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Representation Theory for Strange Attractors

Robert Gilmore and Daniel J. Cross

Physics Department
Drexel University
Philadelphia, PA 19104
robert.gilmore@drexel.edu, daniel.j.cross@drexel.edu

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ICCSA, Le Havre, France, Tuesday, 30 June 2009, Room D, 14:50

Data

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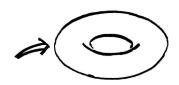




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Embedding

What to do with Data

Step 1: Data \rightarrow Embedding

Step 2: Analyze Reconstructed Attractor

Step 3: What do you learn about:

The Data
The Embedding

???????

Background

Representation Theory for Strange Attractors

Embeddings-

Important theorems

Whitney (1936): $\mathcal{M}^n \to \mathbb{R}^N$: N generic functions -Embedding if $N \geq 2n + 1$.

Takens (1981): $(\mathcal{M}^n, \dot{X} = F(X)) \rightarrow (R^N, Flow)$: One generic function at N measurement intervals. Embedding if $N \ge 2n + 1$.

Wu (1958): All embeddings $\mathcal{M}^n \to \mathbb{R}^N$ are isotopic for N > 2n + 1 and n > 1.

Embeddings and Representations

Representation Theory for Strange Attractors

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Embeddings and Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a representation of an attractor.

Equivalence is by Isotopy.

Preference is for embeddings of lowest possible dimension.

Possible Inequivalence for $n \leq N \leq 2n$.

Geometry, Dynamics, Topology

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Embedding

What do you want to learn?

- \bullet Geometry (Fractals, ...): "Independent" of Embedding
- Dynamics (Lyapunovs, ...) "Independent" of Embedding, but beware of spurious LEs
- Topology: some indices depend on embedding, others (mechanism) do not.

Mechanism

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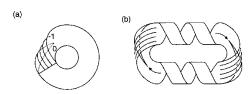
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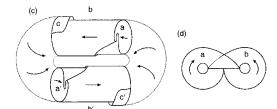
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Revealed by Branched Manifolds





Torus and Genus

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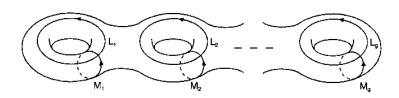
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Classification of 3D Attractors



Program: $\mathcal{M}^3 \to R^3, R^4, R^5, R^6$

Representation Labels

Representation Theory for Strange Attractors

Inequivalent Representations in \mathbb{R}^3

Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

> **Parity** Ρ

Global Torsion N

Knot Type KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Another Visualization

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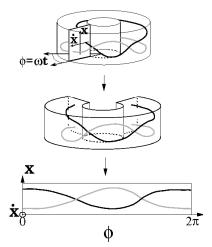
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Cutting Open a Torus



Two Phase Spaces: R^3 and $D^2 \times S^1$

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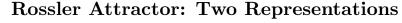
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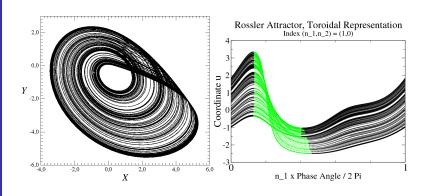
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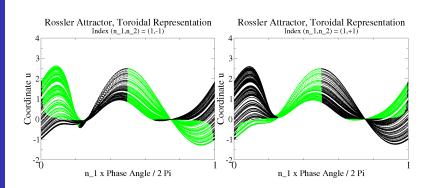


Other Diffeomorphic Attractors

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Rossler Attractor:

Two More Representations with $n = \pm 1$



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Rotating a Driven Attractor

$$\frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X,Y) \\ F_2(X,Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix}$$

$$\left[\begin{array}{c} u(t) \\ v(t) \end{array}\right] = \left[\begin{array}{cc} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{array}\right] \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right]$$

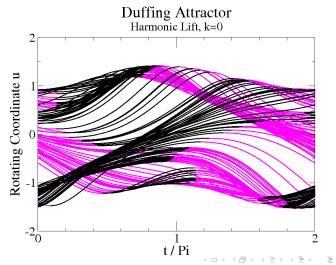
$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix}$$

Diffeomorphisms: $\Omega = n \omega_d$

Representations of Duffing Attractor

Representation Theory for Strange Attractors

Duffing Attractor, Toroidal Representation



Representations of Duffing Attractor

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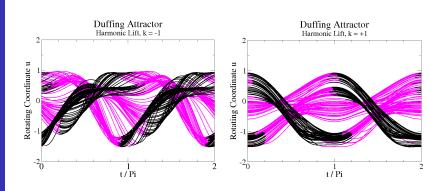
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Duffing Attractor, Rotation by ± 1



Representations of Duffing Attractor

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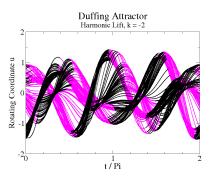
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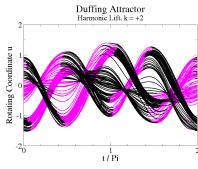
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Duffing Attractor, Rotation by ± 2





Angular Momentum and Energy

$$L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} X dY - Y dX \quad K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) dt$$

$$L(\Omega) = \langle u\dot{v} - v\dot{u}\rangle$$
 $K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2)\rangle$

$$= L(0) + \Omega \langle R^2 \rangle$$

$$= K(0) + \Omega L(0) + \frac{1}{2} \Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (u^2 + v^2) dt$$

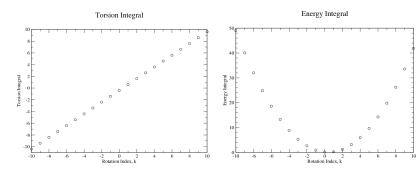
New Measures, Diffeomorphic Attractors

Representation Theory for Strange Attractors



Energy and Angular Momentum

Quantum Number n



Oriented Knot Type

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Knot Representations

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

Repere Mobile: $\mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)$

$$\frac{d}{ds} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

$$(X(t), Y(t)) \to \mathbf{X}(t) = \mathbf{K}(\theta) + X(t)\mathbf{n}(\theta) + Y(t)\mathbf{b}(\theta)$$

$$\frac{\theta}{2\pi} = \frac{t}{T}$$

Creating Isotopies

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Equivalent Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension = 3.

Are these obstructions removed by injections into higher dimensions: R^4, R^5, R^6 ?

Creating Isotopies

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Necessary Labels

	Parity	Knot Type	Global Torsion
R^3	Υ	Y	Y
R^4	-	-	Υ
R^5	_	-	-

There is one *Universal* representation in \mathbb{R}^N , $N \geq 5$. In \mathbb{R}^N the only topological invariant is *mechanism*.

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Parity Isotopy in R^4

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{\text{Inject}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{pmatrix} \xrightarrow{\text{Isotopy}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \cos \theta \\ x^3 \sin \theta \end{pmatrix} \xrightarrow{\text{Project}} \begin{pmatrix} x^1 \\ x^2 \\ -x^3 \end{pmatrix}.$$

Knot Type Isotopy in R^4



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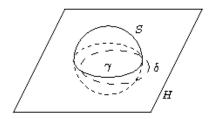
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Global Torsion Isotopy in R^5

$$\begin{bmatrix} s \\ re^{i\phi} \end{bmatrix} \mapsto \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix}$$

Continued Inequivalence in R^4



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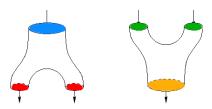
The General Program

- $\bullet \mathcal{M}^n \to R^n$
- Identify all representation labels
- $R^n \to R^{n+1}$: Which labels drop away?
- $\bullet \to n+2, n+3, \ldots, 2n$: Which labels drop away?
- Group Theory: Complete set of Reps separate points.
- Dynamical Systems: Complete set of Reps separate diffeomorphisms.

Aufbau Princip for Bounding Tori

Representation Theory for Strange Attractors

Any bounding torus can be built up from equal numbers of stretching and squeezing units



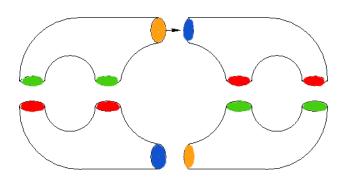
- Outputs to Inputs
- No Free Ends
- Colorless



Aufbau Princip for Bounding Tori

Representation Theory for Strange Attractors

Application: Lorenz Dynamics, g=3

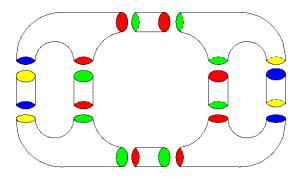


q-1 Pairs of "trinions"

Indices for Tori in \mathbb{R}^3

Representation Theory for Strange Attractors

Insert A Flow Tube at Each Input



 $3 \times (q-1)$ Local Torsion integers: Isotope in R^5 Isotope in \mathbb{R}^4 Parity:

Isotope in \mathbb{R}^4 Knot Type:

La Fin

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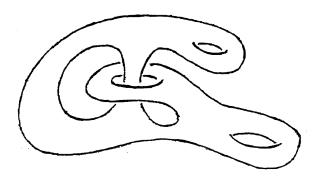
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Merci Bien pour votre attention.

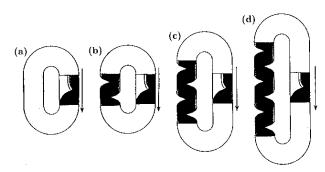


Determine Topological Invariants

Representation Theory for Strange Attractors

What Do We Learn?

- BM Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ...
- Independent of Embedding: Mechanism



Perestroikas of Strange Attractors

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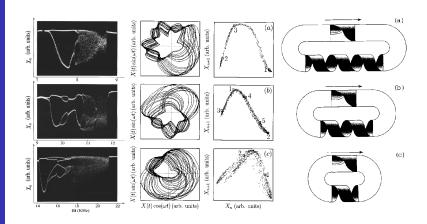
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Evolution Under Parameter Change



Lefranc - Cargese

