Homework 1

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1 Stellar Life

All stars start off as a cloud of dust. Over time the dust starts to clump together and form what is called a protostar. As the protostar becomes more massive the gravitational pressure on the outside shell becomes larger causing it to become more dense. This increase in density in turn causes the temperature to increase. Seeing as the mass of the protostar is larger it can pull in more dust from the surrounding cloud, and this cycle continues.

At some point the density and temperature reach a point where hydrogen starts to fuse into helium. From the fusion process many photons are produced and this in turn creates radiation pressure outwards that balances the force of gravity pulling inwards. Only at this point is the object called a star. All stars that fuse hydrogen in their core are referred to as main sequence stars. These stars can range is mass anywhere from 0.1 $M_\odot$ up to 100 $M_\odot$. During this phase of the stars life the core is well modeled as an ideal gas.

As this process continues the temperature and density of the core continue to rise. As the star becomes more dense the electrons start to get packed closer and closer together but because of the Pauli exclusion principle they can not get infinitely close. Instead they start filling up the available energy levels starting with the smallest. Since the lowest energy levels are filled up there will be some electrons that will have large energies, and in turn large momentums. As they move around they will exert a pressure outward. At some point this pressure will become larger then the ideal gas pressure that was in the core and the core will become degenerate. Once this happens all fusion in the core stops and the star starts to cool down. At this point it becomes known as a white dwarf.

2 Statement of the Problem

In the following paper we will explore the properties of white dwarfs and answer 3 major questions:

1. What is the electron degeneracy pressure?
2. When this pressure is balanced with gravity what is the mass-radius equation?
3. Where is the $H \leftrightarrow p^+ + e^-$ equilibrium below the Fermi Sea?

2.1 Outline of process

The approach used to find the degeneracy pressure uses the pressure integral. Using this three cases are addressed: (1) non-relativistic electrons, (2) ultra-relativistic electrons, and (3) relativistic electrons. With these results in hand this pressure is balanced with the gravitational pressure pulling the star together. From this a mass-radius relation is derived for all three cases. After this the Saha equation is then applied to the degenerate gas. This results is an equation relating the degree of ionization of the hydrogen in the star as a function of its temperature.

3 Electron vs. Gravitational Pressure

3.1 Electron Degeneracy Pressure

The starting place for calculating the electron degeneracy pressure is the pressure integral:

$$P = \frac{1}{3}\int_0^\infty v\ p\ n(p)dp \tag{1}$$

where v is the velocity of the particle, p is the momentum, and $n(p)dp$ is the number per unit volume with momenta in the interval (p,p+dp).

For the degenerate electron gas $n(p)dp$ is determined by the Heisenberg uncertainty principle:

$$n(p)dp = \frac{2}{\Delta V} = \frac{2}{h^3}4\pi p^2 dp \tag{2}$$

The integral of this expression from 0 to p_0 (the maximum momentum) leads to the total number (n_e) of electrons in the star:

$$n_e = \int_0^{p_0} n(p)dp = \frac{8\pi}{3h^3}p_0^3 \Rightarrow p_0 = (\frac{3h^3 n_e}{8\pi})^{\frac{1}{3}} \tag{3}$$

The last bit of information that is needed now is the value of n_e. The value can be written in terms of the density of the star, the atomic mass of the ions making up the star, and the number of protons in the ions (assuming the star is neutral):

$$n_e = \frac{\rho}{m_H}\frac{Z}{A} = \frac{\rho}{\mu_e m_H},\ \ \mu_e \equiv \frac{A}{Z} \tag{4}$$

μ_e is the average number of free electron per nucleon and for metal-poor stars, $\mu_e \simeq 2$.

3.1.1 Non-relativistic Electrons

For the non-relativistic electron gas the velocity is just p/m_e. Plugging this into (1):

$$P_e = \frac{1}{3}\int_0^{p_0} \frac{p}{m_e}p\frac{8\pi}{3h^3}p^2 dp = \frac{8\pi}{3h^3 m_e}\frac{p_0^5}{5} = k_1'(\frac{\rho}{\mu_e})^{\frac{5}{3}} = k_1\rho^{\frac{5}{3}} \tag{5}$$

where k_1' is all the constants collected together and is equal to $1.00\times 10^7 \frac{Nm^{-2}}{(kgm^{-3})^{5/3}}$. This equation shows that the pressure goes as $1/m$, meaning that the particles with the smallest mass will have the largest pressure. So when it comes to the white dwarf the electrons will have a much higher pressure then either the protons or the neutrons.

3.1.2 Ultra-relativistic Electrons

At some point the electrons will become so tightly packed that there random movements become on the order of the speed of light. For simplicity we will take the case where $v \simeq c$, now the pressure is:

$$P_{e,u-rel} = \frac{1}{3}\int_0^{p_0} cp\frac{8\pi}{3h^3}p^2 dp = \frac{8\pi c}{3h^3}\frac{p_0^4}{4} = k_2'(\frac{\rho}{\mu_e})^{\frac{4}{3}} = k_2\rho^{\frac{4}{3}} \tag{6}$$

where k_2' is another constant equal to $1.24 \times 10^{11}\frac{Nm^{-2}}{(kgm^{-3})^{4/3}}$.

3.1.3 Plan old relativistic Electrons

It is possible to expand on the expression for pressure found in the previous section. If you write the relativistic velocity down and plug it into (1) the result can be integrated in closed form:

$$\begin{aligned} v &= \frac{p/m_3}{\sqrt{1+(p/m_e c)^2}} && (7)\\ P_{e,rel} &= \frac{8\pi}{3h^3}\int_0^{p_0}\frac{p^4/m_3}{\sqrt{1+(p/m_e c)^2}}dp && (8)\\ \text{let } x &= \frac{p_0}{m_e c} && (9)\\ \Rightarrow P_{e,rel} &= \frac{\pi c^5 m_e^4}{3h^3}[x\sqrt{1+x^2}(2x^2-3) - 3\sinh^{-1}(x)] && (10) \end{aligned}$$

This expression comes in handy if you expand (10) in the limit of large x:

$$\begin{aligned} P_{e,rel} &\simeq \frac{\pi c^5 m_e^4}{3h^3}[2x^4 - 2x^2] && (11)\\ &= k_2\rho^{\frac{4}{3}} - \frac{c^3 m_e^2}{2h}(\frac{\pi}{3})^{\frac{1}{3}}(\frac{\rho}{m_H\mu_e})^{\frac{2}{3}} \equiv k_2\rho^{\frac{4}{3}} - k_3\rho^{\frac{2}{3}} && (12) \end{aligned}$$

The leading term is the same as (6) and the next term is the first order correction.

3.2 Gravitational Pressure

Assuming that the density of the star is constant the gravitational self potential is:

$$U_G = -\frac{3}{5}\frac{GM^2}{R} \tag{13}$$

This can be used to find the gravitational pressure:

$$P_G = -\frac{\partial U_G}{\partial V} = -\frac{GM^2}{5}(\frac{4\pi}{3})^{1/3}V^{-4/3} = -(\frac{4\pi}{3})^{1/3}\frac{G}{5}M^{2/3}\rho^{4/3} \tag{14}$$

3.3 Balancing the Pressures

3.3.1 Non-relativistic

When in equilibrium both the electron pressure and the gravitational pressure are equal and opposite: $P_e = -P_G$. Doing this results in:

$$k_1\rho^{5/3} = (\frac{4\pi}{3})^{1/3}\frac{G}{5}M^{2/3}\rho^{4/3} \tag{15}$$

$$\Rightarrow k_1(\frac{3M}{4\pi})^{1/3}\frac{1}{R} = (\frac{4\pi}{3})^{1/3}\frac{G}{5}M^{2/3} \tag{16}$$

$$\Rightarrow R = \frac{5k_1}{G}(\frac{3}{4\pi})^{2/3}\frac{1}{M^{1/3}} \tag{17}$$

$$\Rightarrow V = \frac{3}{4\pi}(\frac{5k_1}{G})^3\frac{1}{M} \tag{18}$$

From this equation it is clear that the volume of the white dwarf is inversely proportional to the mass, i.e. if the star is twice as massive it will have half the volume.

3.3.2 Ultra-relativistic

When the star is massive enough to be held up by relativistic election pressure the equilibrium equation becomes: $P_{e,u-rel} = -P_G$. This results in:

$$k_2\rho^{4/3} = (\frac{4\pi}{3})^{1/3}\frac{G}{5}M^{2/3}\rho^{4/3} \tag{19}$$

$$\Rightarrow M = (\frac{5k_2}{G})^{3/2}\sqrt{\frac{3}{4\pi}} \equiv M_{ch} \tag{20}$$

$$\Rightarrow M_{ch} \simeq 1.7M_{\odot} \tag{21}$$

This clearly shows that when as velocity of the electrons approaches c, the mass of the star approaches a constant. This is know as the Chandrasekhar Limit. More detailed calculations that do not take the density of the star to be constant show that this limit is closer to $1.44M_{\odot}$.

3.3.3 Plan old relativistic

Now I am going to use the result from the ultra-relativistic case to find the first order correction to the star's radius (seeing how it fell out before). Applying (12) this leads to:

$$k_2\rho^{\frac{4}{3}} - k_3\rho^{\frac{2}{3}} = (\frac{4\pi}{3})^{1/3}\frac{G}{5}M^{2/3}\rho^{4/3} \tag{22}$$

$$\Rightarrow R = \sqrt{\frac{k_2}{k_3}(1-(\frac{M}{M_{ch}})^{2/3})}(\frac{3M}{4\pi})^{1/3} \tag{23}$$

$$\Rightarrow R = \frac{h}{2cm_e}(\frac{3}{2\pi})^{2/3}(\frac{M}{m_H\mu_e})^{1/3}\sqrt{1-(\frac{M}{M_{ch}})^{2/3}} \tag{24}$$

From (24) it is clear that as the mass approaches M_{ch} the radius approaches zero.

4 Hydrogen to Ion ratio

The next topic of interest for the white dwarf is to look at the equilibrium for $H \leftrightarrow p^+ + e^-$ as a function of temperature. The Saha equation lets us look at this ratio in terms of partition functions:

$$\frac{[n_p][n_e]}{[n_H]} = \frac{Z_p Z_e}{Z_H} \tag{25}$$

The partition function for each particle can be broken into two different parts, (1) the kinetic energy term and (2) the internal energy term. For the kinetic energy term we have to take the sum over all energy states that are available to the system. Since all energy levels below the Fermi energy are filled it takes on the form:

$$\begin{aligned} \sum_{p=-\infty}^{-p_0} e^{\frac{-p^2}{2mkT}} + \sum_{p=p_0}^{\infty} e^{\frac{-p^2}{2mkT}} &\simeq \frac{1}{h^3}\left(\int_{-\infty}^{-p_0} e^{\frac{p^2}{2mkT}} d^3p + \int_{p_0}^{\infty} e^{\frac{p^2}{2mkT}} d^3p\right) & (26) \\ &= \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \mathrm{erfc}^3\left(\frac{p_0}{\sqrt{2mkT}}\right) & (27) \end{aligned}$$

Where erfc is the complementary error function. The internal partition functions are as follows:

$$\begin{aligned} Z(int)_{p^+} &= 2 & (28) \\ Z(int)_{e^-} &= 2 & (29) \\ Z(int)_H &\simeq 4e^{\frac{-E_g}{kT}} & (30) \end{aligned}$$

And for hydrogen $E_g = -13.6$ ev.

When you combine these partition function in the Saha equation the kinetic energy terms from the proton and the hydrogen will cancel out since their masses are for all intents and purposes are equal. Therefor the resulting equation is:

$$\frac{[n_p][n_e]}{[n_H]} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \mathrm{erfc}^3\left(\frac{p_0}{\sqrt{2m_e kT}}\right) e^{\frac{E_g}{kT}} \equiv A(p_0, T) \tag{31}$$

At this point we write the left hand side with the reaction coordinate x, that way $n_{p^+} = n_{e^-} = xn_e$ and $n_H = (1-x)n_e$. Now (31) becomes:

$$\frac{x^2}{1-x} = \frac{A(p_0, T)}{n_e} \tag{32}$$

To use the equation all that is needed is the Fermi momentum (p_0). As an example lets take a white dwarf with $M = 0.3M_\odot$. At this mass the electrons will not be relativistic so we can use (18) to find the volume. From here we find the density by dividing mass by volume and use (4) to find the number of electrons per unit volume and then use (3) for find the fermi momentum. Figure 1 shows the reaction coordinate x as a function of T for this star. This plot is similar to the one for a non-degenerate star. Seeing as the internal temperature of a white dwarf is on the order of 10^7 this plot shows that there will be some mixture for atoms to ions in the star. Figure 2 shows the same plot but this time for $M = 1.0M_\odot$ (using (24) for the mass-radius relation). The higher density of this star pushes the ionization temperature up by an order of magnitude.

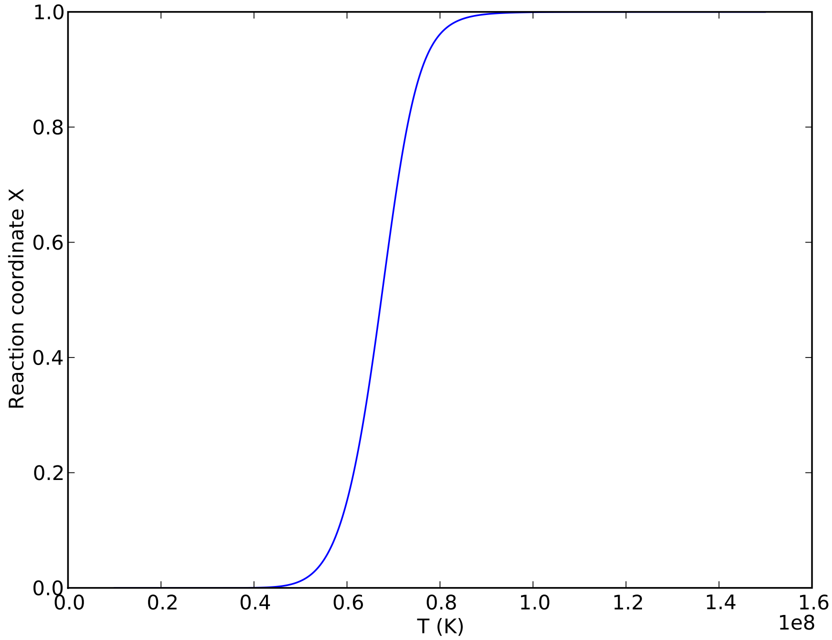

Figure 1: The reaction coordinate x as a function T for a white dwarf with $M = 0.3M_{\odot}$.

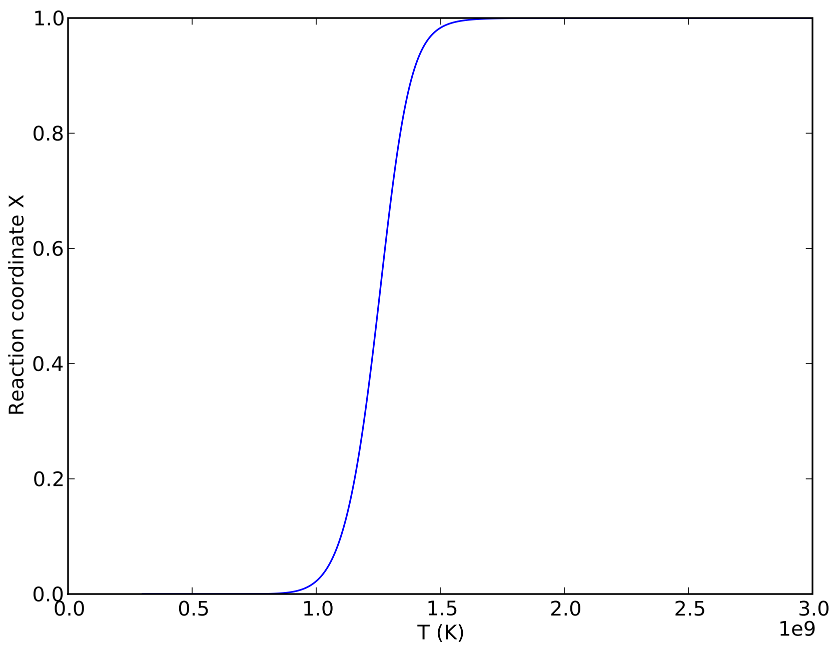

Figure 2: The reaction coordinate x as a function T for a white dwarf with $M = 1.0M_{\odot}$.

5 Conclusion

In this paper the degeneracy pressure of a white dwarf was calculated. For non-relativistic degeneracy pressure it was found that $R \propto M^{-1/3}$. This means that as the star gets more massive it gets smaller. From the ultra-relativistic case a characteristic mass was derived $M_{ch} \simeq 1.7 M_{\odot}$. Since the radius fell out of this equation it was necessary to go to the next order correction to find the desired relation, upon doing this the radius was found to go to zero as the mass approaches M_{ch}. Lastly the Saha equation for the degenerate star was found and a equation for the degree of ionization of hydrogen was found as a function of temperature.

References

[1] R. K. Pathria, *Statistical Mechanics.*

[2] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Dover, New Youk, 1939.

[3] D. Prialnik, *An Introduction to the Theory of Stellar Structure and Evolution*, Cambridge Univ. Press, Cambridge 2000.