The Topology of Chaos

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Outline

1. Overview
2. Experimental Challenge
3. Topology of Orbits
4. Topological Analysis Program
5. Basis Sets of Orbits
6. Quantizing Chaos
7. Summary
J. R. Tredicce

Can you explain my data?

I dare you to explain my data!
Where is Tredicce coming from?

Feigenbaum:
\[ \alpha = 4.66920\ 16091 \ldots \]
\[ \delta = -2.50290\ 78750 \ldots \]
Laser with Modulated Losses
Experimental Arrangement
Our Hope

Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
Result

There is now a classification theory.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to $\mathbb{R}^3$ only — for now
Topology Enters the Picture

The 4 Levels of Structure

- Basis Sets of Orbits
- Branched Manifolds
- Bounding Tori
- Extrinsic Embeddings
Laser Experimental Arrangement
Oscilloscope Traces

(a) Oscilloscope trace 1
(b) Oscilloscope trace 2
(c) Oscilloscope trace 3
(d) Oscilloscope trace 4
(e) Oscilloscope trace 5
Bifurcation Schematics
Some Attractors

Coexisting Basins of Attraction
Many Experiments

Bifurcation Perestroikas

(a) 

(b) 

(c)
Experimental Data: LSA

Lefranc - Cargese
Experimental Data: LSA
Mechanism

Stretching & Squeezing in a Torus
Time Evolution

Rotating the Poincaré Section around the axis of the torus
Rotating the Poincaré Section around the axis of the torus

Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.
A Chemical Experiment

The Belousov-Zhabotinskii Reaction
Chaos

Motion that is

- Deterministic: \( \frac{dx}{dt} = f(x) \)
- Recurrent
- Non Periodic
- Sensitive to Initial Conditions
Strange Attractor

The $\Omega$ limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor
UPOs Outline Strange attractors

BZ reaction
UPOs Outline Strange attractors

Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.
Organization of UPOs in $R^3$: Gauss Linking Number

$$LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(r_A - r_B) \cdot dr_A \times dr_B}{|r_A - r_B|^3}$$

# Interpretations of LN $\sim$ # Mathematicians in World
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.
Evolution in Phase Space

One Stretch-&-Squeeze Mechanism

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Overview-05
Overview-06
Experimental-01
Experimental-02
Experimental-03

(b) 

(c) 

(a) 

(d) 

squeeze 

boundary layer 

stretch
Motion of Blobs in Phase Space

Stretching — Squeezing

[Schematic diagram showing the motion of blobs in phase space, with labels indicating stretching and squeezing processes.]
Collapse Along the Stable Manifold

Birman - Williams Projection

Identify $x$ and $y$ if

$$\lim_{t \to \infty} |x(t) - y(t)| \to 0$$
Fundamental Theorem

Birman - Williams Theorem

If:

Then:
Birman - Williams Theorem

If: Certain Assumptions

Then:
Birman - Williams Theorem

If: Certain Assumptions

Then: Specific Conclusions
Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- on $R^n$ is dissipative, $n = 3$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.
- Generates a hyperbolic strange attractor $SA$

IMPORTANT: The underlined assumptions can be relaxed.
Conclusions, B-W Theorem

- The projection maps the strange attractor $SA$ onto a 2-dimensional branched manifold $BM$ and the flow $\Phi_t(x)$ on $SA$ to a semiflow $\Phi(x)_t$ on $BM$.

- UPOs of $\Phi_t(x)$ on $SA$ are in 1-1 correspondence with UPOs of $\Phi(x)_t$ on $BM$. Moreover, every link of UPOs of $(\Phi_t(x), SA)$ is isotopic to the correspond link of UPOs of $(\Phi(x)_t, BM)$.

Remark: “One of the few theorems useful to experimentalists.”
A Very Common Mechanism

Rössler: Attractor Branched Manifold
A Mechanism with Symmetry

Lorenz:

Attractor

Branched Manifold
Examples of Branched Manifolds

Inequivalent Branched Manifolds

(a)  
(b)  
(c)  
(d)
Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units

subject to the conditions:

• Outputs to Inputs
• No Free Ends
Dynamics and Topology

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Rossler System

(a) Rossler Equations

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*}
\]

(b) Time series for the Rossler System

(c) Phase portrait of the Rossler System

(f) Jacobian matrix of the Rossler System:

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix}
\]
Dynamics and Topology

Lorenz System

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(a) Lorenz Equations
\[
\frac{dx}{dt} = -\sigma x + \sigma y \\
\frac{dy}{dt} = Rz - y - xz \\
\frac{dz}{dt} = -bz + xy
\]

(b) x

(c) y

(d) z

(f)

0 0
0 0
+1 -1
Poincaré Smiles at Us in $R^3$

- Determine organization of UPOs ⇒
- Determine branched manifold ⇒
- Determine equivalence class of $SA$
Topological Analysis Program

Locate Periodic Orbits
Create an Embedding
Determine Topological Invariants (LN)
Identify a Branched Manifold
Verify the Branched Manifold

Model the Dynamics
Validate the Model
Locate UPOs

Method of Close Returns
Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding
An Embedding and Periodic Orbits

**Figure 5.** Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese
### Compute Table of Expt’l LN

**Table 7.2** Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov-Zhabotinskii data

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*All indices are negative.*
Determine Topological Invariants

**Compare w. LN From Various $BM$**

**Table 2.1** Linking numbers for orbits to period five in Smale horseshoe dynamics.

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</table>
Determine Topological Invariants

Guess Branched Manifold

Figure 7. “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese
Determine Topological Invariants

**Identification & ‘Confirmation’**

- \( B\mathcal{M} \) Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion
Determine Topological Invariants

What Do We Learn?

- $BM$ Depends on Embedding
- Some things depend on embedding, some don’t
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism
Evolution Under Parameter Change

Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown); there is a variation in the topological organization across one chaotic tongue [39, 41].

Lefranc - Cargese
Perestroikas of Strange Attractors

Evolution Under Parameter Change

Lefranc - Cargese
Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese
Our Hope → Now a Result

Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.
Orbits Can be “Pruned”

There Are Some Missing Orbits

Lorenz

Shimizu-Morioka
Linking Numbers, Relative Rotation Rates, Braids

Orbit Forcing

A \Rightarrow B

B \Rightarrow C

A \Rightarrow C
Forcing Diagram - Horseshoe

An Ongoing Problem

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Forcing Diagram - Horseshoe

FORCING OF HORSESHOE ORBITS TO PERIOD 3

u - SEQUENCE ORDER

WO: 1/2 3/7 1/3 7/2 1/5 1/6 1/7 1/8

PD: 2 4 6 8 10 12 14 16

QOD: f

(a)
An Ongoing Problem

Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required
Creating New Attractors

Rotating the Attractor

\[ \frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X, Y) \\ F_2(X, Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix} \]

\[ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \]

\[ \frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = RF(R^{-1}u) + Rt + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix} \]

\[ \Omega = n \omega_d \quad q \Omega = p \omega_d \]

Global Diffeomorphisms

Local Diffeomorphisms

(p-fold covers)
Another Visualization

Cutting Open a Torus

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Global Torsion

(a)

(b)
Two Phase Spaces: $R^3$ and $D^2 \times S^1$

Rossler Attractor: Two Representations

$R^3$

$D^2 \times S^1$

Rossler Attractor, Toroidal Representation
Index $(n_1, n_2) = (1,0)$
Other Diffeomorphic Attractors

Rossler Attractor:
Two More Representations with $n = \pm 1$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (1,-1)$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (1,+1)$
Rossler Attractor:

Two Two-Fold Covers with $p/q = \pm 1/2$

Subharmonic, Locally Diffeomorphic Attractors
Subharmonic, Locally Diffeomorphic Attractors

Rossler Attractor:

Two Three-Fold Covers with $p/q = -2/3, -1/3$

![Graph showing Rossler Attractor, Toroidal Representation](image)
Rossler Attractor:

And Even More Covers (with \( \frac{p}{q} = +1/3, +2/3 \))
New Measures

Angular Momentum and Energy

\[
L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau X \, dY - Y \, dX \\
K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{2} \dot{X}^2 + \dot{Y}^2 \, dt
\]

\[
L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle
\]

\[
= L(0) + \Omega \langle R^2 \rangle
\]

\[
K(\Omega) = \langle \frac{1}{2} (\dot{u}^2 + \dot{v}^2) \rangle
\]

\[
= K(0) + \Omega L(0) + \frac{1}{2} \Omega^2 \langle R^2 \rangle
\]

\[
\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (X^2 + Y^2) \, dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (u^2 + v^2) \, dt
\]
New Measures, Diffeomorphic Attractors

Energy and Angular Momentum

Diffeomorphic, Quantum Number n
New Measures, Subharmonic Covering Attractors

Energy and Angular Momentum

Subharmonics, Quantum Numbers p/q
Summary

1 Question Answered ⇒

2 Questions Raised

We must be on the right track!
Our Hope

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
Our Result

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to $R^3$ only — for now
The Classification Theory has 4 Levels of Structure
Four Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
Four Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings
Four Levels of Structure
Poetic Organization

LINKS OF PERIODIC ORBITS organize
BOUNDING TORI organize
BRANCHED MANIFOLDS organize
LINKS OF PERIODIC ORBITS
Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g - 1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan’s Theorem for Lie Groups
- Topological Continuation – Group Continuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors
Unanswered Questions

We hope to find:

- Robust topological invariants for $\mathbb{R}^N$, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of $\chi^2$ test for NLD
- Better forcing results: Smale horseshoe, $D^2 \rightarrow D^2$, $n \times D^2 \rightarrow n \times D^2$ (e.g., Lorenz), $D^N \rightarrow D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy