Dynamo Theory

A Weak-coupling Approach

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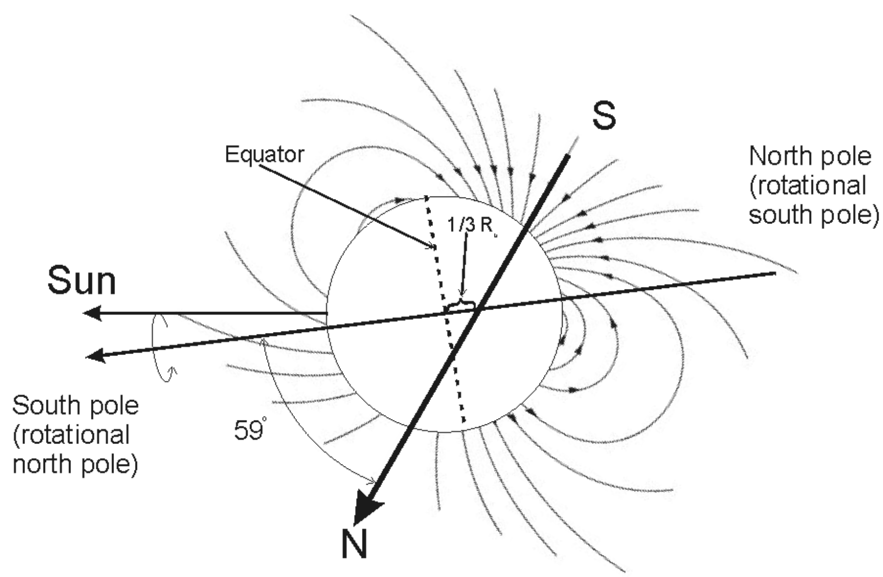
1 Introduction

Before Pioneer 10 directly measured the magnetic field of Jupiter, the only objects in the solar system known to have magnetic fields were the Earth and the Sun.[1] Based on observational evidence, it was expected that Jupiter had a substantial magnetic field. Furthermore, based on the similarity of Saturn with Jupiter, and similar observations of Saturn's radio emmissions, it was expected that Saturn would also possess a magnetic field. This was observed directly by Pioneer 11.[2] More unexpected was the observation of Uranus' magnetic field, as there was no previously existing observational evidence. Furthermore, the observations of the magnetic fields of the ice giants, by Voyager 2, revealed that the magnetic fields are substantially offset from the rotational axes of these planets.[4, 3]



Schematic of the Uranian magnetic field.

Planetary magnetic fields are thought to be created by the dyamo effect. Complex motions of electrically conductive fluids inside a planet induce substantial magnetic fields. In the case of the Earth, for example, the magnetic field is thought to be created in the iron-rich core.[5] Gas giants may owe their magnetic fields to the presence of conductive metallic hydrogen . The least understood of the planets, however, are the ice giants. There is some computational evidence to suggest that, rather than being produced in their depths, their mangetic fields are created in convection in the upper parts of the atmosphere.[9] The equations describing the dynamo effect, in non-dimensional form, are shown below.

$$\tilde{\rho}\,\frac{\mathrm{D}\tilde{\boldsymbol{u}}}{\mathrm{D}t} = \nabla\cdot\tilde{\boldsymbol{\sigma}} + \tilde{\boldsymbol{f}} + N\,(\nabla\times\tilde{\boldsymbol{B}})\times\tilde{\boldsymbol{B}} \tag{1}$$

$$\frac{\partial\tilde{\rho}}{\partial t} + \nabla\cdot(\tilde{\rho}\,\tilde{\boldsymbol{u}}) = 0 \tag{2}$$

$$\frac{\partial\tilde{\boldsymbol{B}}}{\partial t} = \frac{1}{\mathrm{Rm}}\nabla^2\tilde{\boldsymbol{B}} + \nabla\times(\tilde{\boldsymbol{u}}\times\tilde{\boldsymbol{B}}) \tag{3}$$

$$\nabla \cdot \tilde{\boldsymbol{B}} = 0 \tag{4}$$

$$\frac{\partial \tilde{T}}{\partial t} = \frac{k}{\tilde{\rho}} \nabla^2 \tilde{T} + \epsilon \tag{5}$$

In a review of the current state of the understanding of planetary magnetic fields, Stevenson writes "Dynamo simulations require clever ideas as well as merely brute force improvement of the parameter regime."[5] Many mathemtical approaches have been taken to dynamo theory; however, I am particularly interested in applying a weak-coupling expansion to (1)–(5), and investigating the viability of such an approach.

2 Dimensionless numbers in MHD

Fluid mechanics is known to make use of a large number of dimensionless numbers that characterize the flow in question. Magnetohydrodynamics is no exception. One such number, which appears in (3), is the **magnetic Reynolds number**, Rm.[7] Note that, in the limit, $\mathrm{Rm} \to 0$, (3) has the form of the diffusion equation. In the limit $\mathrm{Rm} \to \infty$, the magnetic field is subject to Alfvén's frozen-in theorem. As it were, the magnetic Reynolds number characterizes the ratio of induction of the magnetic field by fluid motion.

$$\mathrm{Rm} \equiv \frac{\boldsymbol{v}_0 \, L}{\sigma \, \mu}$$

A relatively large magnetic reynolds number is required to sustain a dynamo—the Childress limit establishes this as $\mathrm{Rm} > \pi$.

Another dimensionless number in MHD is the **magnetic Prandtl number**, which characterizes the relative importance of viscocity and magnetic diffusion. Many naturally occuring systems occur at low magnetic Prandtl number, and so this has been the subject of other studies.[6]

$$\mathrm{Pm} \equiv \frac{\mathrm{Rm}}{\mathrm{Re}} = \frac{\nu}{\sigma \, \mu}$$

Also in (1) we find the constant N, called the **magnetic interaction parameter**. This constant, of course, determines how strongly the fluid motion is coupled to the magnetic forces. At the heart of the weak-coupling approach is the assumption that $N \ll 1$. This requires that $|B_0| \ll \sqrt{\frac{\rho_0 \, \nu \, \mathrm{Re}}{\sigma}} / L$ or $|B_0| \ll \sqrt{\rho_0 \, \mu \, \mathrm{Rm}} / L$. From these, we can see that N may exist in the weak-coupling regime if, there are either

i. High Reynolds number—i.e. turbulent flow.

ii. High magnetic Reynolds number. This condition is required for a self-sustaining dynamo.

$$N \equiv \frac{B_0^2 \, \sigma \, L}{\rho_0 \, \boldsymbol{v}_0} = \frac{B_0^2 \, \sigma \, L^2}{\rho_0 \, \nu} \frac{1}{\mathrm{Re}} = \frac{B_0^2 \, L^2}{\rho_0 \, \mu} \frac{1}{\mathrm{Rm}}$$

3 Weak Coupling Expansion

Previosly, the weak-coupling expansion has been used to study the behavior of viscoelastic fluids, in which the Navier-Stokes equation contains a term corresponding to the elastic stress in the fluid.[10, 8] By expanding the equations of motion in terms of the coupling constant we are able to solve the Navier-Stokes, unperturbed by forces that are not caused by the stresses inside the fluid. This can then be used in the other equations of motion to calculate the lowest-order term of the elastic stress tensor, which in turn can be used to calculate the first order correction to the velocity field, etc. Expanding in terms of N and comparing terms of like order, we obtain a new ensemble of equations:

$$\frac{\mathrm{D}\tilde{\boldsymbol{u}}_0}{\mathrm{D}t} = \nabla \cdot \tilde{\boldsymbol{\sigma}}_0 + \tilde{\boldsymbol{f}}_0$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot (\tilde{\rho} \, \tilde{\boldsymbol{u}}_0) = 0$$

$$\frac{\partial \tilde{\boldsymbol{B}}_0}{\partial t} = \frac{1}{\mathrm{Rm}} \nabla^2 \tilde{\boldsymbol{B}}_0 + \nabla \times \left(\tilde{\boldsymbol{u}}_0 \times \tilde{\boldsymbol{B}}_0\right)$$

$$\begin{aligned}\nabla\cdot\tilde{\boldsymbol{B}}_0 &= 0\\ \frac{\mathrm{D}\tilde{\boldsymbol{u}}_1}{\mathrm{D}t} &= \nabla\cdot\tilde{\boldsymbol{\sigma}}_1 + \boldsymbol{J}\times\boldsymbol{B}_0\\ \frac{\partial\tilde{\rho}}{\partial t} + \nabla\cdot(\tilde{\rho}\,\tilde{\boldsymbol{u}}_1) &= 0\\ &\vdots\\ \frac{\partial\tilde{T}}{\partial t} &= \frac{k}{\tilde{\rho}}\nabla^2\tilde{T} + \epsilon\end{aligned}$$

This provides a powerful prescription not only for obtaining numberical solutions to the problem—provided a weak-coupling regime is physicical—but also provides a semi-analyitical framework for understanding the creation of planetary magnetic fields. In certain contexts, it should be possible to find solutions to the Navier-Stokes equation, by construction, and then determine the behavior of the magnetic field on that basis. For example, the velocity field in laminar Couette flow can be found easily.

4 Irrotational flow in a cylinder

Let's consider a very simple example. In cylindrical coordinates, an irrotational flow will be represented by $\tilde{\boldsymbol{u}}_0 = \hat{\boldsymbol{\theta}}/r$. We will be somewhat unconcerned as to how this flow can be realized—though, a Couette flow should be able to produce this velocity field. Again, we will consider the case that $\mathrm{Rm}\gg 1$; therefore, we will neglect the diffusive term. In this case, (3) reduces to an, essentially, advective equation. We will also consider the cylinder to be infinitely long. In this case, we can not only exploit the azimuthal symmetry, but also longitudinal symmetry.

$$\begin{aligned}\frac{\mathrm{D}\tilde{\boldsymbol{B}}_0}{\mathrm{D}t} &= \left(\tilde{\boldsymbol{B}}_0\cdot\nabla\right)\tilde{\boldsymbol{u}}_0\\ \nabla\cdot\tilde{\boldsymbol{B}}_0 &= 0\end{aligned} \tag{6}$$

Equation (6), despite efforts to simplify the problem, is still somewhat untractable. However, we can ask under what conditions will the magnetic field be stable. By allowing the time derivatives of $\tilde{\boldsymbol{B}}_0$ to be zero, we find that $(\tilde{\boldsymbol{u}}_0\cdot\nabla)\tilde{\boldsymbol{B}}_0 = \left(\tilde{\boldsymbol{B}}_0\cdot\nabla\right)\tilde{\boldsymbol{u}}_0$.

$$\begin{aligned}\frac{\partial\tilde{B}_0^r}{\partial\theta} &= 0\\ \frac{\partial\tilde{B}_0^\theta}{\partial\theta} &= -2\,\tilde{B}_0^r\\ \frac{\partial\tilde{B}_0^z}{\partial\theta} &= 0\end{aligned}$$

These conditions imply that—as we expect in an azimuthally symmetric system—that the radial and longitudinal components of the magnetic field do not depend on θ. However, we also have to concede that $\frac{\partial\tilde{B}_0^\theta}{\partial\theta}$, based on the condition that $\tilde{B}_0^\theta(\theta=0) = \tilde{B}_0^\theta(\theta=2\pi)$. This, furthermore, implies that $\tilde{B}_0^r = 0$. The requirement that $\tilde{\boldsymbol{B}}_0$ is solenoidal implies that $\tilde{B}_0^z$ is a function of only r, and $\tilde{B}_0^\theta$ is a function of r and, possibly, z. We can now ask whether this configuration is stable. Suppose that we add a small contribution $\tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{\beta}}\,\theta$. This small perturbation evolves in time as

$$\frac{\partial\tilde{\boldsymbol{\beta}}}{\partial t} = \frac{1}{r^2\,\theta}\begin{pmatrix} -\theta & 0 & 0\\ -2\,\theta^2 & -\theta & 0\\ 0 & 0 & -\theta\end{pmatrix}\tilde{\boldsymbol{\beta}} \tag{7}$$

The eigenvalues of this matrix are all, clearly, $-1/r^2$. Because they are negative, this is a stable equilibrium. However, this condition is relatively mum on the details about $\tilde{\boldsymbol{B}}_0$. For example, $\tilde{B}_0^z$ may be either positive or negative, or whether it can spontaneously change between the two. However, what is clear is that this flow geometry leads to a helically shaped magnetic field.

A first order correction can be obtained for the velocity field, $\tilde{\boldsymbol{u}}_1$, can be obtained from the equation

$$\frac{\mathrm{D}\tilde{\boldsymbol{u}}_1}{\mathrm{D}t} = \nabla \cdot \tilde{\boldsymbol{\sigma}}_1 - \left\{ \tilde{B}_0^z \frac{\partial \tilde{B}_0^z}{\partial r} + \frac{\tilde{B}_0^\theta}{r} \frac{\partial \left(r\, \tilde{B}_0^\theta \right)}{\partial r} \right\} \hat{\boldsymbol{r}} + \tilde{B}_0^z \frac{\partial \tilde{B}_0^\theta}{\partial z} \hat{\boldsymbol{\theta}} - \tilde{B}_0^\theta \frac{\partial \tilde{B}_0^\theta}{\partial z} \hat{\boldsymbol{z}} \tag{8}$$

Less interesting–at present, at least–is the solution, compared to the fact that the stresses the fluid is subjected to is substantially more complicated. The resulting first-order correction to the velocity field could contain components that are not directed only in the $\hat{\boldsymbol{\theta}}$, which could introduce a correction to the magnetic field that is unstable.

5 Numberical difficulties

Consider another simple example: one-dimensional advection, subject to an initial condition.

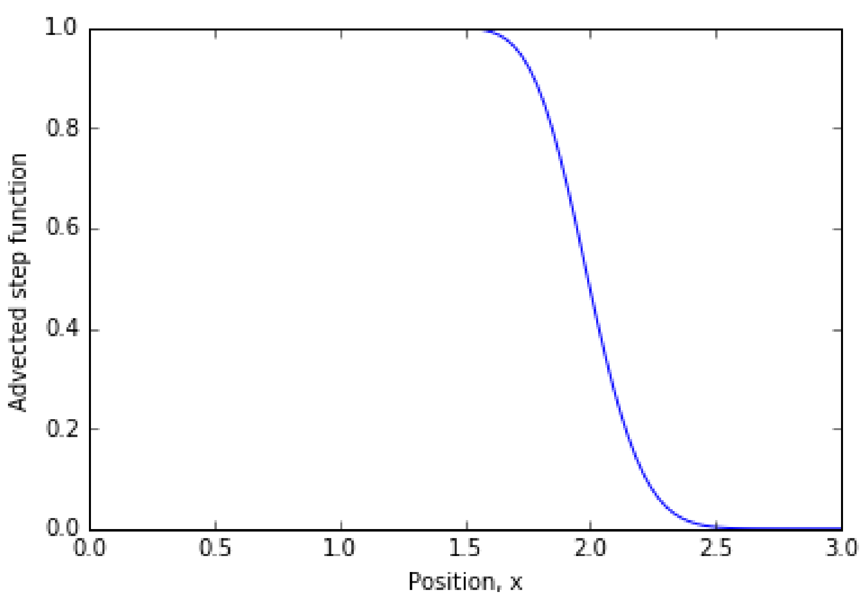
$$\begin{gathered} \frac{\partial q}{\partial t} + v \frac{\partial q}{\partial x} = 0 \\ q(x, 0) = Q(x) \end{gathered}$$

This problem can be solved exactly, using Fourier transforms, to show that $q(x,\, t) = Q(x - v\, t)$. If we approach the problem numberically, to first-order, we can update the function q by

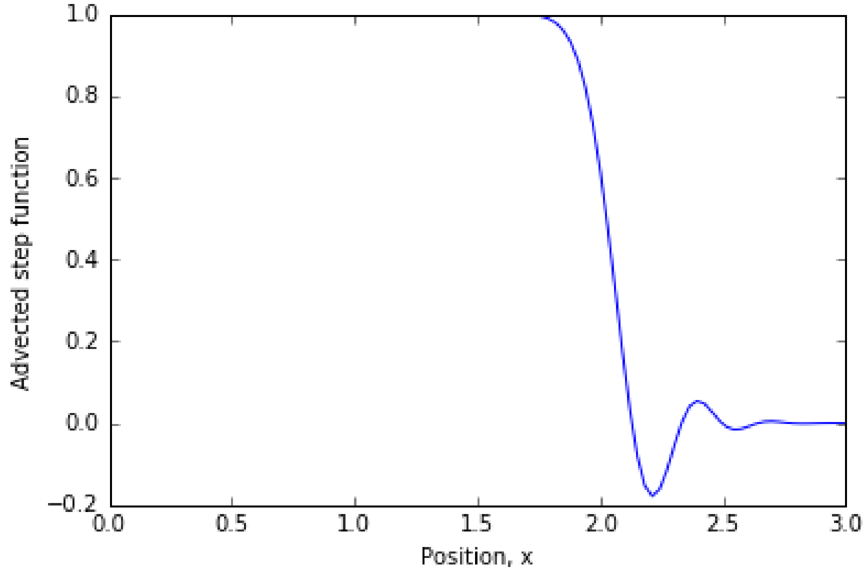
$$q(x_i, t_j + \delta t) = q(x_i, t_j) - \frac{v\, \delta t}{\delta x} \left[q(x_i, t_j) - q(x_i - \delta x, t_j) \right] \tag{9}$$

Suppose that Q is a step function at $x = 1$, and $v = 1$. This function should maintain a sharp, clean edge that should flow down the stream defined by v. Specifically, the step function should have propagated to $x = 2$ at $t = 1$. However, when we implement this solution, the numberical results is somewhat smoothed out. I also tried a method that was second order in δx. The edge was perhaps sharper; however, there was some resulting oscillation following the descent. I would like to spend some time in the future examining better numberical techniques that might be applied to this problem.

$$q(x_i, t_j + \delta t) = q(x_i, t_j) - \frac{v\, \delta t}{\delta x} \left[\frac{3}{2} q(x_i, t_j) - 2\, q(x_i - \delta x, t_j) + \frac{1}{2} q(x_i - 2\, \delta x, t_j) \right] \tag{10}$$



Numberical solution implementing (9)



Numberical solution implementing (10)

6 Conclusion

In this paper, I attempted to outline the solution technique of the weak coupling expansion for the dynamo theory equations. Ultimately, the weak coupling expansion is meant to be a computational tool; however, it is in principle able to yield semi-analytic results. In the example, I attempted to gain some understanding of the magnetic field that results from a simple irrotational flow. There are, perhaps, more illustrative examples that can serve as toy-models for the planetary magnetic fields that I ultimately want to apply this to. However, being a computaitonal method, I hope to devote some additional time to finding a numberical method of a high order in time and spatial differential elements than the naïve finite-differences method I attempted.

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