The Topology of Chaos

Robert Gilmore

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The Topology of Chaos

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Colloquium, Physics Department University of Florida, Gainesville, FL

October 17, 2008

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The Topology of Chaos



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- Overview
- ② Experimental Challenge
- Topology of Orbits
- Topological Analysis Program
- Basis Sets of Orbits
- Quantizing Chaos
- Summary

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J. R. Tredicce

Can you explain my data?

I dare you to explain my data!

Motivation

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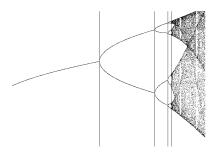
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Where is Tredicce coming from?



Feigenbaum:

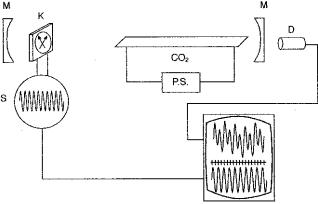
$$\alpha = 4.66920 \ 16091 \dots$$

Experiment

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P.S. Overview-03

Laser with Modulated Losses Experimental Arrangement



Our Hope

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Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Our Result

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Result

There is now a classification theory.

- It is topological
- 2 It has a hierarchy of 4 levels
- 6 Each is discrete
- There is rigidity and degrees of freedom
- **5** It is applicable to R^3 only for now

Topology Enters the Picture

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The 4 Levels of Structure

- Basis Sets of Orbits
- Branched Manifolds
- Bounding Tori
- Extrinsic Embeddings

Experimental Schematic

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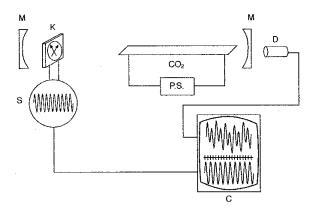
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Laser Experimental Arrangement



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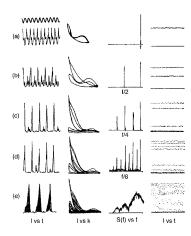
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Oscilloscope Traces



Results, Single Experiment

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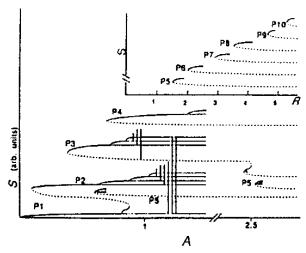
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Bifurcation Schematics



Some Attractors

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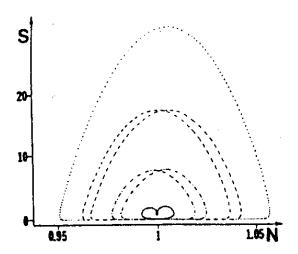
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Coexisting Basins of Attraction



Many Experiments

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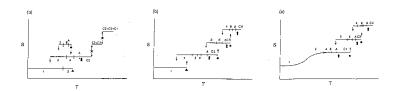
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Real Data

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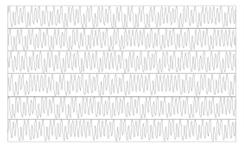
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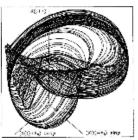
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Experimental Data: LSA





Lefranc - Cargese

Real Data

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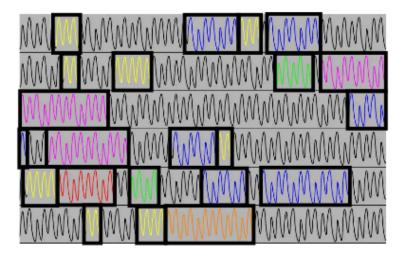
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Mechanism

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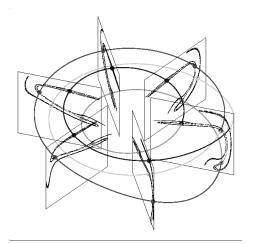
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Stretching & Squeezing in a Torus



Time Evolution

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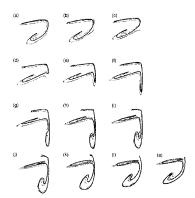
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Rotating the Poincaré Section around the axis of the torus



Time Evolution

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Rotating the Poincaré Section around the axis of the torus

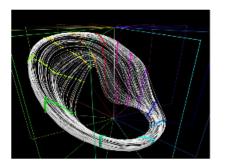




Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

Experimental Schematic

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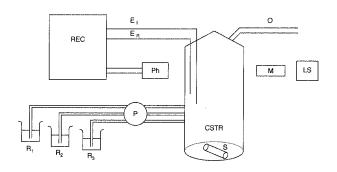
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Experimenta 02 A Chemical Experiment

The Belousov-Zhabotinskii Reaction



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Chaos

Motion that is

- **Deterministic:** $\frac{dx}{dt} = f(x)$
- Recurrent
- Non Periodic
- Sensitive to Initial Conditions

Strange Attractor

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Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- "Abundant"
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

Skeletons

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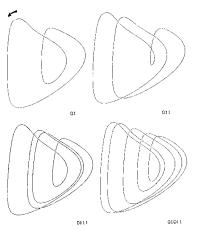
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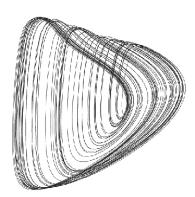
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UPOs Outline Strange attractors





Skeletons

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UPOs Outline Strange attractors

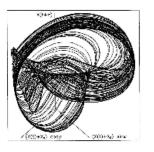




Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.

Dynamics and Topology

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Organization of UPOs in R³: Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq \#$ Mathematicians in World

Linking Numbers

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Linking Number of Two UPOs

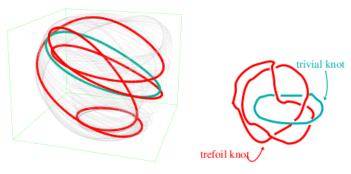


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

Evolution in Phase Space

(d)

squeeze

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boundary

(a)

One Stretch-&-Squeeze Mechanism

(c)

stretch

(b)

Motion of Blobs in Phase Space

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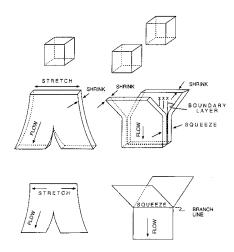
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Stretching — Squeezing



Collapse Along the Stable Manifold

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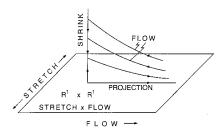
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Birman - Williams Projection

Identify \boldsymbol{x} and \boldsymbol{y} if

$$\lim_{t \to \infty} |x(t) - y(t)| \to 0$$



Fundamental Theorem

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Birman - Williams Theorem

If:

Then:

Fundamental Theorem

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Birman - Williams Theorem

If: Certain Assumptions

Then:

Fundamental Theorem

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Birman - Williams Theorem

If: Certain Assumptions

Then: Specific Conclusions

Birman-Williams Theorem

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Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- on R^n is dissipative, $\underline{n=3}$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.
- Generates a <u>hyperbolic</u> strange attractor SA

IMPORTANT: The underlined assumptions can be relaxed.

Birman-Williams Theorem

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Conclusions, B-W Theorem

- The projection maps the strange attractor \mathcal{SA} onto a 2-dimensional branched manifold \mathcal{BM} and the flow $\Phi_t(x)$ on \mathcal{SA} to a semiflow $\overline{\Phi}(x)_t$ on \mathcal{BM} .
- UPOs of $\Phi_t(x)$ on \mathcal{SA} are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on \mathcal{BM} . Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

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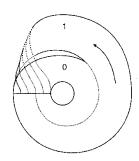
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Rössler:

Attractor Branched Manifold





A Mechanism with Symmetry

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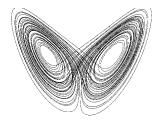
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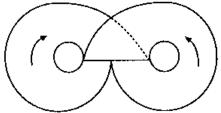
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Lorenz:

Attractor

Branched Manifold





Examples of Branched Manifolds

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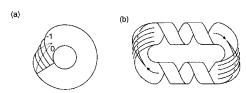
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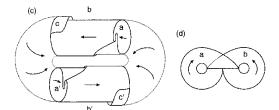
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Inequivalent Branched Manifolds





Aufbau Princip for Branched Manifolds

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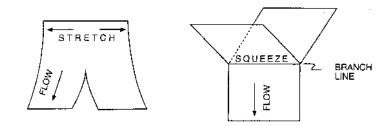
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Any branched manifold can be built up from stretching and squeezing units



subject to the conditions:

- Outputs to Inputs
- No Free Ends



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Rossler System



 $\frac{dx}{dt} = -y - \epsilon$

 $\frac{dy}{di} = x + ay$

 $\frac{dz}{dt}=b+z(z-a)$





(f)

 $\begin{bmatrix}
 -1 & 0 \\
 0 & 0
 \end{bmatrix}$

 $\left[\begin{array}{cc} 0 & \pm 1 \end{array}\right]$





(d)



Dynamics and Topology

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Lorenz System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + \alpha y$$

$$\frac{dy}{dt} = Rx \cdot y \cdot xz$$

$$\frac{dz}{dt} = -bz + xy$$

$$\left(+1,-1\right)$$

(b)















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Poincaré Smiles at Us in R³

- Determine organization of UPOs \Rightarrow
- Determine branched manifold ⇒
- Determine equivalence class of SA

Topological Analysis Program

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Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

Identify a Branched Manifold

Verify the Branched Manifold

Model the Dynamics

Validate the Model



Locate UPOs

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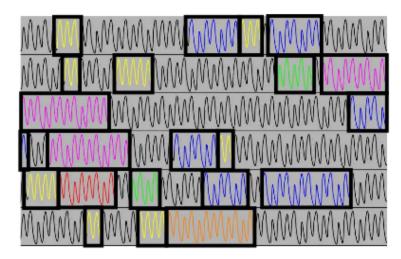
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Method of Close Returns



Embeddings

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Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological[†]

None Good

We Demand a 3 Dimensional Embedding

Locate UPOs

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An Embedding and Periodic Orbits

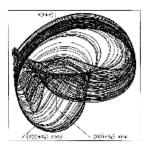




Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.

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Linking Number of Orbit Pairs

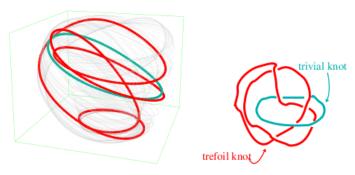


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

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Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov-Zhabotinskii data^a

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8Ь
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8Ь	01 011 011	3	6	8	12	15	16	21	24	21

All indices are negative.

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Compare w. LN From Various BM

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	19	1 <i>f</i>	21	3 <i>f</i>	39	41	4_2f	$4_{2}9$	5 ₃ f	539	5 ₂ f	529	5 ₁ f	518
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

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Guess Branched Manifold

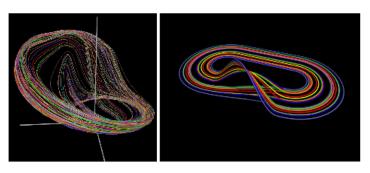


Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

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Identification & 'Confirmation'

- ullet \mathcal{BM} Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

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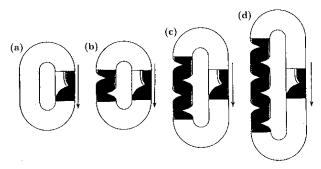
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What Do We Learn?

- BM Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Perestroikas of Strange Attractors

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Evolution Under Parameter Change

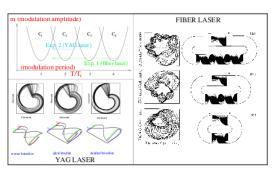


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment; global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown); there is a variation in the topological organization across one chaotic tongue [39, 41].

Perestroikas of Strange Attractors

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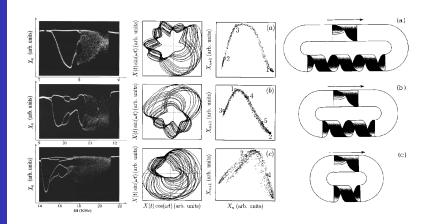
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Evolution Under Parameter Change



Lefranc - Cargese



An Unexpected Benefit

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Analysis of Nonstationary Data

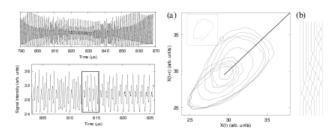


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

Our Hope \rightarrow Now a Result

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Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Orbits Can be "Pruned"

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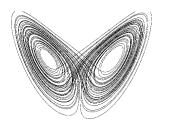
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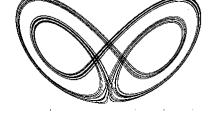
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There Are Some Missing Orbits





Lorenz

Shimizu-Morioka



Linking Numbers, Relative Rotation Rates, Braids

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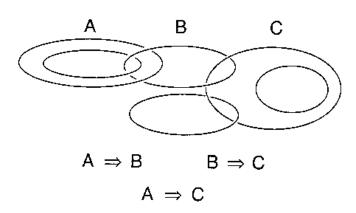
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Orbit Forcing



An Ongoing Problem

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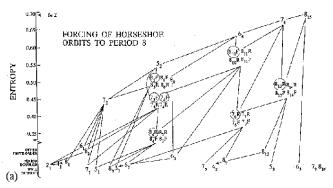
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Forcing Diagram - Horseshoe



u - SEQUENCE ORDER



An Ongoing Problem

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Status of Problem

- Horseshoe organization active
- More folding barely begun
- Circle forcing even less known
- Higher genus new ideas required

Creating New Attractors

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Rotating the Attractor

$$\frac{d}{dt} \left[\begin{array}{c} X \\ Y \end{array} \right] = \left[\begin{array}{c} F_1(X,Y) \\ F_2(X,Y) \end{array} \right] + \left[\begin{array}{c} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{array} \right]$$

$$\left[\begin{array}{c} u(t) \\ v(t) \end{array}\right] = \left[\begin{array}{cc} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{array}\right] \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right]$$

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix}$$

$$\Omega = n \ \omega_d$$

$$q \Omega = p \omega_d$$

Global Diffeomorphisms

Local Diffeomorphisms (p-fold covers)





Another Visualization

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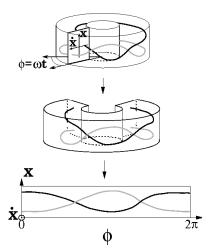
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Cutting Open a Torus



Satisfying Boundary Conditions

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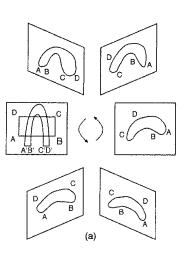
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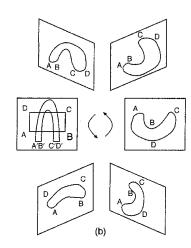
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Global Torsion





Two Phase Spaces: R^3 and $D^2 \times S^1$

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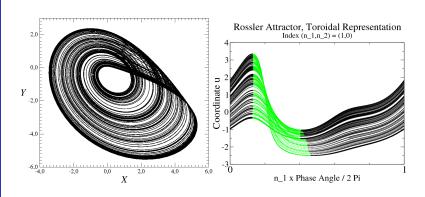
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Rossler Attractor: Two Representations

 R^3 $D^2 \times S^1$



Other Diffeomorphic Attractors

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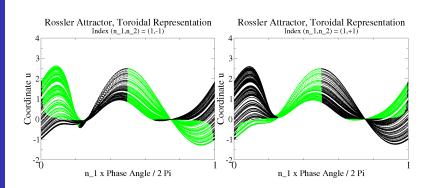
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Rossler Attractor:

Two More Representations with $n = \pm 1$



Subharmonic, Locally Diffeomorphic Attractors

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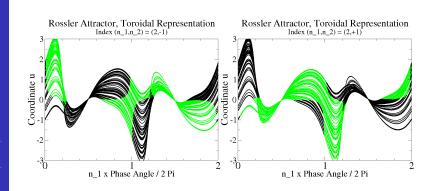
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Rossler Attractor:

Two Two-Fold Covers with $p/q = \pm 1/2$



Subharmonic, Locally Diffeomorphic Attractors

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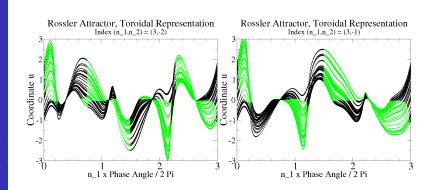
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Rossler Attractor:

Two Three-Fold Covers with p/q = -2/3, -1/3



Subharmonic, Locally Diffeomorphic Attractors

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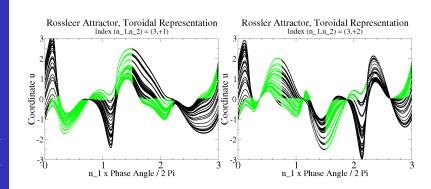
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Rossler Attractor:

And Even More Covers (with p/q = +1/3, +2/3)



New Measures

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Angular Momentum and Energy

$$L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} X dY - Y dX \qquad K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) dt$$

$$L(\Omega) = \langle u\dot{v} - v\dot{u}\rangle$$
 $K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2)\rangle$

$$= L(0) + \Omega \langle R^2 \rangle$$

$$= K(0) + \Omega L(0) + \frac{1}{2} \Omega^2 \langle R^2 \rangle$$

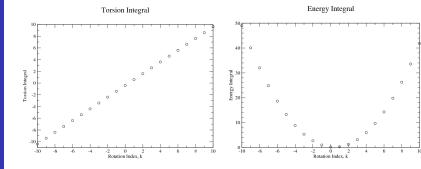
$$\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (u^2 + v^2) dt$$

New Measures, Diffeomorphic Attractors

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Energy and Angular Momentum

Diffeomorphic, Quantum Number n





New Measures, Subharmonic Covering Attractors

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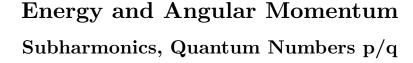
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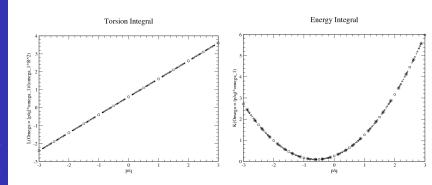
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Summary

1 Question Answered \Rightarrow

2 Questions Raised

We must be on the right track!

Our Hope

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Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

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Result

There is now a classification theory for low-dimensional strange attractors.

- 1 It is topological
- 2 It has a hierarchy of 4 levels
- Each is discrete
- There is rigidity and degrees of freedom
- **1** It is applicable to R^3 only for now

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The Classification Theory has 4 Levels of Structure

Basis Sets of Orbits

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Experimental

- Basis Sets of Orbits
- ② Branched Manifolds

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- Basis Sets of Orbits
- ② Branched Manifolds
- 8 Bounding Tori

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- Basis Sets of Orbits
- ② Branched Manifolds
- Bounding Tori
- Extrinsic Embeddings

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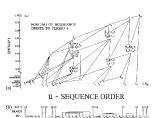
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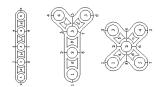
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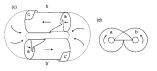
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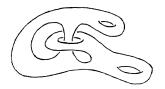


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Topological Components

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Poetic Organization

organize
BOUNDING TORI
organize
BRANCHED MANIFOLDS
organize
LINKS OF PERIODIC ORBITS



Answered Questions

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Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of g-1 disks
- Systematic methods for cover image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation Group Continuuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors



Unanswered Questions

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We hope to find:

- Robust topological invariants for \mathbb{R}^N , N>3
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \to D^2$, $n \times D^2 \to n \times D^2$ (e.g., Lorenz), $D^N \to D^N$, N>2
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy