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PHYSICS 517 QUANTUM MECHANICS II

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'NON – HILBERTIAN' QUANTUM MECHANICS ON THE FINITE GALOIS FIELD $\mathbb{Z}_q$

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1. Introduction

Quantum mechanics, as we have known for the past 88 years, rests upon the Hilbert vector space. In a recent paper published in January 8th 2013, Lay Nam Chang, Zachary Lewis, Djordje Minic, and Tatsu Takeuchi from Virginia Tech university proposed the idea of constructing an alternative 'Non – Hilbertian' "discrete quantum mechanics on the vector space $\mathbb{Z}_q^N$ over the finite Galois field $GF(q) = \mathbb{Z}_q$" (Source 1). This is an attempt to improve our understanding on mathematical foundations of quantum mechanics. Just as Non – Euclidean geometry has helped us understood the curvature structure of space – time in general relativity, the replacement of Hilbert space with the finite Galois field as the space on which quantum state vectors are defined could potentially lead to invaluable insights into quantum mechanics at its most fundamental level. However, in order for the new algebraic system defined on $\mathbb{Z}_q^N$ to be qualified as a truly quantum mechanical system, it needs to be shown that "no theory of local hidden variables can reproduce all the predictions of the system" (Sources 1 and 3). A common way to do this is to show that our algebraic construction violates the Clause – Horne – Shimony – Holt (CHSH) inequality which set an upper bound on the correlation between two distant events. If the local hidden variable, which connect two distant events, really exist then this correlation upper bound is found to be 2 (Sources 1 and 3). The mathematical structure of quantum mechanics on Hilbert space forces this upper bound to be $2\sqrt{2} > 2$ which means that quantum mechanics on Hilbert space violate the CHSH inequality. So, violation of the CHSH inequality is a confirmation test to show that whether a system is quantum mechanical or not (Source 1). However, our new system on finite Galois field turns out not to violate the CHSH inequality, and the correlation upper bound for this system turns out to be the classical hidden variable value 2. Nevertheless, the authors were able to show some results derived from the mathematical structure of the new system on $\mathbb{Z}_q$ that contradicts with the predictions of classical local hidden variables theory. Thus they conclude that "the new system cannot be fully described by any local hidden variable theory" (Source 1). Therefore, the new system is qualified as a quantum mechanical system.

2. Algebraic construction of the new quantum theory

The heart of this new quantum construction is the idea of replacing the Hilbert space with a finite field $H_q = \mathbb{Z}_q^N$ as the 'space' in which the N - dimensional state vectors $|\psi\rangle$ of the quantum system are defined (1). Here according to Galois theorem, $\mathbb{Z}_q$ is the unique finite field of order q known as "Galois field GF(q) where $q = p^n$ with p a prime number and $n \in \mathbb{N}$" (Sources 1 and 2). When n=1, GF(q) = GF(p) = $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$ (1 and 2). The outcome of a measurement made on the state $|\psi\rangle$ is defined to be dual vector $\langle x|$ in the dual space H_q^*. "The probability of obtaining a result $\langle x|$ when a measurement of a certain observable O is made on the state $|\psi\rangle$ is defined as":

$$P(\langle x|\psi\rangle) = \frac{|\langle x|\psi\rangle|^2}{\sum_i |\langle i|\psi\rangle|^2} \text{ (1 and 2)}$$

Where the sum in the denominator runs over the set of all possible results of the observable O. Here, the number $|\langle x|\psi\rangle|$, where $\langle x|\psi\rangle \in \mathbb{Z}_q$, is defined using the following function:

$$|\underline{k}| = \begin{cases} 0, & if \ \underline{k} = \underline{0} \\ 1, & if \ \underline{k} \neq \underline{0} \end{cases} \text{ (Eqn 1) (1 and 2)}$$

The underline bar is used to denote elements in $\mathbb{Z}_q$. The 'state space' H_q is given the finite projective geometry $PG(N-1, q) = (\mathbb{Z}_q^N \backslash \{\underline{\mathbf{0}}\})/(\mathbb{Z}_q \backslash \{\underline{0}\})$ where $\underline{\mathbf{0}}$ is the $N - vector$ $(\underline{0}, ..., \underline{0})$ in $\mathbb{Z}_q^N$. "The projective geometry PG(N – 1, F) of an N – vector space V over a field F is the geometry whose points, lines, planes,… are the vector subspaces of V of dimensions 1, 2, 3 …" (Source 4). "The group of all possible basis transformations of the space H_q is the projective linear group $PGL(N, q) = GL(N, q)/Z(N, q)$ where $GL(N, q)$ is the general linear group of $\mathbb{Z}_q^N$ and $Z(N, q)$ is the center of $GL(N, q)$" (Source 1). "The general linear group $GL(N, q)$ is the group of all non – singular linear transformations of $\mathbb{Z}_q^N$. The representation of the group $GL(N, q)$ is the group of invertible N×N matrices with entries in $\mathbb{Z}_q$" (Source 1).

The absolute value function defined above for the elements of $\mathbb{Z}_q$ is the only one which satisfies the product preserving condition $|\underline{k} \ \underline{l}| = |\underline{k}||\underline{l}|$ required in order for "the probabilities of product observables on product states to be factorizable into product of individual observable on individual state" (Source 2)

$$P(\langle xy|\psi\phi\rangle) = \frac{|(\langle x|\times\langle y|)(|\psi\rangle\times|\phi\rangle)|^2}{\sum_{zw}|(\langle z|\times\langle w|)(|\psi\rangle\times|\phi\rangle)|^2} = \frac{|\langle x|\psi\rangle\langle y|\phi\rangle|^2}{\sum_{zw}|\langle z|\psi\rangle\langle w|\phi\rangle|^2} = \frac{|\langle x|\psi\rangle|^2|\langle y|\phi\rangle|^2}{\sum_{zw}|\langle z|\psi\rangle|^2|\langle w|\phi\rangle|^2} =$$
$$\frac{|\langle x|\psi\rangle|^2}{\sum_z|\langle z|\psi\rangle|^2}\frac{|\langle y|\phi\rangle|^2}{\sum_w|\langle w|\phi\rangle|^2} = P(\langle x|\psi\rangle)P(\langle y|\phi\rangle) \text{ (Sources 1 and 2)}$$

The quantum system built on the above specified algebraic structure is called Galois field quantum mechanics, abbreviated as GQM. If our GQM model is built over the vector space $\mathbb{Z}_q^N$, then we denote it as GQM(N, q). "Spin – like systems with two possible outcomes ± 1 for the

spin operator can be constructed on the vector space $V_q = \mathbb{Z}_q^2$ as GQM(2, q). Two – particle spin – like systems can be constructed as GQM(4, q) on the space $V_q \times V_q = \mathbb{Z}_q^2 \times \mathbb{Z}_q^2 = \mathbb{Z}_q^4$". For each system of GQM(N, q), we have $q^N - 1$ non – zero quantum states (Source 1). In this paper I will demonstrate in detail the case q = 2 because the mechanisms of the cases of higher values of q is exactly the same as that of the case q = 2.

3. Hidden variable theory and CHSH inequality

Einstein, Podolsky, and Rosen proposed a paradox, known collectively as the EPR paradox, which led them to conclude that quantum theory was incomplete unless a local hidden variable existed. Their idea became known as the local hidden variable theory (Source 3). The local hidden variable theory states that distant events have no instantaneous (or faster than light) effects on local events. "However, in 1964, John Stewart Bell showed that in a Gedanken experiment of Bohm, no physical theory of local hidden variables can ever reproduce all of the predictions of quantum mechanics" (Source 3). In 1969, John Clauser, Michael Horne, Abner Shimony, and Richard Holt proposed an experimental test to confirm Bell's theorem. This experiment led directly to the Clauser – Horne – Shimony – Holt inequality (CHSH inequality) which set an upper bound to how much correlated two distant events could be if local hidden variable theory were true (Source 3). This upper bound turned out to be 2. Violation of the CHSH inequality, i.e. a correlation upper bound greater than 2, is a necessary step to show that a system does not behave according to local hidden variable theory (Source 1). The mathematical structure of the regular quantum theory on Hilbert space requires this correlation upper bound to be $2\sqrt{2} > 2$ (Source 1). Therefore, quantum mechanics on Hilbert space violates the CHSH inequality. This leads to the conclusion that quantum mechanics cannot by described by the local hidden variable theory. So, in order for an algebraic system to be qualified as a quantum system, it needs to be shown, either through violation of CHSH inequality or some other way, that no local hidden variable theory can describe the system (Source 1). This is the argument used in this paper to show that the quantum system defined over the finite Galois field $\mathbb{Z}_q$ is indeed a truly quantum system (Source 1).

In the experiment proposed by Clause, Horne, Shimony, and Holt, a source simultaneously ejects two identical particles in opposite directions. One particle enters the apparatus A and the other enters the apparatus B. A and B are two – channel systems where the two channels are +1 and -1. So, a particle entering an apparatus will have to choose whether to go through channel +1 or -1. Let a and b be adjustable apparatus parameters for systems A and B, so A(a) and B(b) will have values ± 1 depending on which channel the particle decides to go through once it has entered an apparatus (Source 3). There is a coincidence monitor which increments by 1 every time there is a pair of particle that have A(a) = B(b). Suppose, the two particles share some common information, then this common information is the set of locally hidden variables carried within each particle (Source 3). The set of locally hidden variables are denoted collectively as λ. So, the result of channel selection by a particle within an apparatus, either A or B, is described by

$A(a, \lambda)$ and $B(b, \lambda)$. Once again, $A(a, \lambda)$ and $B(b, \lambda)$ take on values of ± 1 depending on which channel of the apparatus the particle selects (Source 3). "The probability of the result of channel selection by each particle does not depend on the apparatus parameter a or b because each particle is ejected from a source which is physically independent from the apparatus. Thus, the probability of channel selection depends only λ. Therefore, the probability distribution of channel selection $\rho(\lambda)$ is defined over the domain of the common information λ" (Source 3).

"The correlation function is defined as $P(a, b) = \int_\Gamma A(a, \lambda)B(b, \lambda)\rho(\lambda)d\lambda$ where Γ is the domain of λ" (Source 3). Using triangle inequality, we have:

$|P(a, b) - P(a, c)| \leq \int_\Gamma |A(a, \lambda)B(b, \lambda) - A(a, \lambda)B(c, \lambda)|\rho(\lambda)d\lambda = \int_\Gamma |A(a, \lambda)B(b, \lambda)| [1 - B(b, \lambda)B(c, \lambda)]\rho(\lambda)d\lambda$ (1) (Source 3)

Because A and B takes on values $\pm 1 \Rightarrow |A(a, \lambda)B(b, \lambda)| = 1$.

So: $|P(a, b) - P(a, c)| \leq \int_\Gamma [1 - B(b, \lambda)B(c, \lambda)]\rho(\lambda)d\lambda = 1 - \int_\Gamma B(b, \lambda)B(c, \lambda)\rho(\lambda)d\lambda$ (*) (Source 3)

The correlation function P takes on values between 0 and 1 with 1indicating 100% correlation and 0 indicating no correlation. Suppose that $P(b', b) = 1 - \delta \; where \; 0 \leq \delta \leq 1 \Rightarrow \delta = 1 - P(b', b)$ ($Source$ 3). We can now divide the Γ domain into two parts $\Gamma_+ \; and \; \Gamma_-$ such that $\Gamma_\pm = \{\lambda | A(b', \lambda) = \pm B(b, \lambda)\}$, then $\int_{\Gamma_-} \rho(\lambda)d\lambda = \frac{1}{2}\delta$ (Source 3). Also, we have:

$\int_\Gamma B(b, \lambda)B(c, \lambda)\rho(\lambda)d\lambda = \int_\Gamma A(b', \lambda)B(c, \lambda)\rho(\lambda)d\lambda - 2\int_{\Gamma_-} A(b', \lambda)B(c, \lambda)\rho(\lambda)d\lambda \geq$

$P(b', c) - 2\int_{\Gamma_-} |A(b', \lambda)B(c, \lambda)|\rho(\lambda)d\lambda$ (Source 3)

Because: $|A(b', \lambda)B(c, \lambda)| = 1 \Rightarrow 2\int_{\Gamma_-} |A(b', \lambda)B(c, \lambda)|\rho(\lambda)d\lambda = 2\int_{\Gamma_-} \rho(\lambda)d\lambda = 2\frac{1}{2}\delta = \delta$

$\Rightarrow \int_\Gamma B(b, \lambda)B(c, \lambda)\rho(\lambda)d\lambda \geq P(b', c) - \delta = P(b', c) + P(b', b) - 1$(**) (Source 3)

From (*) and
(**)$\Rightarrow |P(a, b) - P(a, c)| \leq 2 - P(b', c) - P(b', b) \Rightarrow$
$\begin{cases} P(a, b) - P(a, c) \leq 2 - P(b', c) - P(b', b) \\ P(a, b) - P(a, c) \geq -2 + P(b', c) + P(b', b) \end{cases}$ (Source 3)

But since: $P(b', c) + P(b', b) \geq 0 \geq -P(b', c) - P(b', b) \Rightarrow -2 + P(b', c) + P(b', b) \geq -2 - P(b', c) - P(b', b)$

$\Rightarrow -2 - P(b', c) - P(b', b) \leq P(a, b) - P(a, c) \leq 2 - P(b', c) - P(b', b)$

$\Rightarrow |P(a, b) - P(a, c) + P(b', c) + P(b', b)| \leq 2$ (Source 3)

This is the most popular form of the CHSH inequality. From the above derivation, we can see where the upper bound of 2 came from if a local hidden variable existed. The most common schematic for the CHSH experiment is shown below:
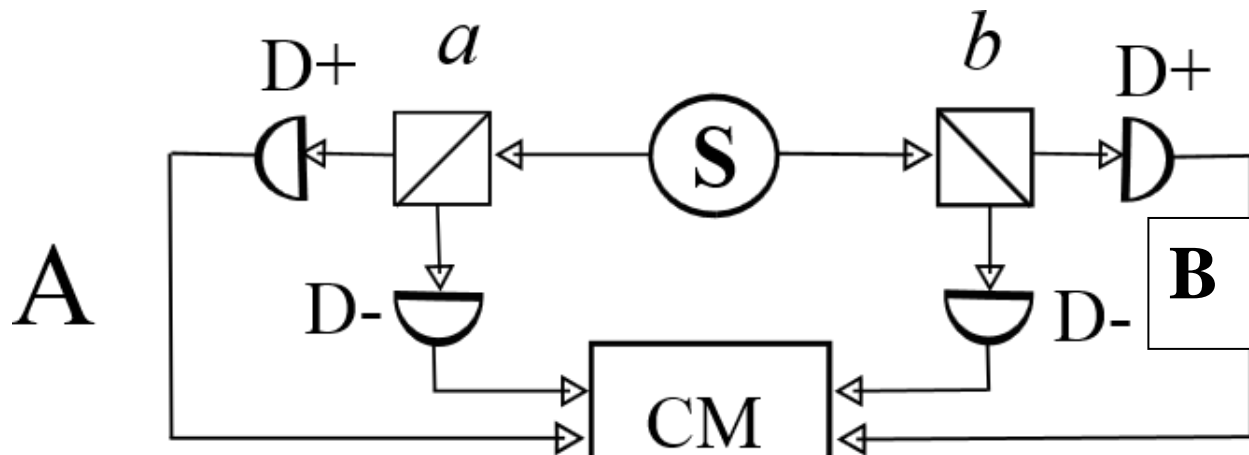

FIGURE 1. Basic schematic of CHSH experiment. A pair of particles is ejected simultaneously from a common source S into opposite directions. Each particle will enter a two – channel ± 1 system A or B with an adjustable apparatus parameter a or b. A detector is placed in each channel to detector the particle, detector D^+ is used to detect the particle if it enters channel +1 of the apparatus and detector D^- is used in channel -1 of the apparatus. The coincidence monitor CM will increment by 1 if the two particles choose the same type of channel (i.e. either both +1 or both -1) in each apparatus (Source 5).

For the quantum system defined on the vector space $\mathbb{Z}_q^N$ presented in this paper, the CHSB bound is not the common value $2\sqrt{2}$ for regular quantum system on Hilbert space but turns out to be 2 which is classical hidden variable value (Source 1). However, the authors argued that the derivation of the CHSH bound value of $2\sqrt{2}$ for quantum system in Hilbert space relies heavily on the inner product operation which the vector space $\mathbb{Z}_q^N$ does not have because of the cyclicity of the Galois field (Source 1). Despite the fact that the quantum system on Galois field does not violate the CHSH inequality, the authors were able to derive some results that contradict the predictions of classical hidden variable theory. Therefore, the newly defined quantum system cannot be described by any theory of hidden variables, and thus is a truly quantum system.

4. Quantum mechanics on $\mathbb{Z}_2$

The simplest quantum mechanical system on $\mathbb{Z}_q$ is the $2-$dimensional system on $\mathbb{Z}_2$, denoted as GQM(2,2). There are only two elements in $\mathbb{Z}_2 = \{\underline{0}, \underline{1}\}$ with the multiplication and addition tables:

+	$\underline{0}$	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{1}$
$\underline{1}$	$\underline{1}$	$\underline{0}$

(Source 1)

×	$\underline{0}$	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$

(Source 1)

There are only $2^2 - 1 = 3$ possible non$-$zero quantum states in this system:

$|a\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |b\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, |c\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (Source 1)

Any state of among these 3 states can be written as a combination of the other 2:

$|a\rangle = |b\rangle + |c\rangle; \ |c\rangle = |a\rangle + |b\rangle; \ |b\rangle = |c\rangle + |a\rangle$ (Source 1)

Therefore, we can choose any 2 vectors to be the basis for the vector space $\mathbb{Z}_2^2$ of the $\mathbb{Z}_2$ quantum mechanics (Source 1). The group of all possible basis transformations for this vector space is $S_3 \cong PGL(2,2)$ (Source 1). The dual vectors corresponding to the above 3 non$-$zero state vectors are defined as:

$\langle \bar{a}| = [\underline{0}\ \underline{1}]; \langle \bar{b}| = [\underline{1}\ \underline{0}]; \langle \bar{c}| = [\underline{1}\ \underline{1}]$ (Source 1)

We also have the definition:

$\langle \bar{r}|s\rangle = \begin{cases} \underline{0}, if\ r = s \\ \underline{1}, if\ r \neq s \end{cases}$ (Eqn 2) (Source 1)

So from (Eqn 1) and (Eqn 2)$\implies |\langle \bar{r}|s\rangle| = 1 - \delta_{rs}$ (Eqn 3) (source 1)

Just like for the vector space $\mathbb{Z}_2^2$, we can choose any 2 of the above 3 non – zero dual vectors to form the basis for the dual vector space $(\mathbb{Z}_2^2)^*$. We have 3 dual vectors, and we need to select 2 of them to form the basis for the dual space, so have $\binom{3}{2} = 3$ possible selections for the basis. An observable for this quantum system is defined to be one the 3 possible selections for the pair of dual vectors above. So, we have:

Observable $A_{rs} = \{\langle \bar{r}|, \langle \bar{s}|\}$ where $\langle \bar{r}|$ *and* $\langle \bar{s}|$ are two of the 3 non – zero dual vectors $\langle \bar{a}|, \langle \bar{b}|, \langle \bar{c}|$ defined above (Source 1.)

Also, $\langle \bar{r}|$ represents the outcome +1 and $\langle \bar{s}|$ represents the outcome of -1 when observable A_{rs} is measured and $A_{rs} = -A_{sr}$ (Source 1).

Since we have 3 possible selections of the pair of dual vectors, we have 3 observables $A_{ab}, A_{bc}, and\ A_{ca}$. Let us denote $X = A_{bc}, Y = A_{ca}, and\ Z = A_{ab}$ (Source 1).

X, Y, and Z transform into each other under the permutation group $S_3 = \{(ab), (bc), (ca), (abc), (acb), (Id)\}$ where the cycle (Id) represents the identity permutation, i.e. no permutation, which is trivial so we will not consider it here (Source 1). In this situation, (abc) means a gets changed into b, b gets changed into c, and c gets changed into a; so the subscript indices of an observable, say A_{ab}, will be changed into bc under the cycle (abc). So, $(abc)A_{ab} = A_{bc}$. Below is the list of all non – trivial transformations of the observables X, Y, and Z (Source 1):

	X	Y	Z
(ab)	-Y	-X	-Z
(bc)	-X	-Z	-Y
(ca)	-Z	-Y	-X
(abc)	+Y	+Z	+X
(acb)	+Z	+X	+Y

"The six cycles in S_3 can be mapped into the six rotations in the dihedral group D_3 which rotate the triangle abc in the figure below into itself." (Source 1)
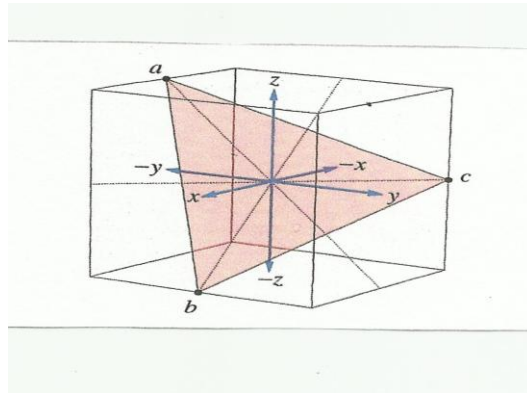

FIGURE 2. The six rotation axes that map the triangle Δabc onto itself (Source 1)

As mentioned previously, in this $\mathbb{Z}_2$ quantum mechanics, there are only two eigenvalues ± 1 for all the observables in this system. So, let us take the observable A_{ab} for an example, it is defined as $A_{ab} = \{\langle \overline{a}|, \langle \overline{b}|\}$. When the observable A_{ab} operates on a certain state $|\psi\rangle$, the outcome value could only be +1 or -1, with $\langle \overline{a}|$ representing $\langle +|$ and $\langle \overline{b}|$ representing $\langle -|$. So, the probability of getting an outcome of +1 when A_{ab} operates on $|\psi\rangle$ is:

$$P(A_{ab}; \langle +|\psi\rangle) = \frac{|\langle +|\psi\rangle|^2}{|\langle +|\psi\rangle|^2 + |\langle -|\psi\rangle|^2} = \frac{|\langle \overline{a}|\psi\rangle|^2}{|\langle \overline{a}|\psi\rangle|^2 + |\langle \overline{b}|\psi\rangle|^2} \text{ (Eqn 4)}$$

Where $|\langle \overline{a}|\psi\rangle|$ and $\left|\langle \overline{b}|\psi\rangle\right|$ are calculated according to (Eqn 3):

$$|\langle \overline{a}|\psi\rangle| = 1 - \delta_{a\psi} \text{ and } \left|\langle \overline{b}|\psi\rangle\right| = 1 - \delta_{b\psi} \text{ (Eqn 5)}$$

And the outcome expectation value of the observable A_{ab} when operating on $|\psi\rangle$ is:

$$\langle A_{ab}\rangle_\psi = (+1) \times P(A_{ab}; \langle +|\psi\rangle) + (-1) \times P(A_{ab}; \langle -|\psi\rangle) \text{ (Eqn 6)}$$

Since we only have 3 states $|a\rangle, |b\rangle, |c\rangle$, the outcome probabilities and when the observable A_{ab} operates on them are calculated based on (Eqn 4) and (Eqn 5) as follows:

$$P(A_{ab}; \langle +|a\rangle) = \frac{|\langle \overline{a}|a\rangle|^2}{|\langle \overline{a}|b\rangle|^2 + \left|\langle \overline{b}|a\rangle\right|^2} = 0 \text{ (Source 1)}$$

$$P(A_{ab}; \langle -|a\rangle) = \frac{\left|\langle \overline{b}|a\rangle\right|^2}{|\langle \overline{a}|b\rangle|^2 + \left|\langle \overline{b}|a\rangle\right|^2} = 1 \text{ (Source 1)}$$

$$P(A_{ab}; \langle +|b\rangle) = \frac{|\langle \overline{a}|b\rangle|^2}{|\langle \overline{a}|b\rangle|^2 + \left|\langle \overline{b}|b\rangle\right|^2} = 1 \text{ (Source 1)}$$

$$P(A_{ab}; \langle -|b\rangle) = \frac{|\langle \bar{b}|b\rangle|^2}{|\langle \bar{a}|b\rangle|^2 + |\langle \bar{b}|b\rangle|^2} = 0 \text{ (Source 1)}$$

$$P(A_{ab}; \langle +|c\rangle) = \frac{|\langle \bar{a}|c\rangle|^2}{|\langle \bar{a}|c\rangle|^2 + |\langle \bar{b}|c\rangle|^2} = \frac{1}{2} \text{ (Source 1)}$$

$$P(A_{ab}; \langle -|c\rangle) = \frac{|\langle \bar{b}|c\rangle|^2}{|\langle \bar{a}|c\rangle|^2 + |\langle \bar{b}|c\rangle|^2} = \frac{1}{2} \text{ (Source 1)}$$

The expectation values of A_{ab} when operating on $|a\rangle, |b\rangle, |c\rangle$ are calculated based on (Eqn 6) as follow:

$$\langle A_{ab}\rangle_a = (+1) \times (0) + (-1) \times (1) = -1 \text{ (Source 1)}$$

$$\langle A_{ab}\rangle_b = (+1) \times (1) + (-1) \times (0) = +1 \text{ (Source 1)}$$

$$\langle A_{ab}\rangle_c = (+1) \times \left(\frac{1}{2}\right) + (-1) \times \left(\frac{1}{2}\right) = 0 \text{ (Source 1)}$$

Like wise we can compute the outcome probabilities and expectation values for A_{bc} and A_{ca}, and the list of all the probabilities and expectation values are shown below:

Observable	State	P(+)	P(-)	Expectation Value
A_{ab}	$|a\rangle$	0	1	-1
	$|b\rangle$	1	0	+1
	$|c\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	0
A_{bc}	$|a\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	0
	$|b\rangle$	0	1	-1
	$|c\rangle$	1	0	+1
A_{ca}	$|a\rangle$	1	0	+1
	$|b\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	0
	$|c\rangle$	0	1	-1

(Source 1)

5. Two – particle system in $\mathbb{Z}_2$ quantum mechanics

Each particle in $\mathbb{Z}_2$ quantum mechanics exist in the space $\mathbb{Z}_2^2$, so the space for 2 – particle system in $\mathbb{Z}_2$ quantum mechanics is $\mathbb{Z}_2^4$. This Galois quantum mechanics for the 2 – particle system is denoted as GQM(4, 2). There are $2^4 - 1 = 15$ non – zero quantum states which divide themselves into 9 pure states and 6 entangled states. The nine pure states are defined as follow:

$$|aa\rangle = |a\rangle \times |a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|ab\rangle = |a\rangle \times |b\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|ac\rangle = |a\rangle \times |c\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|ba\rangle = |b\rangle \times |a\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|bb\rangle = |b\rangle \times |b\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|bc\rangle = |b\rangle \times |c\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|ca\rangle = |c\rangle \times |a\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|cb\rangle = |c\rangle \times |b\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|cc\rangle = |c\rangle \times |c\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

The six entangled states are defined as follow:

$$|S\rangle = |aa\rangle + |bb\rangle + |cc\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

This state transforms into itself under all permutations of S_3

The other 5 entangled states are:

$$|(ab)\rangle = |ab\rangle + |ba\rangle + |cc\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|(bc)\rangle = |aa\rangle + |bc\rangle + |cb\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ (Source 1)}$$

$$|(ca)\rangle = |ac\rangle + |bb\rangle + |ca\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|(abc)\rangle = |ab\rangle + |bc\rangle + |ca\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

$$|(acb)\rangle = |ac\rangle + |cb\rangle + |ba\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ (Source 1)}$$

Similar to the case of $1-$particle system, we can define dual vectors in the dual space $(\mathbb{Z}_2^4)^*$ for the $2-$particle system. However, we can only define the dual vectors corresponding to the 9 pure state vectors:

$$\langle \bar{a}| \times \langle \bar{a}| = [0 \quad 0 \quad 0 \quad 1] \text{ (Source 1)}$$

$$\langle \bar{a}| \times \langle \bar{b}| = [0 \quad 0 \quad 1 \quad 0] \text{ (Source 1)}$$

$$\langle \bar{a}| \times \langle \bar{c}| = [0 \quad 0 \quad 1 \quad 1] \text{ (Source 1)}$$

$\langle\overline{b}|\times\langle\overline{a}| = [\underline{0} \quad \underline{1} \quad \underline{0} \quad \underline{0}]$ (Source 1)

$\langle\overline{b}|\times\langle\overline{b}| = [\underline{1} \quad \underline{0} \quad \underline{0} \quad \underline{0}]$ (Source 1)

$\langle\overline{b}|\times\langle\overline{c}| = [\underline{1} \quad \underline{1} \quad \underline{0} \quad \underline{0}]$ (Source 1)

$\langle\overline{c}|\times\langle\overline{a}| = [\underline{0} \quad \underline{1} \quad \underline{0} \quad \underline{1}]$ (Source 1)

$\langle\overline{c}|\times\langle\overline{b}| = [\underline{1} \quad \underline{0} \quad \underline{1} \quad \underline{0}]$ (Source 1)

$\langle\overline{c}|\times\langle\overline{c}| = [\underline{1} \quad \underline{1} \quad \underline{1} \quad \underline{1}]$ (Source 1)

Also similar to the case of 1 – particle system, we can define the observables for the 2 – particle system as the selections of basis for the dual space $(\mathbb{Z}_2^4)^*$:

$A_{rs}A_{tu} = \{\langle\overline{r}|\times\langle\overline{t}|, \langle\overline{r}|\times\langle\overline{u}|, \langle\overline{s}|\times\langle\overline{t}|, \langle\overline{s}|\times\langle\overline{u}|\}$ where $rs, tu \in \{ab, bc, ca\}$ (Source 1)

So there are a total of 9 observables. Suppose we have a state
$|\psi\rangle = |\psi_1\rangle \times |\psi_2\rangle$, then $A_{rs}A_{tu}|\psi\rangle = (A_{rs}|\psi_1\rangle) \times (A_{tu}|\psi_2\rangle)$

So an operator of a 2 – particle system state consists of one operator on the state of the 1 first particle, and another operator on the state of the second particle. Let the set of operators on the first particle be $\{X_1, Y_1, Z_1\}$ and the set of operators on the second operators be $\{X_2, Y_2, Z_2\}$ where $\{X_1, Y_1, Z_1\} = \{X_2, Y_2, Z_2\} = \{A_{bc}, A_{ca}, A_{ab}\}$.

Suppose we have a 2 – particle operator $A_{ab}A_{bc} = \{\langle\overline{a}|\times\langle\overline{b}|, \langle\overline{a}|\times\langle\overline{c}|, \langle\overline{b}|\times\langle\overline{b}|, \langle\overline{b}|\times\langle\overline{c}|\}$ operating on a state $|\psi\rangle = |\psi_1\rangle \times |\psi_2\rangle$. The vector $\langle\overline{a}|$ represent the +1 outcome and $\langle\overline{b}|$ represents the -1 outcome when A_{ab} is measured, but $\langle\overline{b}|$ represents +1 when A_{bc} is measured and $\langle\overline{c}|$ represents -1 when A_{bc} is measured. So, $\langle\overline{a}| = \langle+|$ and $\langle\overline{b}| = \langle-|$ for the first particle and $\langle\overline{b}| = \langle+|$ and $\langle\overline{c}| = \langle-|$ for the second particle $\Rightarrow A_{ab}A_{bc} = \{\langle+|\times\langle+|, \langle+|\times\langle-|, \langle-|\times\langle+|, \langle-|\times\langle-|\}$. Probabilities for the outcomes are calculated as follow:

$P(A_{ab}A_{bc}; \langle+ - |\psi\rangle) = \frac{|\langle+-|\psi_1\psi_2\rangle|^2}{\sum_{ij}|\langle ij|\psi_1\psi_2\rangle|^2} = \frac{|(\langle+|\times\langle-|)(|\psi_1\rangle\times|\psi_2\rangle)|^2}{\sum_{ij}|(\langle i|\times\langle j|)(|\psi_1\rangle\times|\psi_2\rangle)|^2} = \frac{|\langle+|\psi_1\rangle|^2|\langle-|\psi_2\rangle|^2}{\sum_i\sum_j|\langle i|\psi_1\rangle|^2|\langle j|\psi_2\rangle|^2} =$

$\frac{|\langle+|\psi_1\rangle|^2}{|\langle+|\psi_1\rangle|^2+|\langle-|\psi_1\rangle|^2}\frac{|\langle-|\psi_2\rangle|^2}{|\langle+|\psi_2\rangle|^2+|\langle-|\psi_2\rangle|^2} = P(A_{ab}; \langle+|\psi_1\rangle)P(A_{bc}; \langle-|\psi_2\rangle)$ (Eqn 7)

Similarly,

$P(A_{ab}A_{bc}; \langle+ - |\psi\rangle) = P(A_{ab}; \langle+|\psi_1\rangle)P(A_{bc}; \langle-|\psi_2\rangle);$

$P(A_{ab}A_{bc}; \langle- + |\psi\rangle) = P(A_{ab}; \langle-|\psi_1\rangle)P(A_{bc}; \langle+|\psi_2\rangle);$

$P(A_{ab}A_{bc}; \langle+ + |\psi\rangle) = P(A_{ab}; \langle+|\psi_1\rangle)P(A_{bc}; \langle+|\psi_2\rangle);$

$P(A_{ab}A_{bc}; \langle -- |\psi\rangle) = P(A_{ab}; \langle -|\psi_1\rangle)P(A_{bc}; \langle -|\psi_2\rangle);$

The outcome expectation when $A_{ab}A_{bc}$ operates on $|\psi\rangle = |\psi_1\rangle \times |\psi_2\rangle$ is calculated as:

$\langle A_{ab}A_{bc}\rangle_\psi = \sum_{ij} ij P(A_{ab}A_{bc}; \langle ij|\psi\rangle) = \sum_{ij} ij P(A_{ab}; \langle i|\psi_1\rangle)P(A_{bc}; \langle j|\psi_2\rangle) = (\sum_i i P(A_{ab}; \langle i|\psi_1\rangle)) (\sum_j j P(A_{bc}; \langle j|\psi_2\rangle)) = \langle A_{ab}\rangle_{\psi_1} \langle A_{bc}\rangle_{\psi_2}$ (Eqn 8)

Based on the above rules specified in (Eqn 7) and (Eqn 8), we can calculate all the outcome probabilities and expectation values for all the 9 observables of the 2 – particle system. The results are tabulated below:

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
X_1X_2	$|S\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$|(ab)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$|(bc)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$|(ca)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$|(abc)\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	$|(acb)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
X_1Y_2	$|S\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$|(ab)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$|(bc)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$|(ca)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$|(abc)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$|(acb)\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
X_1Z_2	$\lvert S\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert(ab)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert(bc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert(ca)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert(abc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert(acb)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
Y_1X_2	$\lvert S\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert(ab)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert(bc)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert(ca)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert(abc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert(acb)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
Y_1Y_2	$\lvert S\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$\lvert(ab)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert(bc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert(ca)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert(abc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert(acb)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
$Y_1 Z_2$	$\lvert S\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert (ab)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert (bc)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert (ca)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert (abc)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$\lvert (acb)\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
$Z_1 X_2$	$\lvert S\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert (ab)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert (bc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert (ca)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert (abc)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$\lvert (acb)\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
Z_1Y_2	$\lvert S\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert (ab)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert (bc)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert (ca)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert (abc)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert (acb)\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1

(Source 1)

Observable	State	P(+ +)	P(+ -)	P(- +)	P(- -)	Expectation Value
Z_1Z_2	$\lvert S\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$\lvert (ab)\rangle$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	$\lvert (bc)\rangle$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	$\lvert (ca)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	$\lvert (abc)\rangle$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	$\lvert (acb)\rangle$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$

(Source 1)

6. Contradictions with classical local hidden variable theory

According to the above probabilities tables, $P(X_1; \langle +\lvert S\rangle)P(Z_2; \langle -\lvert S\rangle) = P(X_1Z_2; \langle + -\lvert S\rangle) = 0$

The interpretation of the above equation is that for the 2 particles in the state $\lvert S\rangle$, if the outcome value when the operator X_1 operates on the 1$^{\text{st}}$ particle of the state $\lvert S\rangle$ is +1, i.e. $X_1 = +1$, then the outcome value when the operator Z_2 operates on the 2$^{\text{nd}}$ particle of the state $\lvert S\rangle$ CANNOT be -1, i.e. $Z_2 \neq -1$, which implies that Z_2 MUST BE +1 because it can only be either +1 or -1. So, the probability equation $P(X_1Z_2; \langle + -\lvert S\rangle) = 0$ says that $X_1 = +1\ IMPLIES\ Z_2 = +1, or\ X_1 = +1 \Rightarrow Z_2 = +1$ (I). Conversely, $P(X_1Z_2; \langle + -\lvert S\rangle) = 0$ also says that $Z_2 = -1 \Rightarrow X_1 = -1$.

Similarly from the probability $P(Y_1Z_2; \langle - +\lvert S\rangle) = 0$ we have $Z_2 = +1 \Rightarrow Y_1 = +1$ (II). And from $P(Y_1X_2; \langle + -\lvert S\rangle) = 0$ we have $Y_1 = +1 \Rightarrow X_2 = +1$ (III)

From (I) (II) and (III), we have the line of implication below:

$X_1 = +1 \Rightarrow Z_2 = +1 \Rightarrow Y_1 = +1 \Rightarrow X_2 = +1$ (❖)

Which means that $X_1 = +1 \Rightarrow X_2 = +1$ (IV)

If a classical local hidden variable theory were true for the this quantum system, the line of implication (❖) must be true, hence the implication (IV) must be true.

But we also have from the first probability table above that $P(X_1X_2; \langle + + |S\rangle) = 0$ which says

$X_1 = +1 \Rightarrow X_2 \neq +1$ or $X_1 = +1 \Rightarrow X_2 = -1$ which contradicts the implication (IV). So, we have just seen one prediction of our quantum system that contradicts the prediction of a local hidden variable theory if it were true on our system. Therefore, we can conclude that no local hidden variable theory can reproduce all the predictions of our quantum mechanical system on $\mathbb{Z}_q^N$. Therefore, the algebraic system constructed on the space $\mathbb{Z}_q^N$ is qualified as a truly quantum mechanical system.

Just to further solidify our reasoning, another contradiction with local hidden variable theory will be shown below:

From $P(X_1Z_2; \langle + - |S\rangle) = 0$, we have $Z_2 = -1 \Rightarrow X_1 = -1$

From $P(X_1Y_2; \langle - + |S\rangle) = 0$, we have $X_1 = -1 \Rightarrow Y_2 = -1$

From $P(Z_1Y_2; \langle + - |S\rangle) = 0$, we have $Y_2 = -1 \Rightarrow Z_1 = -1$

From $P(Z_1X_2; \langle - + |S\rangle) = 0$, we have $Z_1 = -1 \Rightarrow X_2 = -1$

So a classical hidden variable theory would predict $Z_2 = -1 \Rightarrow X_2 = -1$ (□)

From $P(Y_1X_2; \langle + - |S\rangle) = 0$, we have $Y_1 = +1 \Rightarrow X_2 = +1$ (□□)

From (□) and (□□), classical hidden variable theory would predict $Y_1 = +1 \Rightarrow Z_2 \neq -1$ which is equivalent to $P(Y_1Z_2; \langle + - |S\rangle) = 0$

So, classical hidden variable theory predicts that $P(Y_1Z_2; \langle + - |S\rangle) = 0$

But from our probabilities tables we have $P(Y_1Z_2; \langle + - |S\rangle) = \frac{1}{3}$

Therefore, the prediction of our quantum system in this case, once again, contradicts the prediction of local hidden variable theory. Hence, our construction is truly quantum.

7. CHSH bound

Let A_α and B_β be observables that operates on the states of the particles $\alpha\ and\ \beta$ respectively, where $\alpha\ and\ \beta$ take on values 1 and 2 since there are only 2 particle in our system (Source 1). Then the CHSH correlation function for a certain $2-$particle state $|\psi\rangle$ is defined as:

$\langle A_1, A_2; B_1, B_2\rangle_\psi = \langle A_1 B_1\rangle_\psi + \langle A_1 B_2\rangle_\psi + \langle A_2 B_1\rangle_\psi - \langle A_2 B_2\rangle_\psi$ where $\langle A_i B_j\rangle_\psi$ is just the expectation value of the observable $A_i B_j$ when it operates on $|\psi\rangle$ (Source 1)

Going through the list of all the 9 observables and all the states of this $2-$particle system, the authors obtained the following maximum correlation bounds for the state $|S\rangle$

$\langle X_1, Y_2; Y_1, X_2\rangle_S = -2\ and\ \langle X_1, Z_2; Y_1, Z_2\rangle_S = +2$ where once again $\{X_1, Y_1, Z_1\} = \{X_2, Y_2, Z_2\} = \{A_{bc}, A_{ca}, A_{ab}\}$ (Source 1)

The results obtained for other observables and other states are all $\pm\frac{2}{3}$. Therefore, the upper bound for the correlation of our $2-$particle system is 2, which is the CHSH limit for classical hidden variable theory (Source 1).

However, the fact that our system does not violate the CHSH bound does not mean that our system could be fully described by a local hidden variable theory. The contradictions we derived in previous section already showed that this is not the case.

So, violating the CHSH bound implies violating local hidden variable theory. But not violating CHSH bound DOES NOT imply not violating local hidden variable theory.

8. Conclusion

In our construction in this paper the dual vector $\langle\overline{r}|$ that we defined represents an outcome value of either +1 or -1 when an observable O is measured for a certain state $|\psi\rangle$. It does not, however, represent the conjugate transpose of a quantum state. Therefore, $\langle\overline{r}|\psi\rangle$ is an outcome value of the observable O when it is measured on $|\psi\rangle$ NOT the inner product between two states. Hence, (Eqn 2) and (Eqn 3) do not define an inner product. In fact, an inner product is impossible for our system because the Galois field is cyclic (Source 1). It is because of this lack of inner product that our system fails to violate the CHSH bound of 2 for classical hidden variable theory (Source 1). Because the authors argued that the derivation of the CHSN bound value of $2\sqrt{2}$ for the regular quantum mechanics on Hilbert space involves extensive usage of inner product, which is perfectly defined in Hilbert space (Source 1).

Also, the fact that the observables $\{A_{bc}, A_{ca}, A_{ab}\}$ defined in our system can transform into each other under the permutation group $S_3 = \{(ab), (bc), (ca), (abc), (acb), (Id)\}$ and the fact they can only take values of ±1, imply that these observables behave similarly to the spin operator in

Hilbert space quantum mechanics (Source 1). So we can think of our set of observables as the Galois field analog of the Hilbert space quantum mechanics spin operator.

Other than that, I think we have successfully 'synthesized' a mini quantum mechanical model on the finite Galois field $\mathbb{Z}_q$. I think that the idea of removing quantum mechanics from the Hilbert space and trying to rebuild it on a different algebraic structure is a uniquely interesting idea. Because it reveals some unknown aspects of the mathematical foundations of quantum theory. For example, only by examining the Galois field, are we able to find out the role of the inner product in producing the CHSH limit of $2\sqrt{2}$ for quantum theory on Hilbert space. There are many more yet to be discovered algebraic aspects of quantum theory that the new quantum idea promises to offer. In this respect, I think the authors of this Galois field quantum mechanics have found a new route to better understanding of quantum theory.

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