White Dwarfs - Degenerate Stellar Configurations

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Abstract

The end product of low-medium mass stars is the degenerate stellar configuration called a white dwarf. Here we discuss the transition into this state thermodynamically as well as developing some intuition regarding the role of quantum mechanics in this process.

I Introduction and Stellar Classification

It is widely believed that the end stage of the low or intermediate mass star is an extremely dense, highly underluminous object called a white dwarf. Observationally, these white dwarfs are abundant ($\sim 6\%$) in the Milky Way due to a large birthrate of their progenitor stars coupled with a very slow rate of cooling. Roughly 97% of all stars will meet this fate. Given no additional mass, the white dwarf will evolve into a cold black dwarf.

In this paper I hope to introduce a qualitative description of the transition from a main-sequence star (classification which aligns hydrogen fusing stars to a line on the Hertzsprung-Russell Diagram) to a white dwarf through the help of quantum mechanics. In addition, I will also discuss possible mechanisms for the formation of other configurations of exotic matter more dense than the electron degenerate gas.

II Stellar Pressure and Hydrostatic Equilibrium

In the field of Stellar Dynamics, we often define the relation between the pressure exerted by a system of particles of known composition and its ambient temperature and density, P=P(ρ,T,**X**), to be called the equation of state. Assuming stellar gas to be an ideal gas, an assumption seemingly made on careless grounds, will prove to be acceptable solely due to the fact that Coulomb interactions of ionized gases present at such high temperatures are small compared with the kinetic (thermal) energy of the particles. This assumption implies a mixture of free non-interacting particles. One may also note that the pressure of a mixture of different species of particles will be the sum of the pressures exerted by each (this is where photon pressure (P_{Rad}) will come into play). Following this logic we can express the total stellar pressure as:

$$P_{Tot} = P_{Gas} + P_{Rad} = P_{Ion} + P_{e^-} + P_{Rad} \quad (1)$$

At Hydrostatic Equilibrium: Pressure Gradient = Gravitational Pressure.

$$\frac{dP}{dR} = -\frac{GM}{R^2}\rho \quad (2)$$

This is of course assuming a spherically symmetric distribution of mass.

III Ion Pressure

We start with the equation of state for an ideal ion gas:

$$P_{Ion} = n_{Ion}kT \quad (3)$$

Where n_{Ion} is the number of ions per unit volume V. To obtain the total number of ions in this unit volume, we must sum the following relation over all ion species:

$$n = \frac{\rho}{m_H}\frac{X}{A} \quad (4)$$

$$n_{Ion} = \sum_i n_i = \sum_i \frac{\rho}{m_H} \frac{X_i}{A_i} \tag{5}$$

Where m_H is the atomic mass unit (by convention - $\frac{1}{12}$ of a carbon nucleus $\neq$ proton mass or a hydrogen atom), X is the mass fraction of a species, and A is the baryon number of a species. The mean atomic mass of stellar material μ_{Ion} and the ideal gas constant are then defined respectively by:

$$\frac{1}{\mu_{Ion}} \equiv \sum_i \frac{X_i}{A_i} \tag{6}$$

$$R \equiv \frac{k}{m_H} \tag{7}$$

Finally we get:

$$P_{Ion} = \frac{R}{\mu_{Ion}} \rho T \tag{8}$$

IV Electron Pressure

Starting in a similar fashion, the equation of state for an ideal electron gas is:

$$P_{e^-} = n_{e^-} kT \tag{9}$$

Where n_{e^-} is the number of free electrons per unit volume V. Here we concentrate our focus on the stellar interior, where at temperatures of $10^6 K$, hydrogen and helium are completely ionized. The total number of electrons per unit volume is:

$$n_{e^-} = \sum_i Z_i n_i = \frac{\rho}{m_H} \sum_i X_i \frac{Z_i}{A_i} \tag{10}$$

Where Z is the charge on the nucleus. We then define the average number of free electrons per nucleon, $\mu_{e^-}^{-1}$, as:

$$\frac{1}{\mu_{e^-}} = \sum_i X_i \frac{Z_i}{A_i} \tag{11}$$

Which after some substitution will give us:

$$n_{e^-} = \frac{\rho}{\mu_{e^-} m_H} \tag{12}$$

The electron pressure can thus be written as:

$$P_{e^-} = \frac{R}{\mu_{e^-}} \rho T \tag{13}$$

To clarify, our assumptions have been non-interacting gases and complete ionization. So far, we have still failed to mention anything regarding the conditions of stellar interiors such that these effects cannot always be neglected. At a certain point, quantum mechanical and relativistic effects will enter the picture.

V Radiation Pressure

The idea of a pressure is generally associated with a $\frac{Force}{Area}$. This force can then be thought of as the sum of massive particles imparting momenta upon the walls of some container. Although a photon does not have mass, its momentum is $h\nu$. Radiation pressure is due to the transfer of momentum to gas particles whenever absorption and scattering of photons occurs within the gas. In Thermodynamic equilibrium the photon distribution is isotropic and the number of photons with frequencies in the range $(\nu, \nu + d\nu)$ is given by the blackbody distribution:

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{d\nu}{exp^{\frac{h\nu}{kT}} - 1} \tag{14}$$

We can use the pressure integral to obtain an expression for the pressure due to radiation:

$$P = \frac{1}{3} \int_0^\infty \nu p n(p) dp \tag{15}$$

$$P_{Rad} = \frac{1}{3} \int_0^\infty c \frac{h\nu}{c} n(\nu) d\nu = \frac{1}{3} a T^4 \tag{16}$$

Where a is the radiation constant:

$$a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c} \tag{17}$$

As stated previously, each collision between a photon and an atom will excite the atom energetically, transferring momentum in the direction of the incoming photon. Inevitably returning to its original state by photon emission, it recoils in the direction opposite the initial photon. The direction of these emitted photons are random and after a long series of these interactions the random changes in momenta due to emission cancel out and the net change in the atom's momentum is in the direction of the photon.

Putting all pressures together((7),(12) and(15)), we achieve a total pressure of:

$$P = (\frac{1}{\mu_{Ion}} + \frac{1}{\mu_{e^-}}) R \rho T + \frac{1}{3} a T^4 \tag{18}$$

VI Evolution of a Low-Medium Mass Star

Throughout a star's life on the main sequence, the star constantly battles to maintain stability. The rate at which stars fuse elements in their cores is very sensitive to temperature $(e^{(\frac{-E}{kT})})$. If the star were to be compressed even a little, the core would rise in temperature and the rate of fusion would increase considerably. This increased rate of fusion results in a higher production of photons (which can take on the order of 10^5 years to reach the photosphere) and consequently a higher photon pressure. Given enough fuel, this self-sustaining process will continue for the vast majority of the star's life.

Considering the topic at hand we focus our attention to those stars we believe will end up as white dwarfs: 0.8M⊙-8M⊙. Stars with insufficient mass will never reach a core temperature and density high enough to begin the fusion of hydrogen. Stars in excess of roughly $8M\odot$ will eventually produce cores which exceed the Chandrasekhar limit and will result in a supernova leaving behind a neutron star or black hole remnant.

Once the hydrogen in the core has been exhausted, the star undergoes contraction due to lack of energy production in the core. It then shrinks in radius, heating up the core to higher and higher temperatures. At roughly 10^8K the core begins to fuse helium into carbon and oxygen through what is known as the triple-alpha process. This new production of energy will temporarily re-stabilize the star. This phase of stable helium burning is significantly shorter than the main-sequence phase of hydrogen burning. This is essentially due to two reasons:

- The fusion of helium into carbon and oxygen provides roughly $\frac{1}{10}$ the energy per unit mass supplied by hydrogen burning.
- The stellar luminosity is higher by more than an order of magnitude compared with the main-sequence luminosity of the same star.

Inevitably, the Helium fuel source will run out as well, causing another phase of contraction as well as an increase in temperature in the core. This time the remaining hydrogen in the shell outside of the core will proceed to burn, separating the core from its remaining envelope (this is the planetary nebula).

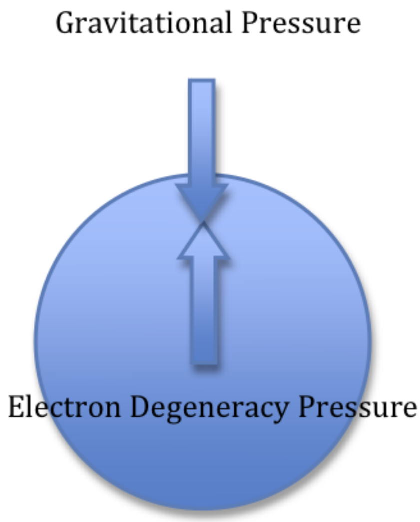

Figure 1: Hydrostatic Equilibrium

Since the core is now completely devoid of an energy source, it begins its final descent into the white dwarf stage.

VII Electron Degeneracy Pressure

The final collapse of the stellar core is halted by the electron degeneracy pressure (Fig.1), an application of the Pauli Exclusion Principle. Since electrons are fermions, no two can occupy identical quantum states. the electrons are forced into higher and higher energy levels causing them to be relativistic. Even at zero temperature, there exists a non-zero energy due to these relativistic electrons. This is referred to as the Fermi Energy.

Since an election can have two spin states $(+\frac{1}{2}, -\frac{1}{2})$ each location in phase space can have at most two electrons. Complete degeneracy occurs when electrons are forced to occupy all available momentum states. Applying the Heisenberg and Pauli Principles to a completely degenerate isotropic electron gas yields the momentum distribution (number of electrons with momenta in the interval $(p, p + dp)$ per unit volume):

$$n_{e^-}(p)dp = \frac{2}{\Delta V} \qquad (19)$$

$$n_{e^-}(p)dp = \frac{2}{h^3} 4\pi p^2 dp \tag{20}$$

After a similar process, we achieve an electron degeneracy pressure of:

$$P_{e^-,deg} = \frac{8\pi}{15 m_{e^-} h^3} p_0^5 \tag{21}$$

$$P_{e^-,deg} = \frac{h^2}{20 m_{e^-}} (\frac{3}{\pi})^{\frac{2}{3}} \frac{1}{(m_H)^{\frac{5}{3}}} (\frac{\rho}{\mu_{e^-}})^{\frac{5}{3}} \tag{22}$$

Where p_0 is the maximal momentum:

$$p_0 = (\frac{3h^3 n_e}{8\pi})^{\frac{1}{3}} \tag{23}$$

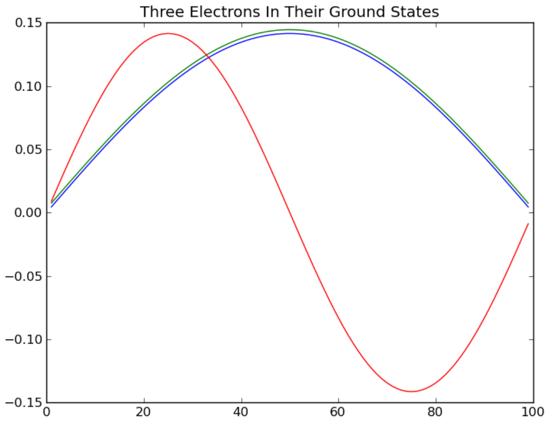

Figure 2: An Infinite Square Well of Length L

Let us now consider visually three electrons in a 1-D square well (Fig. 2). Shown here are the wavefunctions of these electrons in the ground state.

$$m_e v = p = \frac{h}{\lambda} \tag{24}$$

$$E = \frac{p_1^2}{2m_e} + \frac{p_2^2}{2m_e} + \frac{p_3^2}{2m_e} \tag{25}$$

$$E = \frac{h^2}{2m_e} (\frac{1}{(2L)^2} + \frac{1}{(2L)^2} + \frac{1}{L^2}) = \frac{3h^2}{4 m_e L^2} \tag{26}$$

As the length of our box shrinks in size, the energy of the particles will increase:

$$P \propto |\frac{dE}{dL}| = \frac{3h^2}{2 m_e L^3} \tag{27}$$

In this case we are doing work against degeneracy pressure.

According to the Heisenberg Uncertainty Principle as we restrict the location of these electrons by decreasing the size of the box, the uncertainty in their momentum will increase.

$$\Delta V \Delta^3 p \geq h^3 \tag{28}$$

Eventually as the density becomes sufficiently high the range of momenta exceed the range of momenta corresponding to the gas temperature. Once this occurs, equation (28) is minimized to a strict equality. This marks the level at which quantum mechanics must be considered to be of equal importance to classical thermodynamics.

White dwarfs differ from main-sequence stars in another very important way. They are considered to be highly under-luminous. Judged with reference to other stars of comparable mass, they exhibit very low average luminosities. This clearly has mostly to do with a lack of nucleosynthesis, although it may be partly due to another characteristic of the degenerate gas it is composed of. A degenerate electron gas behaves much like a metal, conducting heat very efficiently. One known consequence of this will be a nearly uniform internal temperature - the temperature gradient is very small.

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{F}{4\pi r^2} \tag{29}$$

This is the radiative transfer equation, where a is the radiation constant, $F \equiv Luminosity$, $\frac{F}{4\pi r^2} \equiv Flux$, and:

$$\kappa = \kappa_0 \rho^a T^b \tag{30}$$

VIII Further Degeneracy

Sufficiently dense matter containing protons will also experience proton degeneracy pressure. Protons confined to a small enough region will have large uncertainties in their momenta. Due to a much larger mass than the electron, proton velocities will be smaller, representing only a small correction to the equations of state of an electron degenerate gas.

The equations governing electron degeneracy show us that a maximum mass for a non-rotating white dwarf exists. This mass is the Chandrasekhar limit - approximately 1.4M$\odot$. Although seemingly in the last phase of their lives, carbon-oxygen white dwarfs are capable of further fusion reactions when part of a binary system with a red giant. If the white dwarf accretes mass from its companion, its core temperature will reach the ignition temperature required for carbon fusion. The sudden initiation of nuclear fusion causes a runaway reaction causing what is known as a supernova (Type Ia). Once exceeding the mass limit, gravity overcomes the electron degeneracy pressure and begins to collapse. Depending on how much mass the white dwarf accreted in the process, the end product is a neutron star or a black hole. Not all neutron stars remain in isolation. In dense stellar regions such as globular clusters, neutron stars are able to capture a companion. Through a mass accretion process (as with the white dwarf binary) the neutron star can transition to a black hole.

In either case, further degeneracy pressures become more important than that of the electron. In the case of a neutron star, gravitational pressure is counteracted by neutron degeneracy pressure. As the density increases the Fermi Energy of the electrons will increase to a point where it is energetically favorable for them to combine with protons to produce neutrons. This occurs through an inverse beta-decay process.

Similarly, for neutron stars there exists some upper limit to the mass as well, called the Tolman-Oppenheimer-Volkoff limit. Any more massive than this and we now move into the realm of quark stars (composed of, you guessed it, degenerate quark gas - if you can even call it that), and finally into the realm of black holes. Solving for this final limit requires the use of General Relativity, and is most commonly displayed as the Schwarzschild radius:

$$r_s = \frac{2GM(r)}{c^2} \tag{31}$$

The Schwarzschild radius is a characteristic radius which is associated with every massive body. It describes the volume at which the object must be fully contained in such that there exists no known degeneracy pressure to counteract its self-gravity. Essentially it describes a singularity: a point mass of infinitely large density.

IX Conclusion

In this paper we have discussed hydrostatic equilibrium by piecing together the various components of pressure of a main-sequence star. We then developed the sequence of processes through which low-medium mass stars transition into degenerate matter by making use of the Pauli Exclusion Principle and the Heisenberg Uncertainty Principle. We have also made the distinction as to where quantum mechanics cannot be ignored in stellar calculations. Lastly, a qualitative description has been provided regarding the transition to other degenerate configurations arising from white dwarf and neutron star binary accretion.

X References

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