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The Topology of Chaos Chapter 3: Topology of Orbits

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Chaos

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Chaos

Motion that is

• Deterministic:

$$\frac{dx}{dt} = f(x)$$

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- Recurrent
- Non Periodic
- Sensitive to Initial Conditions

Strange Attractor

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Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- "Abundant"
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

Skeletons

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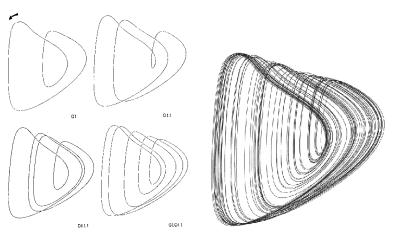
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UPOs Outline Strange attractors



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BZ reaction

Skeletons

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UPOs Outline Strange attractors

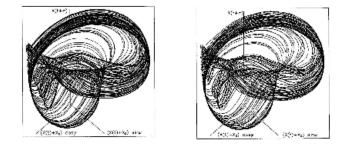


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

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Organization of UPOs in R³: Gauss Linking Number

Dynamics and Topology

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq \#$ Mathematicians in World

Linking Numbers

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Linking Number of Two UPOs

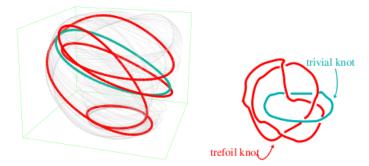


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

Evolution in Phase Space

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One Stretch-&-Squeeze Mechanism

(c) (d) boundary layer stretch squeeze (b) (a)

Motion of Blobs in Phase Space

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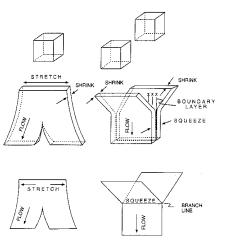
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Stretching — Squeezing



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Collapse Along the Stable Manifold

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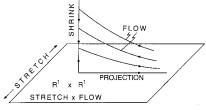
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Birman - Williams Projection

Identify x and y if

 $\lim_{t \to \infty} |x(t) - y(t)| \to 0$



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Fundamental Theorem

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Birman - Williams Theorem

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Then:

Tf:

Fundamental Theorem

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Birman - Williams Theorem

If:

Certain Assumptions

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Then:

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Birman - Williams Theorem

Certain Assumptions

Specific Conclusions

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Then:

Tf:

Eundamental Theorem

Birman-Williams Theorem

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Assumptions, B-W Theorem

A flow $\Phi_t(x)$

• on \mathbb{R}^n is dissipative, $\underline{n=3}$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$

 Generates a <u>hyperbolic</u> strange attractor \mathcal{SA}

IMPORTANT: The underlined assumptions can be relaxed.

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Conclusions, B-W Theorem

- The projection maps the strange attractor SA onto a 2-dimensional branched manifold \mathcal{BM} and the flow $\Phi_t(x)$ on SA to a semiflow $\overline{\Phi}(x)_t$ on \mathcal{BM} .
- UPOs of $\Phi_t(x)$ on SA are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on \mathcal{BM} . Moreover, every link of UPOs of $(\Phi_t(x), SA)$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

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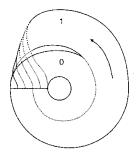
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Attractor Branched Manifold

Rössler:





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A Mechanism with Symmetry

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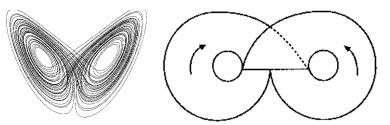
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Attractor

Lorenz: Branched Manifold



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Examples of Branched Manifolds

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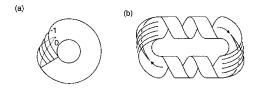
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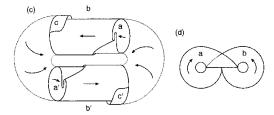
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Inequivalent Branched Manifolds





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Aufbau Princip for Branched Manifolds

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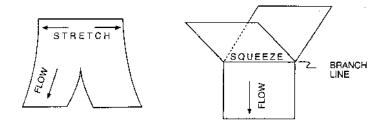
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Any branched manifold can be built up from stretching and squeezing units

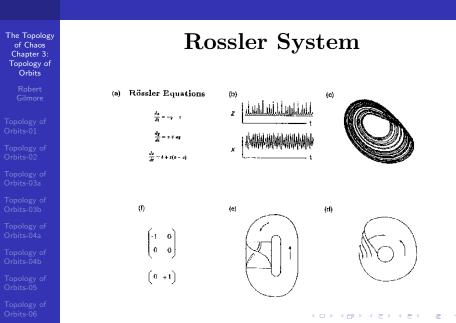


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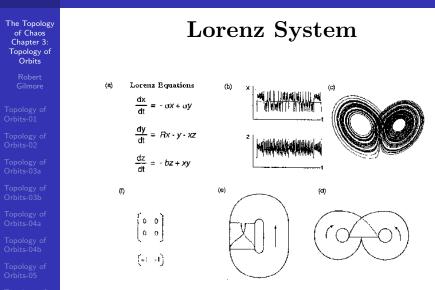
subject to the conditions:Outputs to Inputs

• No Free Ends

Dynamics and Topology



Dynamics and Topology



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Dynamics and Topology

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Poincaré Smiles at Us in R^3

- \bullet Determine organization of UPOs \Rightarrow
- \bullet Determine branched manifold \Rightarrow
- \bullet Determine equivalence class of \mathcal{SA}