}} = -\frac{K}{\hbar}\sqrt{n_1}\sin(\phi_2 - \phi_1) \tag{4}$$

$$i\sqrt{n_1}\dot{\phi}_1 = -\frac{i}{\hbar}\{E_1\sqrt{n_1} + K\sqrt{n_2}\cos(\phi_2 - \phi_1)\}$$

$$i\sqrt{n_1}\dot{\phi}_2 = -\frac{i}{\hbar}\{E_1\sqrt{n_2} + K\sqrt{n_1}\cos(\phi_1 - \phi_2)\} \tag{5}$$

If we assume the superconductors to be identical so that $n_1 = n_2 \equiv n$ and multiply equations (4) by the volume of the superconductor, V. and by the charge of a single cooper pair $2e$, we have an equation for the supercurrent across the insulating barrier

$$I_s = I_c \sin(\gamma) \tag{6}$$

where $\gamma = \phi_2 - \phi_1$ called the phase difference, and $I_c = \frac{4Ke}{\hbar}Vn = \frac{4\pi K}{\Phi_0}Vn$ and is a property of the Josephson Junction. It is common to represent the current in terms of the flux quantum $\Phi_0 = \mathrm{h}/2\mathrm{e}$. It comes from the experimentally verified fact that the magnetic flux from the current in a superconducting ring is quantized. A good explanation of this can be found in [1]. If we take the difference of equations (5) and consider $E_2 - E_1 = 2eU$ where U is the electric potential difference across the junction, we get

$$U = \frac{\Phi_0}{2\pi}\dot{\gamma} \tag{7}$$

Equations (6) and (7) are the fundamental Josephson relations. They state that the current depends on the phase difference between the wavefunction and the electric potential depends on the rate of change of the phase difference. These are very bizarre results. It was long believed the phase of wavefunction was only a mathematical tool and had no real physical meaning. Since Brian Josephson published these relations in his 1962 paper [3], physicists have been mystified by the quantum mechanical underpinnings that Josephson Junctions demonstrate. Much of the motivation for studying Josephson Junctions is the potential answers they shed on the most fundamental questions of quantum mechanics.

III. The Scattering Matrix

In the development of scattering matrices, we are going to concern ourselves only with 1 dimension as it is a convenient and simple starting point, and it is the environment for which we have the necessary tools. The process of calculating scattering matrices stems from the transfer matrix method of quantum mechanics as outlined in [4], and I will try to briefly summarize here. If we have a constant potential of finite width V(x), the solutions to the time-independent Schrödinger are complex linear combinations of a right and left traveling wave of the form $\psi(x) = Ae^{ikx} + Be^{-ikx}$ where $k = \sqrt{2m(E-V)/\hbar^2}$ if E>V. If E<V, then the general solution is $\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$ where $\kappa = \sqrt{2m(V-E)/\hbar^2}$. To find the solution to any piecewise constant potential, we simply match the wavefunction and its first derivative at each boundary. For our purposes here, we are not going to consider energies above the potential,

except at the asymptotic left and right limits where the potential is zero. This will be more clear in a bit.

Here I will just outline the general procedure for matching boundaries without going through the full derivation. This is so that the general reader can follow the method for finding the scattering matrix. Refer to [4] for more detail. To match boundaries, we first start with amplitudes A_R and B_R on the right side of the potential and put them in column form $\begin{bmatrix} A_R \\ B_R \end{bmatrix}$.

Then for each piece of constant potentials, we build the matrix

$M = \begin{bmatrix} \cosh(\kappa d) & -\kappa^{-1}\sinh(\kappa d) \\ -\kappa\sinh(\kappa d) & \cosh(\kappa d) \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ where d is the width of the potential.

We multiply each of these matrices together to build one 'M' matrix for the whole potential $M = M_1 M_2 \ldots M_N = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. Then to find the wavefunction amplitudes on the left, $\begin{bmatrix} A_L \\ B_L \end{bmatrix}$, we build the transfer matrix

$$T = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

$$\begin{aligned} 2\alpha &= m_{11} + \frac{k_R}{k_L} m_{22} + i\left(k_R m_{12} - \frac{m_{21}}{k_L}\right) \\ 2\beta &= m_{11} - \frac{k_R}{k_L} m_{22} - i\left(k_R m_{12} + \frac{m_{21}}{k_L}\right) \end{aligned} \tag{8}$$

and multiply by the right amplitude column vector: $\begin{bmatrix} A_L \\ B_L \end{bmatrix} = T \begin{bmatrix} A_R \\ B_R \end{bmatrix}$.

Now we find the S-matrix. The S-matrix relates input signals to output signals, so it is of the form $\begin{bmatrix} A_R \\ B_L \end{bmatrix} = S \begin{bmatrix} B_R \\ A_L \end{bmatrix}$. We can get the elements of S from the transfer matrix. After following the construction in [4], we get

$$S = \begin{bmatrix} -\frac{t_{12}}{t_{11}} & \frac{1}{t_{11}} \\ \frac{\det(T)}{t_{11}} & \frac{t_{21}}{t_{11}} \end{bmatrix}. \tag{9}$$

IV. Josephson Junction Potential

Before we can calculate the S-Matrix of a Josephson Junction, we need a suitable potential that correctly models a Josephson Junction. Since a JJ has a barrier in the middle where the oxide layer is, an obvious guess is a single block potential. It seems simple enough, but it would be nice if we can show that it works by deriving the Josephson relations (6) and (7) from the potential. We can do this with probability currents. In general, for any wavefunction ψ, the probability current is [5]

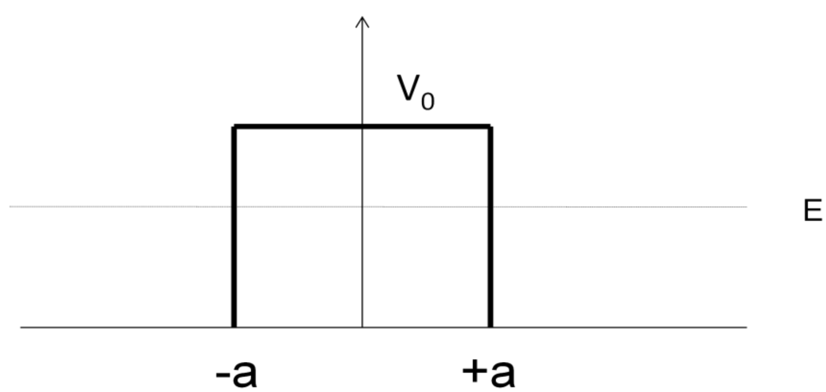

Figure 2 - Potential energy diagram of a Josephson Junction.

$$\boldsymbol{J}_{\wp} = \frac{\hbar}{2im}(\psi^*\nabla\psi - \psi\nabla\psi^*) = Re\left\{\psi^*\frac{\hbar}{im}\nabla\psi\right\} \tag{10}$$

If we consider now our macroscopic wavefunction describing all the electrons in the superconductor, equation (10) would give us “particle” current, meaning the flow rate of particles, if we can speak classically. If we multiply this by the charge, q, of a particle, then we obtain the electric current density. Equation (10) becomes

$$\boldsymbol{J}_S = \frac{q}{m} Re\left\{\psi^* \frac{\hbar}{i} \nabla\psi\right\} . \tag{11}$$

Now we just need to find the macroscopic wavefunction inside the potential. The general solution to the Schrödinger equation for energies below the barrier height is

$$\boldsymbol{\psi}(x) = C_1 \cosh(\kappa x) + C_2 \sinh(\kappa x) \tag{12}$$

If we plug (12) into (11), we get [5]

$$\boldsymbol{J}_S = \frac{q\kappa\hbar}{m} Im\{C_1^* C_2\} \tag{13}$$

To find the coefficients C_1 and C_2, we match the wavefunction inside the barrier, (12), with the wavefunction in the two superconductors (2) at the boundaries of the barrier x= -a, +a. the phases of the wavefunction at the boundaries will be θ_1 and θ_2, respectively. After doing this, we obtain for the coefficients [5]

$$C_1 = \frac{\sqrt{n_1}e^{i\theta_1} + \sqrt{n_2}e^{i\theta_2}}{2\cosh(\kappa a)}$$

$$C_2 = -\frac{\sqrt{n_1}e^{i\theta_1} - \sqrt{n_2}e^{i\theta_2}}{2\sinh(\kappa a)} \tag{14}$$

Plugging (14) into (13), we obtain the current-phase Josephson relation (6) in terms of supercurrent density [5]

$$J_S = J_C \sin(\theta_1 - \theta_2) \qquad (15)$$

where the critical current is $J_C = -\frac{q\hbar\kappa}{m}\frac{\sqrt{n_1 n_2}}{2\sinh(\kappa a)\cosh(\kappa a)}$.

Now that we know the potential pictured in figure 2 correctly models a Josephson Junction, what are typical parameters of real junctions? Are there other potentials that work? The barrier width is simply the thickness of the oxide insulating layer of the Josephson Junction. One group at Cornell University [6] made ultrathin aluminum Oxide JJ with the AlO_x layer ranging from 0.6 to 1.5 nm. The barrier height was 1.2 eV. Another group [7], who studied aluminum nitride barriers with niobium nitride plates (NbN/AlN/NbN), produced junctions with two different barrier heights that depended on the critical current. They published a theoretical equation that relates the critical current density, J_C, with the barrier height, ϕ, and the barrier thickness, d

$$J_C = 3.16 \times 10^{10}\frac{\sqrt{\phi}}{d}\exp(-1.025\sqrt{\phi}d)\ . \qquad (16)$$

In the low and high J_C regions, they found barrier heights of 2.35 eV and 0.88 eV, respectively. They were inconclusive as to why this junction had two different barrier heights. The thicknesses of their barriers ranged from around 1.0 nm to 1.8 nm. The thicker barriers had higher barrier heights, and the transition between low to high barrier heights was around 1.3 nm. This demonstrates how modeling a Josephson Junction with the potential depicted in figure 2 is really just a simple starting point. For example, another group working with Nb,Pb based junction found their potential to be trapezoidal shaped [8].

We can also consider insulators with impurities [1]. These would provide semiconductor-like effects. My question is, would these types of junctions exhibit any kind of resonances? I suspect so, but would need to investigate further. This could possibly lead to other Josephson effects.

V. The S-Matrix for Networked Josephson Junctions

Ultimately, from the network S-matrix, we want to relate the input signals to the output signals at the external ports. Here, we will consider a network with only 2 external ports, so our S-matrix will be a square matrix of dimension two. The process of building such an S-matrix is adapted from references [9] and [10]. In these papers, they use network s-matrices to investigate the magnetic structure of high-T_C superconductors. They model the lattice as a network of small superconducting wires connected at nodes. This fits beautifully for network Josephson Junctions.

The method involves creating two matrices: one that relates all the ports of the wires, and one that relates all the ports of the nodes. In the wire s-matrix, S_w, we want to transform all inputs at each port to all outputs at the opposite port, so for N wires, we will have a 2N dimensional matrix. We put this matrix in this form:

$$\begin{bmatrix} o_1 \\ o_n \\ \vdots \end{bmatrix} = \begin{bmatrix} s_{ee} & & s_{ei} \\ & & \\ s_{ie} & & s_{ii} \end{bmatrix} \begin{bmatrix} i_1 \\ i_n \\ \vdots \end{bmatrix} \tag{17}$$

where s_{ee} relates all and only the external ports. In our case, we will only have two external ports labeled 1 and n, and s_{ee} will be a square dimension two matrix. The sub-matrix s_{ii} works similarly but relates all and only internal ports of the wires. The other two sub-matrices then relate the internal to the external.

The node matrix, Γ, also relates the internal ports to each other, but only for the nodes. The node matrix will have the same dimensions as s_{ii} and needs to have columns and rows in the same order, as we will see. The process of building this matrix is to look at each node separately then put them together. For nodes it is easier to relate how the outputs are directed to the inputs, so the node matrix is in this form:

$$\begin{bmatrix} \vdots \\ i_j \\ \vdots \end{bmatrix} = \Gamma \begin{bmatrix} \vdots \\ o_j \\ \vdots \end{bmatrix}. \tag{18}$$

Once we have these two matrices in the proper form, we can find the total S-matrix of the network by doing the following matrix operations:

$$S_{tot} = s_{ee} + s_{ei}\left(\Gamma^{-1} - s_{ii}\right)^{-1} s_{ie} \tag{19}$$

Now, we will calculate the S-matrix of a simple network, as depicted in figure 3.

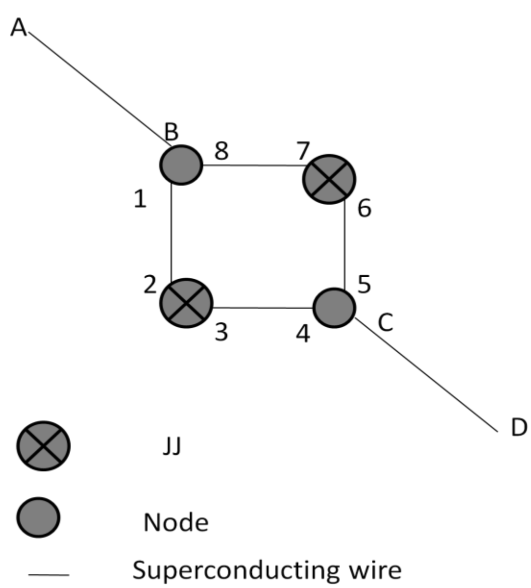

Figure 3 – Schematic of the network for which we calculate the network S-Matrix.

For this network, we have total of six wires and four nodes. I treated the wires as lossless transmitters. This means that the signals at the inputs are exactly the same at the outputs, phase included. Our wire S-matrix is then a 12x12 undirected connectivity matrix with ones for each pair of connected ports and zeros everywhere else. Two of the nodes are Josephson Junctions, depicted as grey circles with a cross. The S-matrix of the Josephson Junction can be found by following the procedure in part III, using the potential in figure 2. For the transfer matrix in the form of equation (8), we get

$$\begin{aligned}\alpha &= \cosh(\kappa a) + i\frac{1}{2}\sinh(\kappa a)\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\\ \beta &= i\frac{1}{2}\sinh(\kappa a)\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)\end{aligned} \tag{20}$$

Following equation (9), we obtain a symmetric S-matrix for a single Josephson Junction of the form

$$\begin{aligned}s_{11} &= s_{22} = Be^{i\rho} = b\\ s_{12} &= s_{21} = Ae^{i\sigma} = a\end{aligned} \tag{21}$$

Where

$$A = \frac{1}{\sqrt{\cosh^2(\kappa a) + \frac{1}{4}\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right)^2 \sinh^2(\kappa d)}}$$

$$\sigma = -\tan^{-1}\left(\frac{1}{2}\frac{\sinh(\kappa d)\left(\kappa^2 - k^2\right)}{\kappa k\cosh(\kappa d)}\right)$$

$$B = \frac{1}{2}\frac{\sinh(\kappa d)\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)}{\sqrt{\cosh^2(\kappa d) + \frac{1}{4}\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right)^2 \sinh^2(\kappa d)}}$$

$$\rho = \tan^{-1}\left(\frac{2k\kappa \cosh(\kappa d)}{(\kappa^2 - k^2)\sinh(\kappa d)} \right) \tag{22}$$

Here, k and κ are defined as before, and d is the thickness of the insulating layer in the Josephson Junction. There are two of these JJ S-matrices relating ports 2 and 3, and 6 and 7. The other two nodes simply connect the wires, and we can choose how it directs all the inputs to the outputs. The only constraint they must be unitary. I chose to treat these nodes like 50/50 beam splitters so that they divide the inputs as symmetric as possible. Therefore, the three-port node s-matrices are of the form

$$\begin{bmatrix} i_1 \\ i_8 \\ i_B \end{bmatrix} = \begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{bmatrix} \begin{bmatrix} o_1 \\ o_8 \\ o_B \end{bmatrix} \tag{23}$$

Where

$$c = \left(\frac{1}{2} \right)^{1/3} e^{2\pi i \frac{1}{3}} \tag{24}$$

I included a phase factor on c so that the nodes would also transmit phase information since Josephson Junctions are phase dependent devices. Our node matrix then is

$$\Gamma = \left[\begin{array}{ccc:ccc:cc:cc} 0 & c & c & & & & & & & \\ c & 0 & c & & & & & & & \\ c & c & 0 & & & & & & & \\ \hdashline & & & 0 & c & c & & & & \\ & & & c & 0 & c & & & & \\ & & & c & c & 0 & & & & \\ \hdashline & & & & & & b & a & & \\ & & & & & & a & b & & \\ \hdashline & & & & & & & & b & a \\ & & & & & & & & a & b \end{array}\right] \tag{25}$$

Now that we have our matrices we can do the matrix operation in (19) to find our network S-matrix for which we obtain

$$S_{tot} = \frac{2c^2}{a^2 - b^2 - c^2 + 2cb}\begin{bmatrix} c-b & a \\ a & c-b \end{bmatrix} \tag{26}$$

VI. Conclusion and Discussion

We quickly notice that the network S-matrix is symmetric as expected because our network is symmetric and all of our nodes are symmetric. In this preliminary work, I made many simplifications. For future work, we include phase altering superconducting wires since phase will be coherent but not uniform all along the wire. As mentioned before, we can also investigate other, possibly more accurate potential energy representations of Josephson Junctions. Also, we can investigate other, more complex networks.

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