Representation Theory for Strange Attractors

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What do we do with Data?

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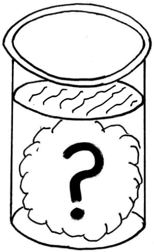

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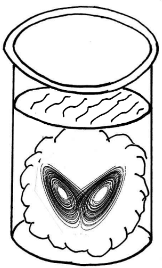

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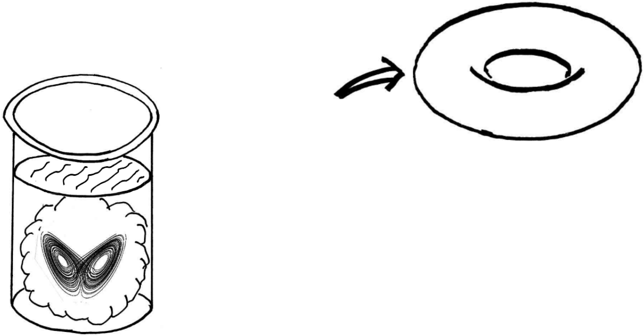

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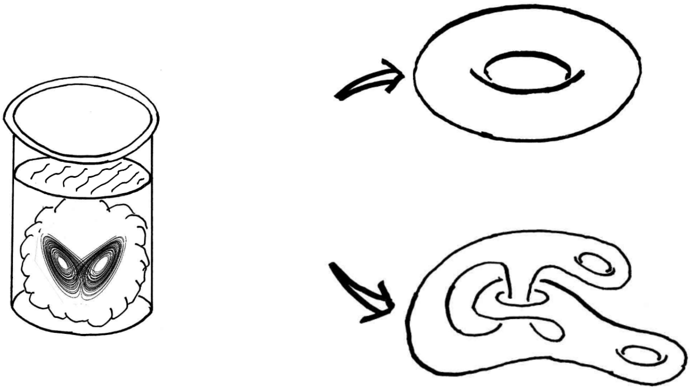


Outline

What to do with Data

Step 1: Data $\rightarrow$ Embedding

Step 2: Analyze Reconstructed Attractor

Step 3: What do you learn about:

The Data
The Embedding

????????

Important theorems

Whitney (1936): $\mathcal{M}^n \rightarrow R^N$:
N generic functions -
Embedding if $N \geq 2n+1$.

Takens (1981) : $(\mathcal{M}^n, \dot{X} = F(X)) \rightarrow (R^N, Flow)$:
One generic function at N measurement intervals.
Embedding if $N \geq 2n+1$.

Wu (1958): All embeddings $\mathcal{M}^n \rightarrow R^N$ are isotopic for $N \geq 2n+1$ and $n > 1$.

Embeddings and Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

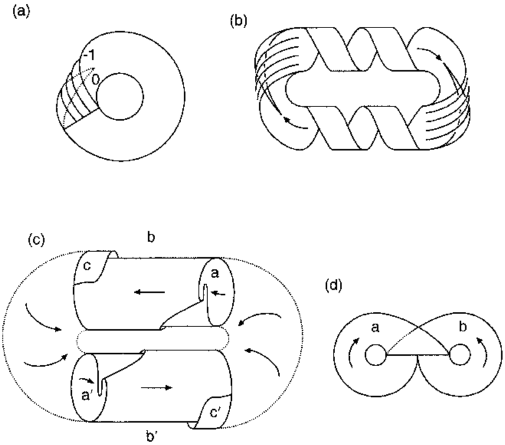
Preference is for embeddings of lowest possible dimension.

Possible Inequivalence for $n \leq N \leq 2n$.

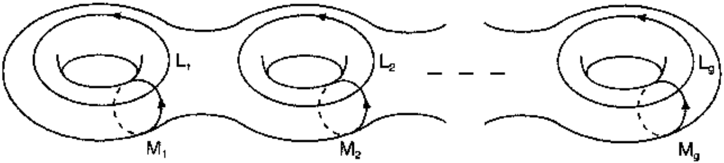
What do you want to learn?

- Geometry (Fractals, ...): "Independent" of Embedding
- Dynamics (Lyapunovs, ...) "Independent" of Embedding, but beware of spurious LEs
- Topology: some indices depend on embedding, others (*mechanism*) do not.

Revealed by Branched Manifolds



Classification of 3D Attractors



Program: $\mathcal{M}^3 \rightarrow R^3, R^4, R^5, R^6$

Inequivalent Representations in R^3

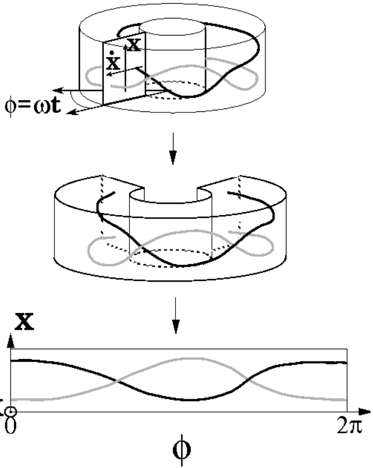
Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

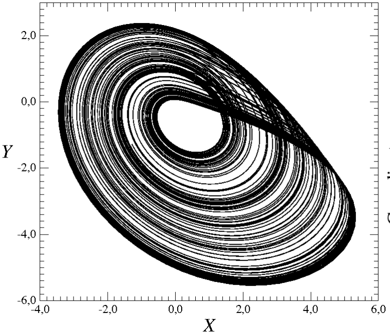
Cutting Open a Torus

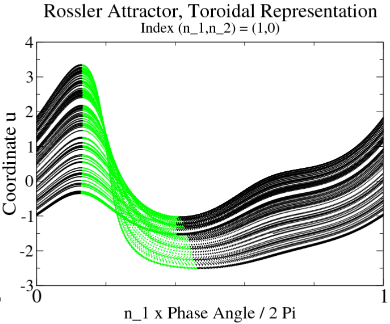


Two Phase Spaces: R^3 and $D^2 \times S^1$

Rossler Attractor: Two Representations

R^3 $\qquad$ $D^2 \times S^1$

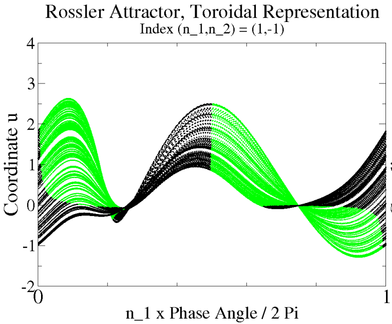


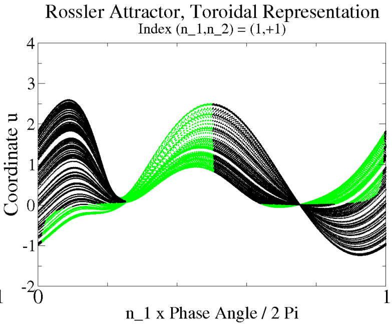


Other Diffeomorphic Attractors

Rossler Attractor:

Two More Representations with $n = \pm 1$





Rotating a Driven Attractor

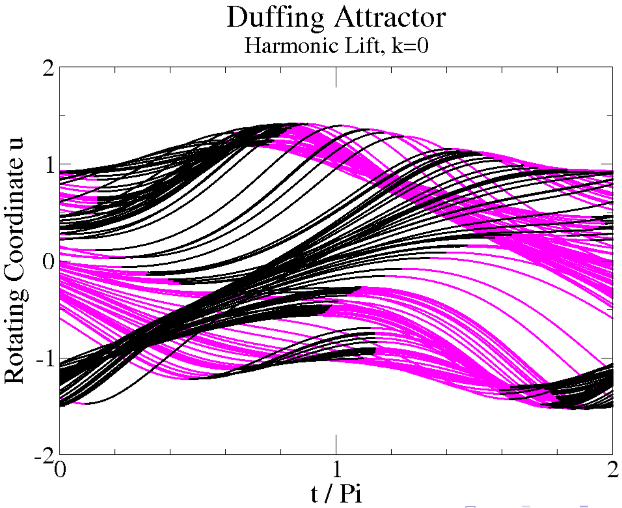
$$\frac{d}{dt}\left[\begin{array}{c} X \\ Y \end{array}\right] = \left[\begin{array}{c} F_1(X,Y) \\ F_2(X,Y) \end{array}\right] + \left[\begin{array}{c} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{array}\right]$$

$$\left[\begin{array}{c} u(t) \\ v(t) \end{array}\right] = \left[\begin{array}{cc} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{array}\right] \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right]$$

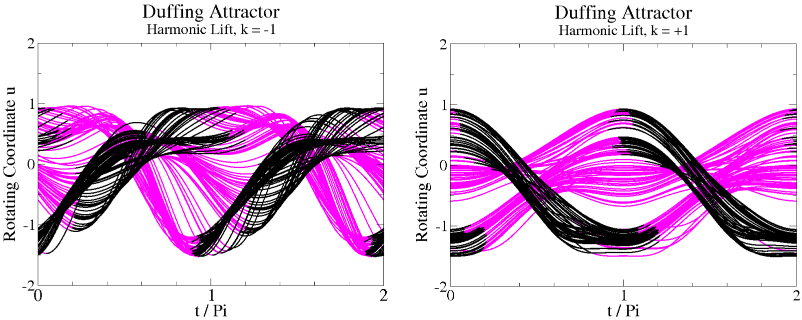
$$\frac{d}{dt}\left[\begin{array}{c} u \\ v \end{array}\right] = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \left[\begin{array}{c} -v \\ +u \end{array}\right]$$

$$\text{Diffeomorphisms} : \quad \Omega = n\ \omega_d$$

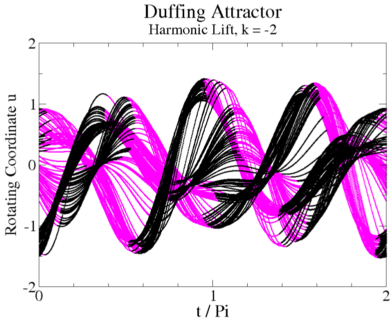
Duffing Attractor, Toroidal Representation

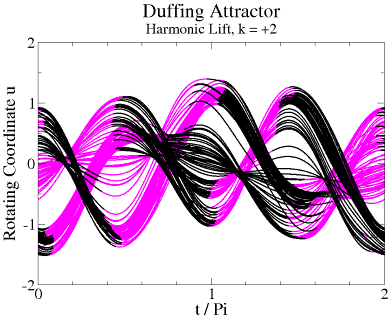


Duffing Attractor, Rotation by ± 1



Duffing Attractor, Rotation by ± 2





Angular Momentum and Energy

$$L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau X dY - Y dX \qquad K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) dt$$

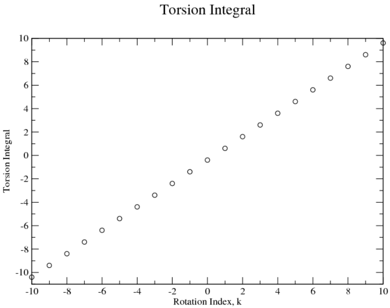
$$L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle \qquad K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \rangle$$

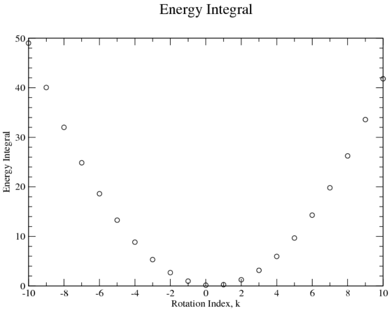
$$= L(0) + \Omega \langle R^2 \rangle \qquad = K(0) + \Omega L(0) + \frac{1}{2}\Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (u^2 + v^2) dt$$

Energy and Angular Momentum

Quantum Number n





Knot Representations

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

$$\text{Repere Mobile: } \mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)$$

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

$$(X(t), Y(t)) \rightarrow \mathbf{X}(t) = \mathbf{K}(\theta) + X(t)\mathbf{n}(\theta) + Y(t)\mathbf{b}(\theta)$$

$$\frac{\theta}{2\pi} = \frac{t}{T}$$

Equivalent Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension = 3.

Are these obstructions removed by injections into higher dimensions: R^4, R^5, R^6 ?

Necessary Labels

	Parity	Knot Type	Global Torsion
R^3	Y	Y	Y
R^4	-	-	Y
R^5	-	-	-

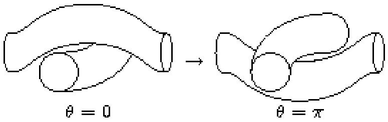
There is one *Universal* representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

$R^3 \rightarrow R^4$

Parity Isotopy in R^4

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{\text{Inject}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{pmatrix} \xrightarrow{\text{Isotopy}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \cos\theta \\ x^3 \sin\theta \end{pmatrix} \underset{\theta=\pi}{\xrightarrow{\text{Project}}} \begin{pmatrix} x^1 \\ x^2 \\ -x^3 \end{pmatrix}.$$

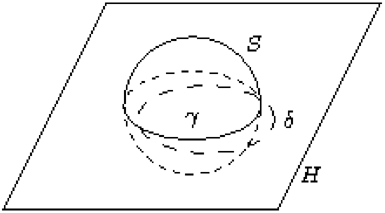
Knot Type Isotopy in R^4



Global Torsion Isotopy in R^5

$$\begin{bmatrix} s \\ re^{i\phi} \end{bmatrix} \mapsto \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix} \rightarrow \left[\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{array}\right] \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix}$$

Continued Inequivalence in R^4



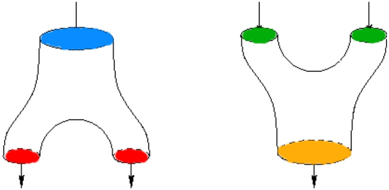
The General Program

- $\mathcal{M}^n \rightarrow R^n$
- Identify all representation labels
- $R^n \rightarrow R^{n+1}$: Which labels drop away?
- $\rightarrow n+2, n+3, \ldots, 2n$: Which labels drop away?

- Group Theory: Complete set of Reps separate points.
- Dynamical Systems: Complete set of Reps separate diffeomorphisms.

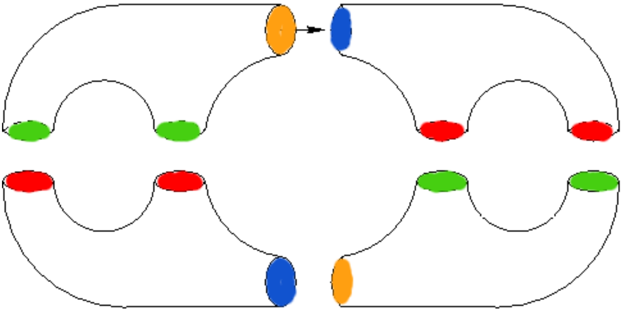
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



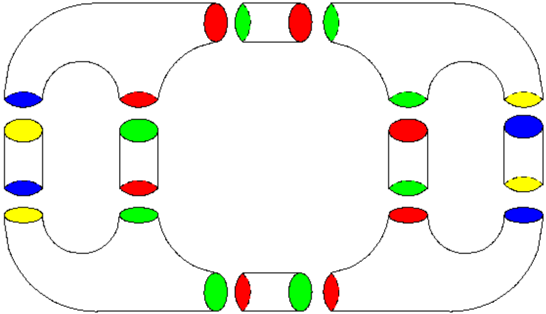
- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Application: Lorenz Dynamics, g=3



$g - 1$ Pairs of "trinions"

Insert A Flow Tube at Each Input



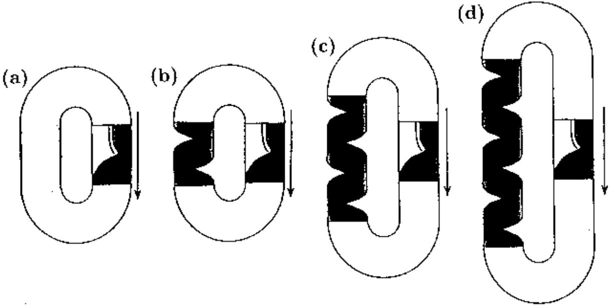
$3 \times (g-1)$ Local Torsion integers: Isotope in R^5
Parity: Isotope in R^4
Knot Type: Isotope in R^4

La Fin

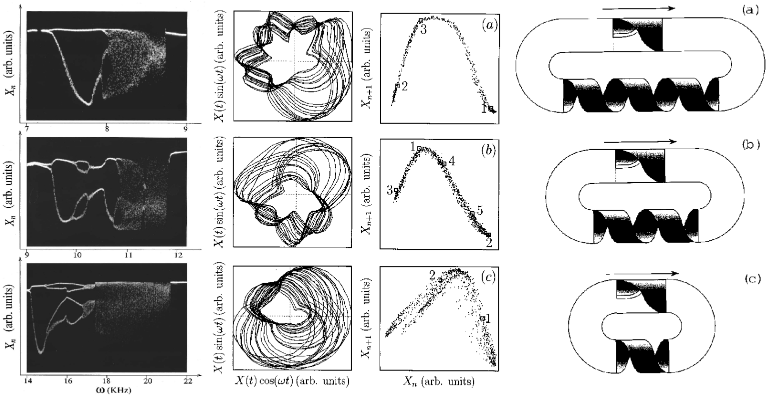
Merci Bien pour votre attention.

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change



Lefranc - Cargese