Scaling in Quantum Mechanics

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We take a page from Landau and Lifshitz [1], (Sec. 10, Mechanical Similarity, pp. 22-24) to investigate scaling in Quantum Mechanics.

I. SCHRÖDINGER EQUATION

The classical hamiltonian for a single particle can often be written in the form

$$\mathcal{H} = \frac{p^2}{2m} + Kr^k \tag{1}$$

Here the homogeneous potential $V = Kr^k$, where K is a coupling constant (for gravitational problems $K = -GMm$, for attracting Coulomb problems $K = -e^2$). For another particle in a similar potential we have $\mathcal{H}' = \frac{p'^2}{2m'} + K'r'^k$. If the two physical systems are related by a scale transformation, solutions for either can be mapped to solutions of the other. Accordingly we assume the scaling relations

$$\begin{array}{rclrclrclrcl} r' &=& \alpha r & p' &=& \alpha^{-1}p & t' &=& \beta t & E' &=& \lambda E \\ m' &=& \gamma m & K' &=& \delta K & \psi' &=& \epsilon\psi & & & \end{array} \tag{2}$$

The relation $p' = \alpha^{-1}p$ follows from the invariance of the commutator (either the classical Poisson bracket or the quantum commutator $[q_i, p_j] = i\hbar$). The time-energy conjugacy similarly requires $\beta\lambda = 1$. The Schrödinger equation for the $'$ system is

$$\left(\frac{p'^2}{2m'} + K'r'^k\right)\psi' = E'\psi' \rightarrow \left(\frac{1}{\alpha^2\gamma}\frac{p^2}{2m} + \alpha^k\delta V\right)\epsilon\psi = \lambda\epsilon\psi \tag{3}$$

Observe first that both sides of this equation are linear in the wavefunction scaling factor ϵ. This comes about because Quantum Mechanics is a *linear* theory. This is not to say that ϵ cannot be computed. It is determined by the normalization condition:

$$\int |\psi|^2 d^3r = 1 \rightarrow \int |\psi'|^2 d^3r' = \underbrace{\alpha^3\epsilon^2}_{}\underbrace{\int |\psi|^2 d^3r}_{} = 1 \tag{4}$$

Both terms on the right hand side must be $+1$, so that $\epsilon = \alpha^{-3/2}$.

Scaling requires the following relations

$$\alpha^{k+2}\gamma\delta = 1 \qquad\qquad \alpha^2\gamma\lambda = 1 \tag{5}$$

II. APPLICATIONS

Example 1: Assume a Coulomb potential, so that $k = -1$ and $\alpha\gamma\delta = 1$. Assume further that m scales but the coupling constant does not: $\gamma \neq 1, \delta = 1$. Then $\alpha\gamma = 1$ and $\alpha\lambda = 1$, so that $\gamma = \lambda = 1/\alpha$. Suppose that the electron is replaced by a mu meson: $e^- \rightarrow \mu^-$ in a hydrogenic atom. With $\alpha = m_\mu/m_e \simeq 207$, these scaling relations tell us that the nonrelativistic energy spectrum for the mesic atom is similar to that of the hydrogen atom, scaled by a factor 207. The atomic size is reduced by this same factor. The ground state μ-mesic orbit radius is approximately 0.529/207 Angstroms.

Example 2: Again with a Coulomb potential, assume K scales but m does not: $\gamma = 1, \delta = Z \neq 1$ for an atomic nucleus containing Z protons. Under these assumptions the scaling relations become $\alpha Z = 1$, $\alpha^2\lambda = 1$, so that $\alpha = 1/Z$ and $\lambda = Z^2$.

Partial Results: Example 1 shows that the energy is proportional to the mass. Example 2 shows that it is proportional to Z^2. Since Z occurs in $\mathcal{H}$ coupled to e^2, the energy is proportional to $(Ze^2)^2$: $E \simeq m(Ze^2)^2$.

Example 3: We could further zero in on the nature of the energy eigenvalues by looking at the scaling of the fine structure constant: $\hbar \rightarrow \hbar' = \phi\hbar$. Following through (a recommended useful exercise) we find $E \simeq 1/\hbar^2$.

Further Results: Including the results of Example 3 with the partial results above, we find $E \simeq m(Ze^2)^2/\hbar^2$. Finally, recalling that $e^2/\hbar c$ is a dimensionless constant ($e^2/\hbar c = \alpha = 1/137.03611... \simeq 0.007... =$ Sommerfeld's fine structure constant) we can write the expression for nonrelativistic Coulomb energy as

$$E \simeq mc^2(Z\alpha)^2 \simeq 510,000eV * Z^2/(137)^2 \tag{6}$$

All that is left to detailed calculations is the structure of the discrete spectrum, $E_N = -mc^2(Z\alpha)^2/(2N^2)$, where $N = 1, 2, \cdots$ is the principal quantum number.

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[1] L. D. Landau and E. M. Lifshitz, *Mechanics*, (J. B. Sykes, J. S. Bacon, and J. S. Bell, translators), Reading, MA: Addison-Wesley, 1960.