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Back From The Future: An Overview of Two State Vector Formalism

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We introduce to the reader a short overview of Two-state Vector Formalism. The motivation of this paper is to investigate an alternate interpretation of quantum mechanics. Moreover, a recent experiment[1] is analyzed within this developed framework

I. INTRODUCTION

The success of quantum mechanics cannot be overstated in its yielding of incredibly precise predictions for a multitude of experiments. However, in the jump from classical to quantum formalism a certain "intuition" has been lost. Deterministic equations of classical mechanics were abandoned for a random interpretation of the physical world. Current dogma finds this concern irrelevant, but this black spot has lead to the development of several modifications to the theory.

When investigating the framework of quantum theory one must begin with Schrödinger.

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathrm{H}|\Psi(t)\rangle \tag{1}$$

We note that wave solutions to this equation will move in one direction while their complex conjugate Ψ^* will move opposite. This symmetry is called *time-reversibility* The first time-symmetric interpretations for quantum mechanics were postulated by Walter Schottky[2]. Later, Satosi Watanabe proposed that a considering only a forward evolving wave equation is incomplete, and for the full picture of information in a quantum system one must also concern themselves with the backwards evolving wave equation. He called this view Double Inferential Vector Formalism. [3]. This work was later rediscovered by Aharanov, Berggman and Lebowitz and further developed as Two-state Vector Formalism (TSVF)[4]. This development is presented below.

II. MECHANICS

A The Two-state Vector

In the normal view of quantum mechanics any system can be described by a state vector: $|\Psi\rangle$. Any quantum state at a forward time t is defined simply by the time evolution operator acting on that initial state.

$$|\Psi(t)\rangle = |\Psi(t=0)\rangle \cdot e^{-\frac{i}{\hbar}Ht} \tag{2}$$

So that the state of this quantum system is an evolution purely dependent on that time t measurement.

In contrast, the framework of TSVF requires a two-state vector to describe the full evolution of the system

$$\langle\Phi|\ |\Psi\rangle$$
$$|\Psi(t)\rangle = |\Psi(t_0)\rangle \cdot e^{-\frac{i}{\hbar}\int_{t_0}^{t} H dt}$$
$$\langle\Phi(t)| = \langle\Phi(t_0)| \cdot e^{\frac{i}{\hbar}\int_{t_0}^{t} H dt}$$

where $|\Psi\rangle$ is the initial quantum state of they system that will propagate forward through time, and $\langle\Phi|$ is the final quantum state that propagates backwards through time.

At time t_0 a quantum system $\mathcal{S}$ is pre-selected for state $|\Psi\rangle$. At time t_f the system is measured and found to be in state $\langle\Phi|$ (post-selected). If we would like to find any resultant measurements at a time t, $t_0 < t < t_f$, then there are two types of measurements to consider: strong measurements and weak measurements.

B Strong Measurements

A strong measurement is colloquially just any measurement that is unambiguous. Assume our quantum system has an observable O, and the quantum system evolves through a time t_f. If we want to make a projective measurement of the observable at some time t, $t_0 < t < t_f$ we calculate the probability of this measurement having an outcome o_i as

$$P(o_i) = \frac{|\langle\Phi|O_i|\Psi\rangle|^2}{\sum_i |\langle\Phi|O_i|\Psi\rangle|^2} \tag{3}$$

This is the Aharonov-Bergmann-Lebowitz (ABL) rule for strong measurements in TSVF[7]. These strong measurements in the formalism can be equated to the projected "collapse" of our wave function in normal quantum mechanics framing.

C Weak Measurements

The main outcome of the TSVF development, though, is the discovery of weak values. If the observable O is preand post-selected by the two-state system $\langle\Phi|\ |\Psi\rangle$, and then weakly couples to another system, that coupling can be described as a *weak value*[5] given as

$$O_w \equiv \frac{\langle\Phi|\, O\, |\Psi\rangle}{\langle\Phi|\Psi\rangle} \tag{4}$$

D Application - Three Box Paradox

A common example system to examine in TSVF is that of the three box paradox[6]. Imagine there are three boxes A, B, C in which we will place a quantum ball. We preselect the system to have the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle + |C\rangle\right)$$

and then allow the system to evolve. We make a projective measurement and find the state to be in

$$|\Phi\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle - |C\rangle\right)$$

Box A is then opened. The probability for the quantum ball to be in A is as follows. The projective measure operators for finding the ball in A are

$$\begin{aligned} A(yes) &= A_Y = |A\rangle\langle A| \\ A(no) &= A_N = |B\rangle\langle B| + |C\rangle\langle C| \end{aligned}$$

So we find that the probability of finding our quantum ball in the first box is

$$\begin{aligned} P(a_{yes}) =& \frac{|\langle\Phi|\, A_Y\, |\Psi\rangle|^2}{|\langle\Phi|\, A_Y\, |\Psi\rangle|^2 + |\langle\Phi|\, A_N\, |\Psi\rangle|^2} \\ =& \frac{|\langle\Phi|\, |A\rangle\langle A|\, |\Psi\rangle|^2}{|\langle\Phi|\, |A\rangle\langle A|\, |\Psi\rangle|^2 + |\langle\Phi|\, (|B\rangle\langle B| + |C\rangle\langle C|)\, |\Psi\rangle|^2} \\ =& \frac{|\frac{1}{3}|^2}{|\frac{1}{3}|^2 + |0|^2} = 1 \end{aligned}$$

Similarly, for finding the ball in B or C we use

$$\begin{aligned} B(yes) &= B_Y = |B\rangle\langle B| \\ B(no) &= B_N = |A\rangle\langle A| + |C\rangle\langle C| \\ C(yes) &= C_Y = |C\rangle\langle C| \\ C(no) &= C_N = |A\rangle\langle A| + |B\rangle\langle B| \end{aligned}$$

and thus,

$$P(b_{no}) = 1, \quad P(c_{yes}) = \frac{1}{5}$$

In layman's terms, if box A is opened the ball would be there. If box B was opened, the ball would be there. If box C was opened, the ball *might* be there.

These statements taken together are nonsensical of course, thus leading to the paradox.

If instead of using these strong measurements on the system, we instead use weak values to investigate the location of the quantum ball we find

$$\begin{aligned} (P_A)_w =& \frac{\langle\Phi|\, A_Y\, |\Psi\rangle}{\langle\Phi|\Psi\rangle} \\ =& \frac{\frac{1}{3}(1+0...)}{\frac{1}{3}(1+1-1)} = 1 \\ (P_B)_w =& \frac{\langle\Phi|\, B_Y\, |\Psi\rangle}{\langle\Phi|\Psi\rangle} \\ =& \frac{\frac{1}{3}(1+0...)}{\frac{1}{3}(1+1-1)} = 1 \\ (P_C)_w =& \frac{\langle\Phi|\, C_Y\, |\Psi\rangle}{\langle\Phi|\Psi\rangle} \\ =& \frac{\frac{1}{3}(-1+0...)}{\frac{1}{3}(1+1-1)} = -1 \end{aligned}$$

III. EXPERIMENT ANALYSIS

Now let us turn our sights on the motivation of this paper. A recent experiment[1] has been performed which may lend credence to the TSVF as the most simple explanation. Let us first describe the set-up

A Mach-Zehnder Interferometer

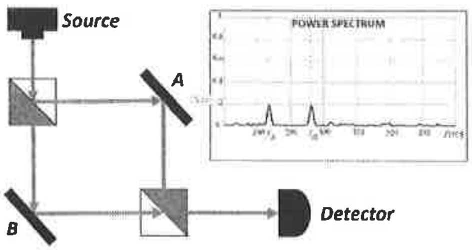
The experiment ustilizes two nested Mach-Zehnder interferometers. Much can be said alone about a Mach-Zehnder interferometer[8], but for brevity I will state it as such. The interferometer consists of two beam splitters and two mirrors. Incoming photon beams are split and directed to one of the mirrors, which will then bounce the beam into a second splitter where they are recombined.

In this experiment the component mirrors in the MZI system are each supplied with a small vibration at fixed frequency. When a photon is reflected off this mirror it vertically shifts the photon.

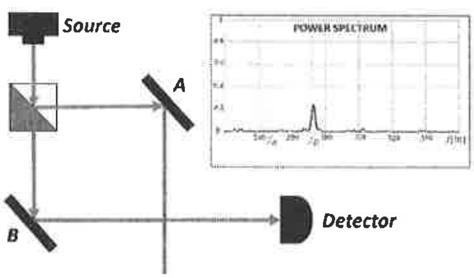
When the photons are detected at the end of the experiment, the detector output signal is analyzed using Fourier techniques and the Power Spectrum of that signal is produced. The result is that one should find well-defined peaks around the frequencies of the component mirrors that a photon has encountered.

B Proof of Concept

The first orientation of the experiment is simple. A beam source is oriented such that a beam splitter directs it towards two separate mirrors and a second beam splitter sends these redirected beams together to a detector. When both beams are redirected to the detector the power spectrum shows both mirror frequencies f_A and f_B. This can be seen in the figure below

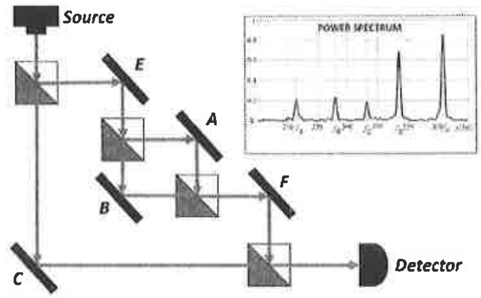


If the second beam splitter is tuned so that only the beam coinciding with mirror B is directed to the detector then only prominent peak in the Power Spectrum is the frequency of that mirror.

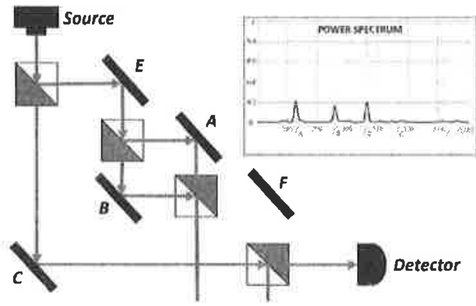


C Peculiar Results

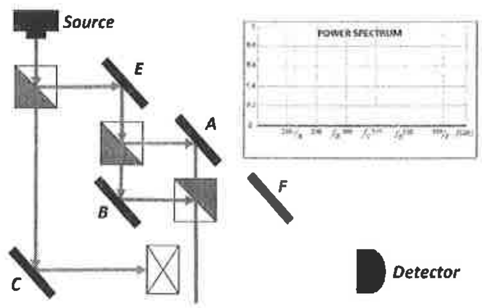
The really interesting results occur when we use the nested orientation is investigated. The first orientation is chosen to be one arm of the interferometer, while the other arm is a simpler system including only one arm. The outer beam splitters are 1:2, which means they will send twice as many photons into the "interior" interferometer as those that travel along the outside path towards mirror C. Again, all mirrors in the interferometer system are driven at unique frequencies and if a photon interacts with a mirror it will be shifted accordingly and this result can be picked out of the power spectrum. This orientation is shown below



Now if the last beam splitter is removed from the system, all the photons that arrive at the detector should only have interacted with mirror C. Surprisingly, while f_C is definitely present in the power spectrum we can also see that f_A and f_B are visible as well. This is quite peculiar as any photons that have interacted in the inner interferometer system have no way to redirect to the detector.



And finally, if the path from mirror C is completely blocked then we arrive at no frequencies in the power spectrum of the detector signal. This is something of a sanity check as we can verify there are no obvious leaks in the system.



Now let's try to make sense of what we are seeing for the second case (interferometer arm 2 blocked). Obviously it makes sense that we see f_C as we can clearly trace out a path a photon would take through the splitters, reflecting off the mirror C and then off towards the detector. But for a photon to "communicate" with mirrors A or B we expect to trace a path, forward through time, where the photon reflects off of either mirror. Except, even if we try to identify some mechanism wherein the photon progresses forward in time to interact with mirror A or B then it would follow the photon also must have interacted with mirror E, thus displaying a corresponding frequency f_E in the output signal.

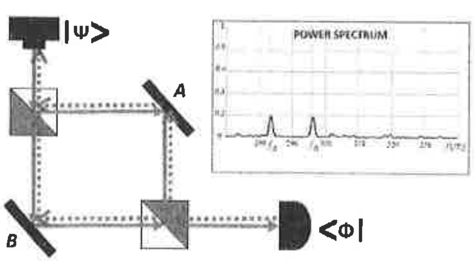
If we were to be so bold as to suggest that there are backwards projected photon states that originate at the detector we still need to account for the apparent invisibility of mirror E. We will see that this non-intuitive result falls out nicely when the system is considered in the framework of Two-state vector formalism.

D TSVF Framework

Using the two-state vector notation, $|\Psi\rangle$ is the system's state starting from the photon emitter and propagating forward through time (and the system) while $\langle\Phi|$ is the state that propagates *backward* through the system. A backward projected state will evolve exactly symmetric to the evolution of the forward state. Therefore when a "forward" photon passes through a beam splitter in a certain orientation we should expect a "backward" photon to evolve along symmetric paths (i.e. if a photon comes in one side of the mirror in a beam splitter, with equal chances of transmitting or reflecting, then a backward photon coming along the opposite direction of one of those paths will experience equal transmission and reflection in its frame). As the photons interact with the mirrors they become weakly coupled, since the vibrational frequency is much smaller than the quantum uncertainty in the photons direction after interaction. In this context, the power spectrum is a weak value measurement, with the frequency being the weak value. So now we can readily explain the presence of the frequency peaks in the power spectrum

Let us examine first the initial configuration, and then work our way up to the more interesting scenario. In the first scenario both mirrors A and B can interact with forward and backward photon states. In the proper notation these will be defined as:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|A\rangle + \frac{1}{\sqrt{2}}|B\rangle$$
$$\langle\Phi| = \frac{1}{\sqrt{2}}\langle A| + \frac{1}{\sqrt{2}}\langle B|$$

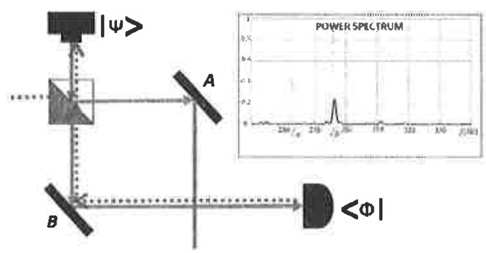


As stated, the power spectrum analysis is a weak value measurement, so we should find equal probability of peak measurements in the system for f_A and f_B. And we do.

$$(P_A)_w = \frac{\langle\Phi|\,|A\rangle\langle A|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{2}(1)}{\frac{1}{2}(1+1)} = \frac{1}{2}$$
$$(P_B)_w = \frac{\langle\Phi|\,|B\rangle\langle B|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{2}(1)}{\frac{1}{2}(1+1)} = \frac{1}{2}$$

Next, we look at the second scenario. As we can see, only mirror B has both backward and forward photon states, so our two state vector is now:

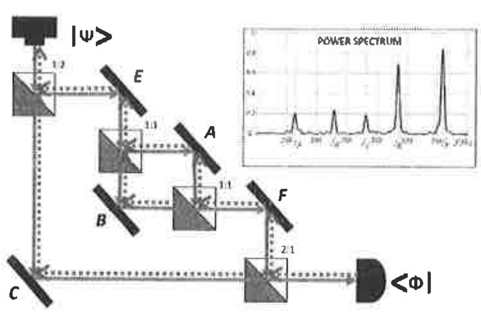
$$|\Psi\rangle = \frac{1}{\sqrt{2}}|A\rangle + \frac{1}{\sqrt{2}}|B\rangle$$
$$\langle\Phi| = \langle B|$$



And the corresponding weak value analysis yields

$$(P_A)_w = \frac{\langle\Phi|\,|A\rangle\langle A|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{0}{\frac{1}{2}(1+0)} = 0$$
$$(P_B)_w = \frac{\langle\Phi|\,|B\rangle\langle B|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{2}(1+0)}{\frac{1}{2}(1+0)} = 1$$

Now that's all well and good, but we didn't need TSVF to interpret this initial configuration. If we now turn our analysis to the nested system, we see the power of this though. For the first configuration of this nested system we can see that all mirrors may interact with forward and backward photons.



To analyze the weak value measurement we will split the system into three distinct times, $t_0 < t_1 < t_2 < t_3 < t_f$. Here t_0 is the time of firing t_f is the detection, t_1 is a time when photons are between mirror C/E and the first splitter armlengths, t_2 is in the nested interferometer, and t_3 is the armlengths from mirror C/F to the outer beam splitter.

Thus, we can probe at t_1 to find

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|C\rangle + \sqrt{\frac{2}{3}}|E\rangle$$
$$\langle\Phi| = \frac{1}{\sqrt{3}}\langle C| + \sqrt{\frac{2}{3}}\langle E|$$

which will yield weak value measurements

$$(P_C)_w = \frac{\langle\Phi|\,|C\rangle\langle C|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1+0)}{\frac{1}{3}(1+2)} = \frac{1}{3}$$
$$(P_E)_w = \frac{\langle\Phi|\,|E\rangle\langle E|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{2}{3}(0+1)}{\frac{1}{3}(1+2)} = \frac{2}{3}$$

time t_3 is, of course, symmetric to this

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|C\rangle + \sqrt{\frac{2}{3}}|F\rangle$$
$$\langle\Phi| = \frac{1}{\sqrt{3}}\langle C| + \sqrt{\frac{2}{3}}\langle F|$$

which will yield weak value measurements

$$(P_C)_w = \frac{\langle\Phi|\,|C\rangle\langle C|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1+0)}{\frac{1}{3}(1+2)} = \frac{1}{3}$$
$$(P_F)_w = \frac{\langle\Phi|\,|F\rangle\langle F|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{2}{3}(0+1)}{\frac{1}{3}(1+2)} = \frac{2}{3}$$

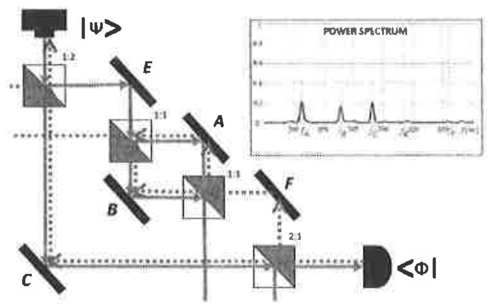
for the middle section, time t_2

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|A\rangle + \frac{1}{\sqrt{3}}|B\rangle + \frac{1}{\sqrt{3}}|C\rangle$$
$$\langle\Phi| = \frac{1}{\sqrt{3}}\langle A| + \frac{1}{\sqrt{3}}\langle B| + \frac{1}{\sqrt{3}}\langle C|$$

$$(P_A)_w = \frac{\langle\Phi|\,|C\rangle\langle C|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1+0+0)}{\frac{1}{3}(1+1+1)} = \frac{1}{3}$$
$$(P_B)_w = \frac{\langle\Phi|\,|C\rangle\langle C|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1+0+0)}{\frac{1}{3}(1+1+1)} = \frac{1}{3}$$
$$(P_C)_w = \frac{\langle\Phi|\,|F\rangle\langle F|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1+0+0)}{\frac{1}{3}(1+1+1)} = \frac{1}{3}$$

Now at first glance this seems to only be half of the picture. Our weak values during time t_2 implies equal probability for f_A, f_B and f_C, and indeed this is what we see. But looking at f_E and f_F we would expect these peaks to be equal to each other and twice as big as f_C. This is explainable by noting within a power spectrum a signal is squared. So f_C:f_E is proportionally 1 to 4.

We have investigated this first orientation, but now we finally get to the fun part. Using the two state formalism we can see from the figure below that the only mirrors in this scenario which interact with both photon directions are A, B, and C.



Because the only mirrors we are concerned with are all within the same "middle section" of the interferometer our two state vector will only be analyzed in this time frame. We will note our two-state vector will look like:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|A\rangle + \frac{i}{\sqrt{3}}|B\rangle + \frac{1}{\sqrt{3}}|C\rangle$$
$$\langle\Phi| = \frac{1}{\sqrt{3}}\langle A| + \frac{i}{\sqrt{3}}\langle B| + \frac{1}{\sqrt{3}}\langle C|$$

And following our same prescription as before we find

$$(P_A)_w = \frac{\langle\Phi|\,|A\rangle\langle A|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(1-0+0)}{\frac{1}{3}(1-1+1)} = 1$$
$$(P_B)_w = \frac{\langle\Phi|\,|B\rangle\langle B|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(0-1+0)}{\frac{1}{3}(1-1+1)} = -1$$
$$(P_C)_w = \frac{\langle\Phi|\,|C\rangle\langle C|\,|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{\frac{1}{3}(0-0+1)}{\frac{1}{3}(1-1+1)} = 1\frac{1}{3}$$

We note that a power spectrum will only show intensities and not the sign of a signal, so though our weak value of f_B is negative it will appear the same as f_C and f_A in the spectrum.

IV. CONCLUSION

We have seen a unique interpretation of quantum mechanics that is time-symmetric. This formalism provides a useful tool for analyzing systems that can be pre- and post- selected for measurement. The two-state vector formalism can more explicitly be used to explain curious results in optical experiments. In particular, photons were shown to "talk" to several interference sources they would not normally see in the generic accepted view of quantum mechanics. As an aside, there have been numerous discussions both for and against the described experiment in the context of TSVF, and the issue does not seem to be currently settled.[9][10].

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