Investigation into the Lamb Shift

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(Dated: May 18, 2010)

Abstract

This paper studies the experiment leading to the discovery of the Lamb Shift. Upon discovery, a discrepancy was formed between accepted theoretical values and experimental values of the energy levels of the $2S_{\frac{1}{2}}$ and the $2P_{\frac{1}{2}}$ states. We discuss and work through a derivation of the Lamb Shift made originally by Hans A. Bethe shortly after its discovery.

I. INTRODUCTION

Much of quantum mechanics has developed based on our understanding of the hydrogen atom. For years, experimentalists studied the hydrogen spectrum, and in 1947 Willis E. Lamb and R. C. Retherford measured an energy shift between the $2S_{\frac{1}{2}}$ and the $2P_{\frac{1}{2}}$ states of the hydrogen atom. The two states were previously thought to be degenerate, as predicted by both Schrödinger's and Dirac's formulations of quantum mechanics. This shift in energy was due to the Lamb Shift. This 'Lamb Shift' was the frequency of the electromagnetic radiation (microwaves) that induced excitations from one state of the hydrogen atom to another. The shift occurs because the atom is interacting with the vacuum surrounding it by absorbing and emitting virtual photons.

In 1928, Dirac discovered an equation that described an electron as having wave-like properties, spin, charge, varying mass, and magnetic moment. He predicted the energy levels of the hydrogen atom with very high precision. Under Coulomb's Law of Attraction, Dirac predicted the $2S_{\frac{1}{2}}$ and the $2P_{\frac{1}{2}}$ energy levels would be equal. Studies were made in attempts to prove or disprove Dirac, but until the late 1940s, no definitive decision was made as to the validity of his theory.

In 1940s, Willis E. Lamb and R. C. Retherford carried out several experiments in microwave spectroscopy in an effort to improve on previous attempts to detect absorption of short-wavelength radio waves in a gas discharge of atomic hydrogen. They noticed that the transition frequency did not match that predicted by traditional Quantum Electrodynamics (QED), the study of electromagnetic fields on atomic structures. The disagreement between Lamb's work and QED was first presented in the summer of 1947 at the Theoretical Physics Conference on Shelter Island, NY; this discrepancy intrigued conference attendee Hans A. Bethe, a theoretical particle physicist from Cornell. Bethe calculated the relativistic corrections to the hydrogen spectrum on the train ride from the conference. In Section II, we discuss Lamb and Retherford's experiment and results. Section III walks through the calculation that Bethe performed on the train returning from the Theoretical Physics Conference in 1947. And, finally, Section IV offers some results and discussions of the Lamb Shift.

II. THE EXPERIMENT

In 1947, Lamb and Retherford experimentally measured the energy difference between the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states. The experiment consists of a beam of ground state hydrogen atoms emerging from a slit in a tungsten oven. After emerging from the oven, the beam is struck by a beam of electrons, which excites about one in 100 million atoms into the $2S_{\frac{1}{2}}$ state (a metastable excited state with a lifetime of about 1 s). The atoms then pass through a region where they are exposed to radio waves and then continue on to a metal surface (tungsten foil). Once the excited atoms hit the foil, they decay into the ground state releasing electrons from the metal. A sketch of the setup is shown in Figure 1. The frequencies of the possible

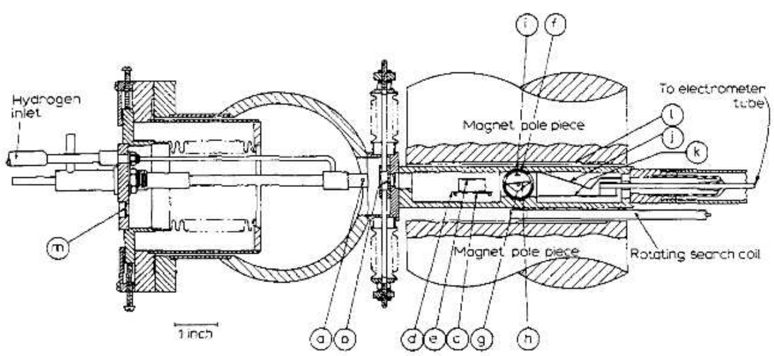

FIG. 1: A sketch of Lamb and Retherford's setup.[1]

transitions depend on magnetic field; thus the entire experiment is carried out in a magnetic field created by a small electromagnet used during World War II for testing magnetrons and reached about 3000 gauss or more.[1] Lamb and Retherford varied this magnetic field and noted that when the field was on, fewer electrons were released from the metal foil and when the field was off, the emission current was higher. The two looked at resonance curves of both Hydrogen and Deuterium. After careful analysis of these resonance curves, Lamb and Retherford determined the energy separation between the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states in zero magnetic field to be $\approx 1000\,\mathrm{Mc/sec}$.[1] Figure 2 shows the comparison between Lamb and Retherford's experimental data and the theoretical curves calculated for the Zeeman effect (assuming Dirac's theory was correct). Dirac's theory predicts that the states are degenerate so the differences shown in the figure correspond to both the difference in energy states as well as the difference between theory and experiment. The figure shows a difference of approximately 1000 Mc/sec between the energy levels. The dotted lines represent Lamb's

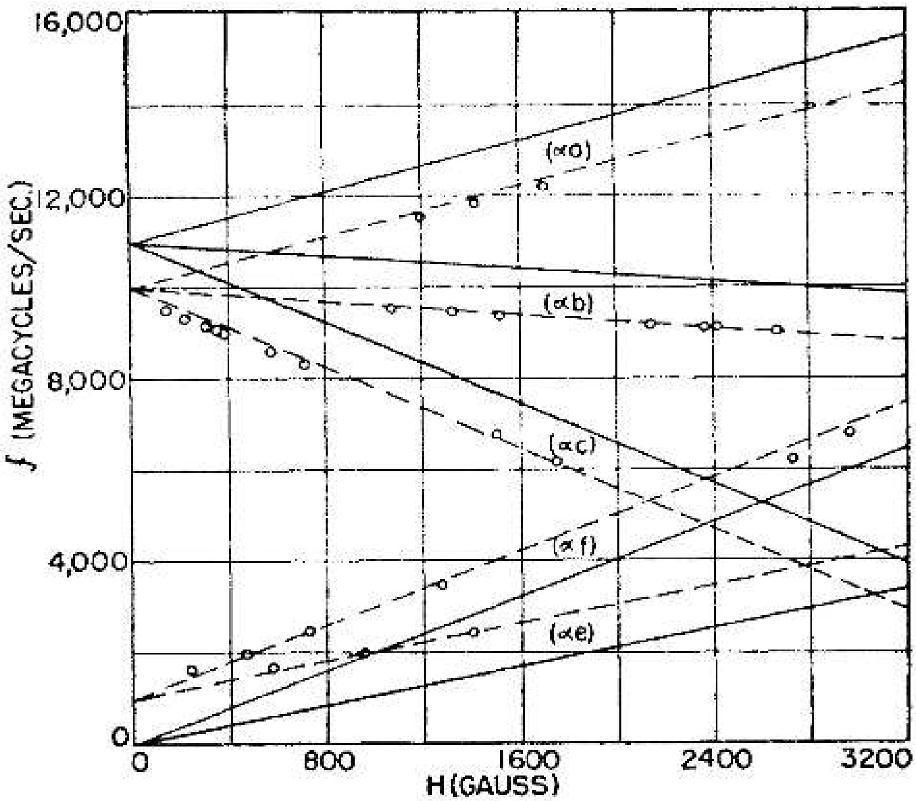

FIG. 2: A comparison of Lamb and Retherford's experimental data with the curves from Dirac's theory.[1]

data and the solid lines correspond to Dirac's Theory.

III. BETHE'S CALCULATION

In order to calculate the energy difference between the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states, we will use second order perturbation theory. We will follow Bethe's derivation of the Lamb Shift, in which he subtracts the 'free' electron self energy from the energy of the bound electron. The general equation for the energy shift to second order is

$$\Delta E_\alpha = \langle \alpha | H_I | \alpha \rangle + \sum_{\alpha \neq \beta} \frac{\langle \alpha | H_I | \beta \rangle \langle \beta | H_I | \alpha \rangle}{E_\alpha^{(0)} - E_\beta^{(0)}}, \tag{1}$$

where the Hamiltonian is

$$H_I = \frac{p^2}{2m} - \frac{Ze^2}{r} - ep \cdot \boldsymbol{A}$$

and

$$\boldsymbol{A} = \sqrt{\frac{\hbar c}{(2\pi)^3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{i=1}^{2} \epsilon_{ki}(\alpha_{ki} + \alpha_{ki}^{\dagger})$$

Here, i represents the two transverse polarizations of the photon, and k is the momentum. When $|\alpha\rangle = |a, 0\rangle$ (a state with no photons present) and $|\beta\rangle = |b, 1k\rangle$ (states with one photon present), we know that $\langle\alpha|H_I|\alpha\rangle = 0$, because $\boldsymbol{A}$ is normal ordered; therefore, its vacuum expectation value is zero.[2] Substituting H_I into Equation (1) and simplifying, we get

$$\Delta E_a = \frac{1}{2\pi} \sum_b \int_0^\infty \frac{d\omega}{E_a - E_b - \omega} \frac{e^2}{4\pi} \frac{\omega}{2\pi m^2} \sum_\lambda d\Omega_k \left| \epsilon_k^\lambda * \cdot \boldsymbol{p_{ba}} \right|^2, \tag{2}$$

where λ is the polarization. Performing a similar calculation for a free electron, we get

$$\Delta E_{free} = \frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} |\boldsymbol{p_a}|^2 \int_0^m d\omega. \tag{3}$$

We see that this takes the form of $-C|\boldsymbol{p_a}|^2$. Hence $E_{free} = \frac{p^2}{2m} - Cp^2 = \left(\frac{1}{2m} - C\right) p^2$. The C term in the energy difference for a free electron is effectively shifting the mass of the electron. The energy shift that Lamb observed, and that Bethe calculated, is the difference between the energy shift of a bound electron and a free electron. Hence

$$\Delta E_{observed} = \Delta E_a - \Delta E_{free} \tag{4}$$

$$= \left(\frac{e^2}{4\pi}\right) \frac{2}{3\pi} \frac{1}{m^2} \mathbb{P} \int_0^\infty d\omega \sum_b \frac{(E_a - E_b)|\boldsymbol{p_{ba}}|^2}{E_a - E_b - \omega} \tag{5}$$

where $\mathbb{P}$ is the principle value integral. We see that there is a singularity at $E_a - E_b = \omega$, so the principle value is

$$\mathbb{P} \int_0^{E_{max}} \frac{d\omega}{x - \omega} = \begin{cases} -log\left(\frac{\epsilon}{x}\right) - \log\left(\frac{E_{max} - x}{\epsilon}\right) \cong -\log\left(\frac{E_{max}}{x}\right) & \text{if } x > 0, \\ \approx -\log\left(\frac{E_{max}}{|x|}\right) & \text{if } x < 0. \end{cases} \tag{6}$$

This gives us

$$\Delta E_{obs} = -\frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} \log\left[\frac{E_{max}}{\langle|E_a - E_b|\rangle}\right] \sum_b (E_a - E_b)|\boldsymbol{p_{ba}}|^2. \tag{7}$$

Gross suggests that we factor the log out of the summation and use a trick from atomic physics to give us the following result:[2]

$$\Delta E_{obs} = \Delta E_{Lamb} = Z\alpha^2 \left(\frac{4}{3m^2}\right) |\psi_a(0)|^2 \ln\left[\frac{E_{max}}{\langle|E_a - E_b|\rangle}\right] \tag{8}$$

where α is the fine structure constant. We know for the $2S$ state, that $\langle |E_a - E_b| \rangle = 226.3eV$. If we assume that $E_{max} = m$ (a reasonable assumption) then we see that the natural log term yields ≈ 7.72, giving

$$\Delta E_{Lamb} = m\alpha^5 \frac{7.72}{6\pi}. \tag{9}$$

We can use this energy difference to find the transition frequency that Bethe found.

$$\begin{aligned} \nu &= \frac{\Delta E}{h} \\ &= \frac{mc^2}{h\hbar}\left(\frac{\alpha^5}{12\pi^2}7.72\right) \\ &\cong 1051MHz \end{aligned}$$

The value that Bethe obtained, is a very good estimate to the experimental value that Lamb and Retherford obtained months before.

IV. RESULTS AND DISCUSSION

When the Lamb Shift was discovered experimentally, it nullified Dirac's theory by exposing its defects and sparked the development of modern Quantum Electrodynamics. Prior to discovery of the Lamb Shift, calculating the mass of an electron produced an infinite, and therefore meaningless, result.[3] Theorists used the experimentally measured mass of an electron to replace the infinite mass and renormalized the quantum electrodynamics equations. The terms introduced in Bethe's calculations cancelled the infinite masses, and produced a finite mass term with which they were able to correctly solve for the Lamb Shift. The measurement of the Lamb Shift gives us a precise way to check the QED theory.[4] The electron emission/absorption of photons, gives researchers a method by which to accurately calculate a value for the electron spin g-factor: $g = 2.0023193043622$. It also produces a means to calculate α, the fine structure constant, to about one part in one million giving us one of the most accurately measured constants in physics.

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[1] Willis E. Lamb Jr., *Nobel Lecture: Fine Structure of the Hydrogen Atom* Dec. 12, 1995

[2] Franz Gross, *Relativistic Quantum Mechanics and Field Theory* (Wiley, New York, 1993), p. 73-81.

[3] Bethe, Hans A., *The Electromagnetic Shift of Energy Levels*, Physical Review, 1947, Vol. 72, Num. 4.

[4] Mitra, A.N., *Hans Bethe, Quantum Mechanics, and the Lamb Shift*, Resonance, 2005,