# The Topology of Chaos Chapter 3: Topology of Orbits 

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## Chaos

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Chaos

## Motion that is

- Deterministic: $\quad \frac{d x}{d t}=f(x)$
- Recurrent
- Non Periodic
- Sensitive to Initial Conditions


## Strange Attractor

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Strange Attractor

The $\Omega$ limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- "Abundant"
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor


## Skeletons

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Topology of Orbits-03a

Topology of Orbits-03b

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## UPOs Outline Strange attractors



01


01011


BZ reaction

## Skeletons

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## UPOs Outline Strange attractors



Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10 .

Lefranc - Cargese

## Dynamics and Topology

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Organization of UPOs in $R^{3}$ :

 Gauss Linking Number$$
L N(A, B)=\frac{1}{4 \pi} \oint \oint \frac{\left(\mathbf{r}_{A}-\mathbf{r}_{B}\right) \cdot d \mathbf{r}_{A} \times d \mathbf{r}_{B}}{\left|\mathbf{r}_{A}-\mathbf{r}_{B}\right|^{3}}
$$

\# Interpretations of $\mathrm{LN} \simeq$ \# Mathematicians in World

## Linking Numbers

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Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

# Linking Number of Two UPOs 



Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

## Lefranc - Cargese

## Evolution in Phase Space

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Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

One Stretch-\&-Squeeze Mechanism


## Motion of Blobs in Phase Space

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Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of
Orbits-04a
Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Stretching - Squeezing



## Collapse Along the Stable Manifold

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Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

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Topology of Orbits-04b

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Topology of Orbits-06

## Birman - Williams Projection

Identify $x$ and $y$ if

$$
\lim _{t \rightarrow \infty}|x(t)-y(t)| \rightarrow 0
$$



## Fundamental Theorem

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Topology of Orbits-02

Topology of
Orbits-03a
Topology of Orbits-03b

Topology of
Orbits-04a
Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Birman - Williams Theorem

## If:

## Then:

Fundamental Theorem

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Birman - Williams Theorem

## If:

## Certain Assumptions

Fundamental Theorem

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

# Birman - Williams Theorem 

If: Certain Assumptions

## Then:

## Specific Conclusions

## Birman-Williams Theorem

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Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Assumptions, B-W Theorem

## A flow $\Phi_{t}(x)$

- on $R^{n}$ is dissipative, $\underline{n=3}$, so that $\lambda_{1}>0, \lambda_{2}=0, \lambda_{3}<0$.
- Generates a hyperbolic strange attractor $\mathcal{S A}$

IMPORTANT: The underlined assumptions can be relaxed.

## Birman-Williams Theorem

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a Orbits-04b

## Conclusions, B-W Theorem

- The projection maps the strange attractor $\mathcal{S A}$ onto a 2 -dimensional branched manifold $\mathcal{B M}$ and the flow $\Phi_{t}(x)$ on $\mathcal{S A}$ to a semiflow $\Phi(x)_{t}$ on $\mathcal{B M}$.
- UPOs of $\Phi_{t}(x)$ on $\mathcal{S A}$ are in 1-1 correspondence with UPOs of $\bar{\Phi}(x)_{t}$ on $\mathcal{B M}$. Moreover, every link of UPOs of $\left(\Phi_{t}(x), \mathcal{S A}\right)$ is isotopic to the correspond link of UPOs of $\left(\bar{\Phi}(x)_{t}, \mathcal{B M}\right)$.

Remark: "One of the few theorems useful to experimentalists."

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

## Rössler:

## Attractor

## Branched Manifold



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The Topology
    of Chaos
    Chapter 3:
    Topology of
        Orbits
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Topology of
Orbits-01
Topology of
Orbits-02
Topology of
Orbits-03a
Topology of
Orbits-03b
Topology of
Orbits-04a
Topology of
Orbits-04b

\section*{Attractor}

\section*{Lorenz:}

\section*{Branched Manifold}


Examples of Branched Manifolds

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of
Orbits-04a
Topology of
Orbits-04b
Topology of Orbits-05

Topology of Orbits-06

\section*{Inequivalent Branched Manifolds}
(a)

(b)

(c)

(d)


\section*{Aufbau Princip for Branched Manifolds}

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Chapter 3:
Topology of
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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

Any branched manifold can be built up from stretching and squeezing units

subject to the conditions:
- Outputs to Inputs
- No Free Ends

\section*{Dynamics and Topology}

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of
Orbits-04a
Topology of
Orbits-04b
Topology of Orbits-05

Topology of Orbits-06

\section*{Rossler System}
(a) Rössler Equations
\[
\begin{aligned}
& \frac{d x}{d t}=-v, \\
& \frac{d y}{d i}=x+a y \\
& \frac{d x}{d i}=d+z(t-c)
\end{aligned}
\]
(9)
\[
\left(\begin{array}{cc}
-1 & 0 \\
0 & 0
\end{array}\right)
\]
\[
(0+1)
\]
(b)

\(x\)

(e)

(c)

(d)


\section*{Dynamics and Topology}

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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

\section*{Lorenz System}
(a)

Loreaz Equations
\[
\begin{aligned}
& \frac{d x}{d t}=-\Delta x+o y \\
& \frac{d y}{d t}=\pi x \cdot y \cdot x z \\
& \frac{d z}{d t}=-b z+x y
\end{aligned}
\]
(f)
\[
\left.\begin{array}{l}
\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right\} \\
(+1 \\
-1
\end{array}\right\}
\]
(b)

(c)

(e)

(d)


Dynamics and Topology

The Topology
of Chaos
Chapter 3: Topology of

Orbits
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Topology of Orbits-01

Topology of Orbits-02

Topology of Orbits-03a

Topology of Orbits-03b

Topology of Orbits-04a

Topology of Orbits-04b

Topology of Orbits-05

Topology of Orbits-06

\section*{Poincaré Smiles at Us in \(R^{3}\)}
- Determine organization of UPOs \(\Rightarrow\)
- Determine branched manifold \(\Rightarrow\)
- Determine equivalence class of \(\mathcal{S A}\)```

