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Quantum Mechanics II
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Photonic Crystal Lattices

In a photonic crystal, we examine the propagation of light through a three-dimensional lattice structure. This propagation is analogous to the diffusion of electrons through a conducting crystal, which can allow certain energies of electrons to diffuse through without reflection and can reject others entirely. This is caused by the relation between the width of the periodic potentials and the wavelength of the electron inside the potential. A similar relation occurs in optics if we replace the electron with a photon and the atoms with macroscopic slices of material of differing dielectric constants which repeat periodically. In this case, the incident wave will both partially transmit and partially reflect at each change in dielectric constant. This change will occur at a phase related to the dielectric constant and, if the change in phase interferes destructively with the incident wave, then the reflections will stack and almost entirely reflect the incident wave. This reflection occurs with very little absorption, as opposed to metal mirrors, whose absorptions can be intolerably large for shorter wavelengths of light.

It is easy to imagine a one dimensional lattice consisting of plates of two different materials stacked one on top of the other; however, this construction does not completely reflect light approaching from any direction or angle. If the light approaching is not normal to the stacking surface of the plates, the reflections will propagate towards the edges of the lattice and escape. To solve this, the lattice is extended to two and three dimensions. With a crystal that can almost completely reflect light with little absorption, many photon manipulations become possible. A couple simple ones include introducing a line of cavities that will carry a photon from beginning to end with little loss in energy, or even a single cavity in the center which would trap the photon for as long as it takes to completely leak out through tunneling.

In order to analyze a crystal lattice, we must employ the four macroscopic Maxwell's equations, which we can write in SI units like this:

$$\nabla \cdot B = 0 \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot D = \rho \qquad \nabla \times H - \frac{\partial D}{\partial t} = J$$

where E and H are the macroscopic electric and magnetic fields (respectively), D and B are the displacement and magnetic induction fields, and ρ and J define the free charge and current density. At this point, we will make a few assumptions and restrictions that will allow us to use these equations to define a photonic crystal system. We first assume that the material we use will be consist of a mixed dielectric medium in which the dielectric constants vary with respect to a distance in the Cartesian plane, r, but not with respect to time. We also assume that the material allows light to pass through, but contains no point sources. This allows us to zero out both ρ and J.

The next step is to relate D to E and B to H. We first look at the relationship between D and E and, if we assume the system to be macroscopic and isotropic for sections at position, r, we can define D(r, ω) as equivalent to E(r, ω) multiplied by a scalar dielectric function ε (r, ω) and the vacuum permittivity ε_0, which is also known as the relative permittivity. Then we ignore any explicit frequency dependence of the dielectric constant, as we will simply choose the dielectric constant which best suits our frequency range of interest.

The last step is to assume that we will use transparent materials, which allows us to consider ε (r) to be both real and positive. These assumptions allow us to define D(r) as follows:

$$D(r) = \varepsilon_0 \varepsilon(r) E(r)$$

A similar treatment results in a similar equation for B:

$$B(r) = \mu_0 \mu(r) H(r)$$

where μ_0 is the vacuum permeability. For our purposes we can assume the relative magnetic permeability $\mu(r)$ to be unity, as it is unlikely to vary for the materials we choose.

These assumptions allow us to restate the Maxwell equations in a slightly simpler manner:

$$\nabla \cdot H(r,t) = 0 \qquad \nabla \times E(r,t) + \mu_0 \frac{\partial H(r,t)}{\partial t} = 0$$

$$\nabla \cdot \left[\varepsilon(r)E(r,t)\right] = 0 \qquad \nabla \times H(r,t) - \varepsilon_0 \varepsilon(r) \frac{\partial E(r,t)}{\partial t} = 0$$

At this point, we expand the fields into sets of harmonics modes. These take the form of spatial patterns multiplied by a complex exponential:

$$H(r,t) = H(r)e^{-i\omega t} \qquad E(r,t) = E(r)e^{-i\omega t}$$

In order to evaluate the profiles of these modes, we input the equations in the Maxwell relations stated earlier. The divergence relations can be rewritten:

$$\nabla \cdot H(r) = 0 \qquad \nabla \cdot \left[\varepsilon(r)E(r)\right] = 0$$

which leads us to conclude that there are no point sources or sinks of displacement and magnetic fields in the material and that the field configurations consist of transverse electromagnetic waves. The remaining two equations can be worked out to relate E(r) to H(r).

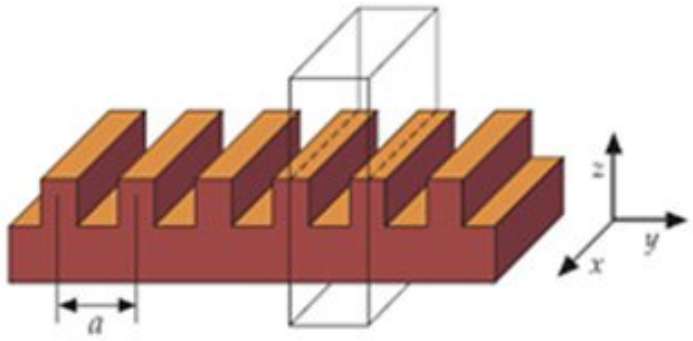
$$\nabla \times E(r) - i\omega\mu_0 H(r) = 0 \qquad \nabla \times H(r) + i\omega\varepsilon_0 \varepsilon(r) E(r) = 0$$

These can be decoupled by dividing the right equation by $\varepsilon(r)$ and then applying the curl. This yields a $\nabla \times E(r)$ in the right equation that we can replace with the left. The result of this is:

$$\nabla \times \left(\frac{1}{\varepsilon(r)} \nabla \times H(r) \right) = \left(\frac{\omega}{c} \right)^2 H(r)$$

here we've replaced the constants μ_0 and ε_0 by the vacuum speed of light, c, by means of the equation $c = 1/\sqrt{\mu_0 \varepsilon_0}$. This equation, along with the diversion constraints established earlier, will allow us to find the modes and frequencies of H(r) for a given dielectric structure $\varepsilon(r)$. We can then use the E(r) to $\nabla \times$ H(r) relation to recover E(r). We can note that this E(r) is already transverse as the divergence of a curl is always equal to zero.

Although we have defined the time dependency of the modes, we still have yet to define the special dependency. We begin by evaluating an example:



for which we have a value, a, which denotes the spacing between repeated intervals, called the lattice constant. From this, we derive the primitive lattice vector **a**=aŷ. With this, it is easy to see that we have repeated dielectric constant values for different values of r: $\varepsilon(r) = \varepsilon(r \pm R)$, where R is an integral multiple, ℓ, of **a** such that R= ℓ a. Now we can identify translation operators in the x and y direction. These take the form of a plane wave, just as H(r,t) and E(r,t) from before.

$$\hat{T}_{d\hat{x}} e^{ik_x x} = e^{ik_x(x-d)} = \left(e^{-ik_x d}\right) e^{-ik_x x}$$
$$\hat{T}_R e^{ik_y y} = e^{ik_y(y-\ell a)} = \left(e^{-ik_y \ell a}\right) e^{-ik_y y}$$

here we can see that if we were to insert a value $k_y + m(2\pi/a)$ in place of the wave vector, k_y, we find that the eigenvalue results in a multiple of $e^{-2\pi i \ell m}$ which, for m and ℓ are constants, is equivalent to one. We can conclude that adding a value of $b = (2\pi/a)$ to the wave vector leaves the state unchanged. This b can be defined in the same manner as a, such that **b**=$b\hat{y}$, which is called the primitive reciprocal lattice vector.

The translation operators we just found are the directional components of an eigenfunction for a certain frequency. By the nature of eigenstates, any linear combination of eigenfunctions corresponding to a singular eigenvalue is in itself an eigenfunction of the state. This allows us to put the translation operators in the form:

$$H_{k_x k_y} = e^{ik_x x} \sum_m c_{k_y,m}(z) e^{i(k_y + mb)y}$$
$$= e^{ik_x x} \cdot e^{ik_y y} \cdot \sum_m c_{k_y,m}(z) e^{imby}$$
$$= e^{ik_x x} \cdot e^{ik_y y} \cdot \mathbf{u}_{k_y}(y,z)$$

where the c's are expansion coefficients that will be determined by the explicit solution and **u**(y,z) is a periodic function in y for which $\mathbf{u}(y+\ell a)=\mathbf{u}(y,z)$ holds true. This result leads us to the general form:

$$H_k(\mathbf{r})=e^{i\cdot\mathbf{k}\cdot\mathbf{r}}\mathbf{u}_k(\mathbf{r})$$

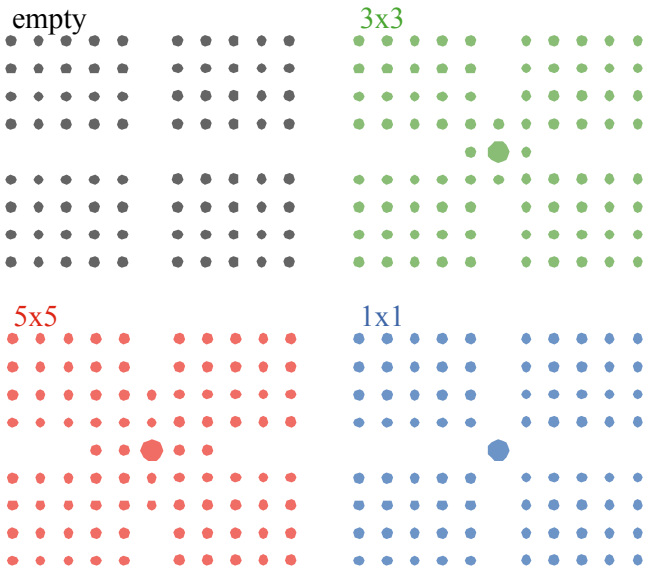
which is the three dimensional system of Bloch states, expressed using the Bloch wave vector $\mathbf{k}=k_1b_1+k_2b_2+k_3b_3$. A Bloch state is described as a state in which the wave vectors k and k+mb are identical. This allows us to reduce our scope to a single cell that covers the ranges of k from $-\pi/a$ to π/a. This region of nonredundant values for k is called the Brillouin zone. If we input the Bloch state equation into the main equation used to define a system above, we can define a new operator that depends on **k**:

$$\nabla\times\left(\frac{1}{\varepsilon(\mathbf{r})}\nabla\times e^{i\cdot\mathbf{k}\cdot\mathbf{r}}\mathbf{u}_k(\mathbf{r})\right)=\left(\frac{\omega}{c}\right)^2 e^{i\cdot\mathbf{k}\cdot\mathbf{r}}\mathbf{u}_k(\mathbf{r})$$

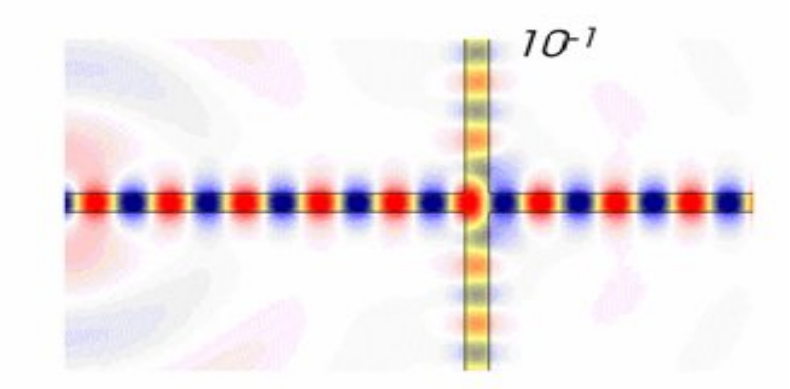
$$(i\mathbf{k}+\nabla)\times\frac{1}{\varepsilon(\mathbf{r})}(i\mathbf{k}+\nabla)\times\mathbf{u}_k(\mathbf{r})=\left(\frac{\omega}{c}\right)^2\mathbf{u}_k(\mathbf{r})$$

By establishing the periodicity of both $\varepsilon(\mathbf{r})$ and $\mathbf{u}_k(\mathbf{r})$, we can evaluate this system for the eigenmodes and eigenvalues and use these to design band pass filters, or optic wire, or many other applications that require low energy absorption, highly reflective material. There are some interesting properties of systems that involve impurities that can aid in some of these applications.

Steven B. Johnson, in his book: "Photonic Crystals," explores a two-dimensional lattice consisting of rods of a defined diameter, spacing, and dielectric constant. We know from our calculations that this will produce a band gap of a certain range of frequencies; however, if we introduce a line of impurities by either increasing or decreasing a row of rods or removing them entirely, we create a waveguide that will lead certain frequencies (or all if the dielectrics are removed) across it. Johnson explored what happens with intersecting waveguides and how to keep light incident on the intersection from diverting from its original path. .He simulated four different cases as shown below:



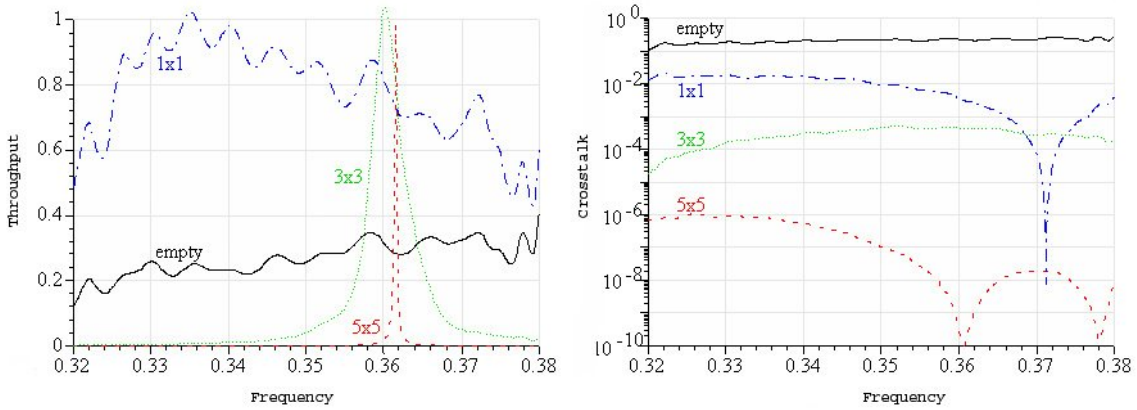
In these, the smaller rods are of a radius 0.2a, where a is the lattice constant (the distance between rod centers). The dielectric material has a dielectric constant of 11.56. Under these conditions, the pure system reflects transverse-magnetic (TM) light of frequencies in the range of 0.286 to 0.421. By opening the paths through the middle of the lattice, we allow light to travel through along the open paths only. The issue Johnson explored here is that when he sent a pulse of TM light down one path, the light would partially diffuse down the other paths as depicted below:



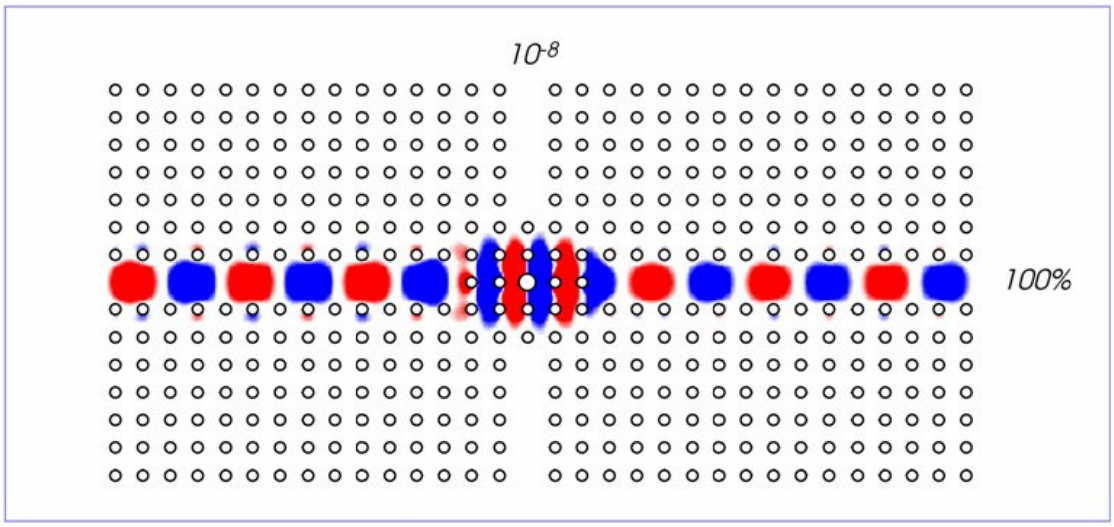
where the blue and red depict the oscillation and the intensity of color reflects the dispersion of light energy. He lists 0.1 as his simulated "crosstalk," or ratio of energy lost to the transverse ports.

His next step is to introduce an impurity in the system in the form of a dielectric with a greater radius. This impurity is designed to have only two degenerate resonant modes within the band gap of the system that correspond to the modes of the waveguides,

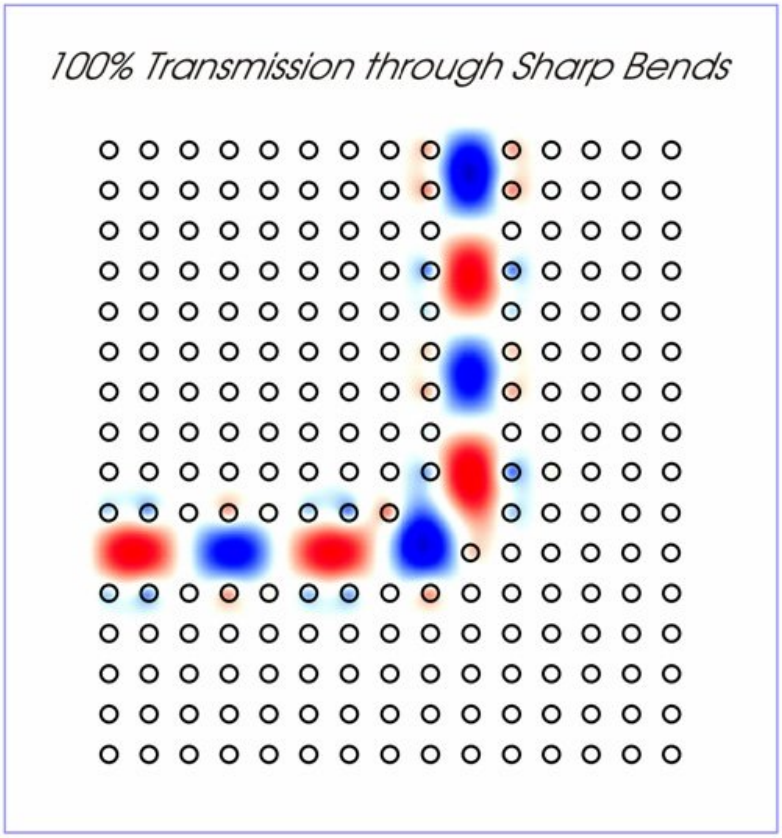
with each mode being orthogonal to the other. This ensures that the incident waves do not couple. The impurity here is designed to a radius of 0.33a, which yields a resonant frequency of 0.361 c/a. He simulated a burst of TM light sent down the path and measured the throughput and crosstalk. He repeated this same simulation for each case depicted above and graphed the results, shown here:



In this graph, we can see that the 5x5 scheme produces a sharp curve of about 1 throughput and 10^{-10} crosstalk at the resonant frequency of the impurity… As expected for the empty intersection, we have a flat, fairly low throughput and a significant amount of crosstalk. The 1x1 has a fairly broad, but high throughput, yet has a high crosstalk outside of around 0.37c/a. The 3x3 has a nice looking peak, but the crosstalk stays relatively flat at around 10^{-4}. If we look closely, we might notice that the graphs for both 1x1 and 3x3 peak at above 1. This is still valid as up to 5% of each throughput and crosstalk can be attributed to reflections from the boundary of the simulation. Considering all of this, we can see that the 5x5 scheme allows almost all of the TM light through the intersection without leaking any significant amount to the transverse ports. We can see how this looks physically:



In this manner, we can design junctions that send signals at the same time with near zero interference. This can only be reproduced in an electrical system by physically placing insulated wires over and under one other. This and a few other analogues have been explored in optical systems and have shown great promise as a possible solution for optic circuitry. One final example would be that Johnson has shown that the symmetry of the waveguides along with single mode in single direction allows us to make sharp bends with no losses.



This is analogous to electron scattering over a potential that increases in frequency as it passes over the potential, but returns to its initial state after passing. These analogues will allow us to replicate electrical systems and maybe even surpass them in speed, efficiency, and complexity.

References

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- Pictures from Johnson SG, < http://ab-initio.mit.edu/photons/tutorial/> Lecture 2