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A Chaotic Walk with Friends

Robert Gilmore

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Birthday Party
CORIA, France

June 20, 2011

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Thank You to All My Friends

My colleagues and my friends — my colleagues *are* my friends — introduced me to and then helped to guide me through this new and delightful field.

Many are assembled here today.

To all I express my thanks for helping to make this such a festive occassion.

Outline

1. Overview
2. Experimental Challenge
3. Topology of Orbits
4. Topological Analysis Program
5. Basis Sets of Orbits
6. Bounding Tori
7. Covers and Images
8. Quantizing Chaos
9. Representation Theory of Strange Attractors
10. Summary

Usual Culprits

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Elia Eschenazi

Jorge Tredicce

Torreador

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J. R. Tredicce

Can you explain my data?

I bet you can't explain my data!

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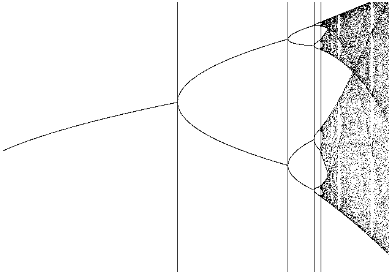
Where is Tredicce coming from?



Feigenbaum:

$$\alpha = \ \ 4.66920\ 16091\ \ldots\ldots$$
$$\delta = -2.50290\ 78750\ \ldots\ldots$$

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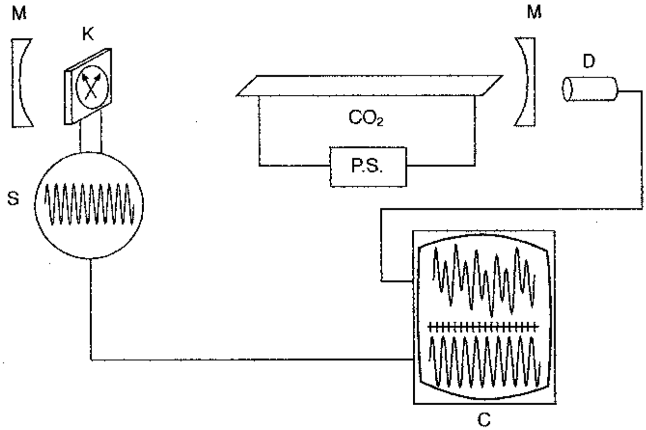
Laser with Modulated Losses Experimental Arrangement



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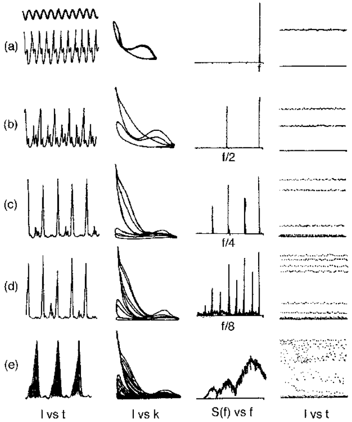
Oscilloscope Traces



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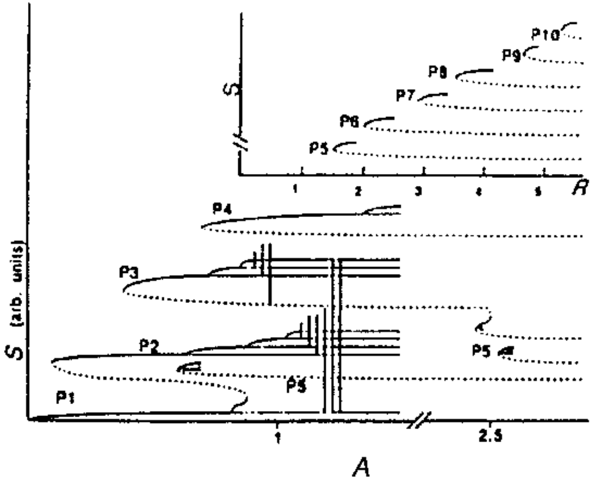
Bifurcation Schematics



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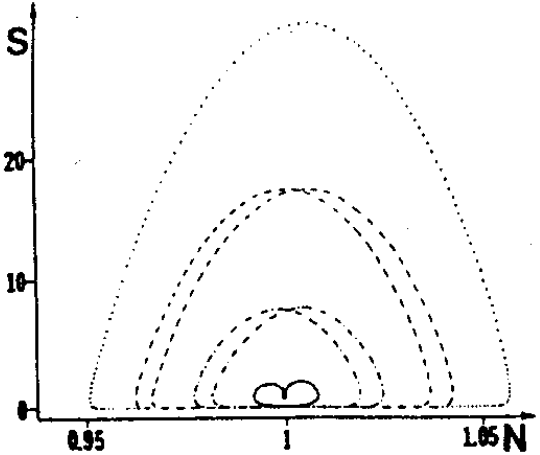
Coexisting Basins of Attraction



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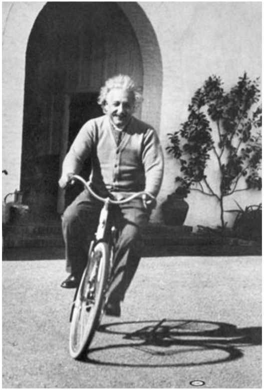
What to Grab Hold of ??



Search for Invariants

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How to Characterize a Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

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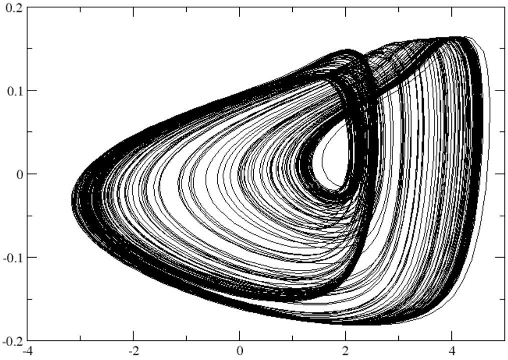
Periodic Orbits are the Key

Joseph Fourier
Linear Systems

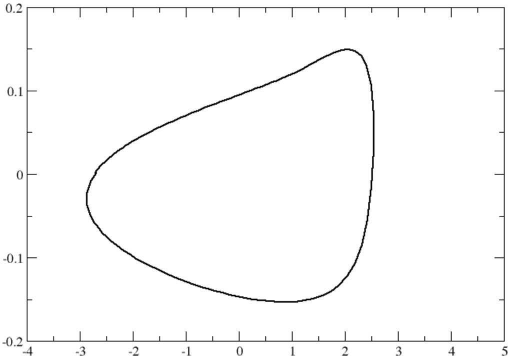
Henri Poincare
Nonlinear Systems

UPOs: Skeletons of Strange Attractors



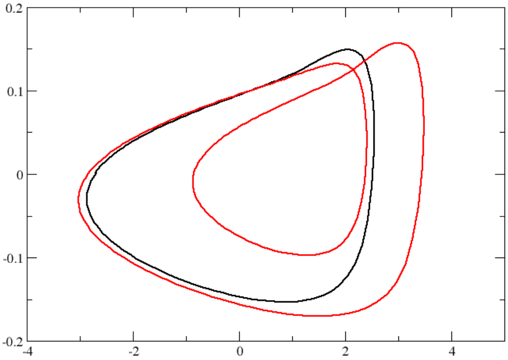
Here is a Strange Attractor (Belousov-Zhabotinskii Reaction)

UPOs: Skeletons of Strange Attractors



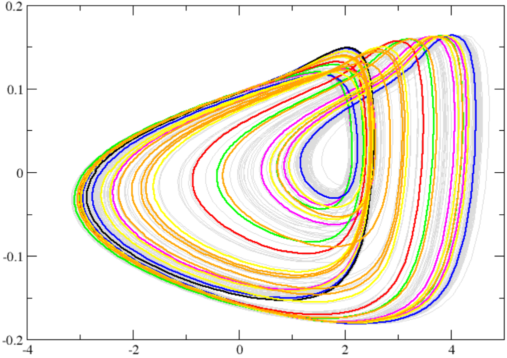
Here is a period-one orbit in the attractor.

UPOs: Skeletons of Strange Attractors



Period-1 and period-2 orbits from the attractor.

UPOs: Skeletons of Strange Attractors



Lots of them.

Ask the Masters: 4

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Quantitative Measures for Periodic Orbits ??

Carl Friedrich Gauss
They Link: Pairwise, 3-Wise, ...

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Organization of UPOs in R^3:

Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

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Linking Number of Two UPOs

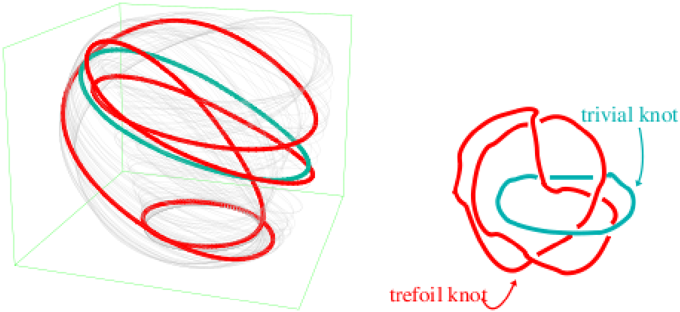

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

Compute Table of Experimental LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

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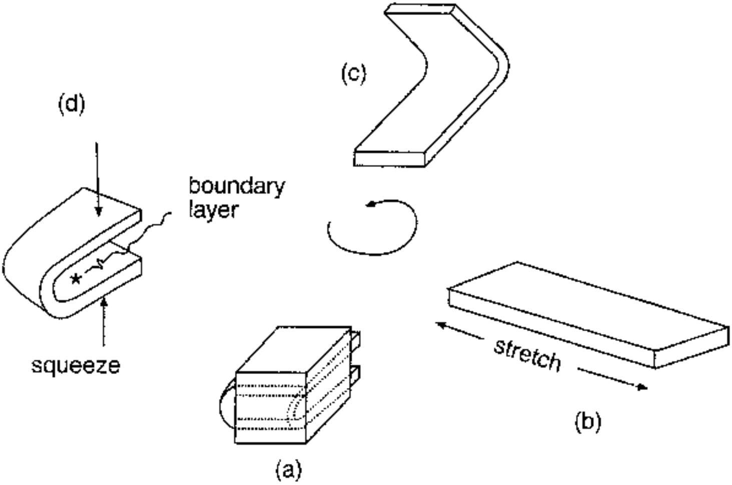
Stretching and Folding



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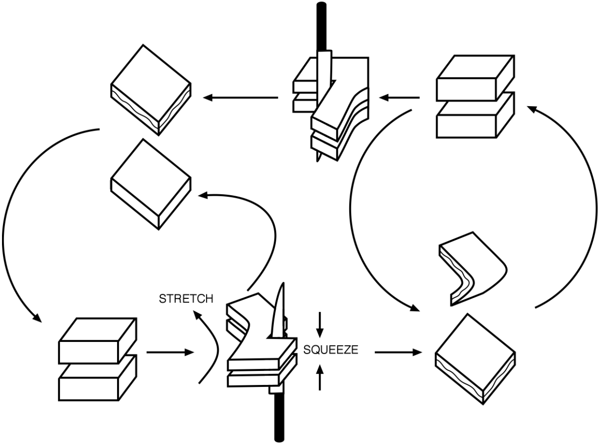
Tearing and Squeezing



Ask the Masters: 5 & 6

Systematics of Linking Number Tables

Joan S. Birman

Robert F. Williams

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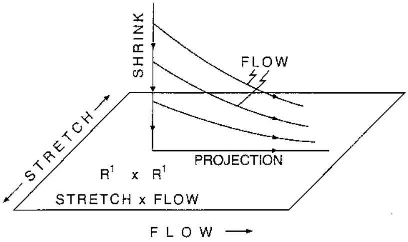
Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



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Birman - Williams Theorem

If:

Then:

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Birman - Williams Theorem

If: **Certain Assumptions**

Then:

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Fundamental Theorem

Birman - Williams Theorem

If: Certain Assumptions

Then: Specific Conclusions

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Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.
- **Generates a** <u>**hyperbolic**</u> **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

• The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.

• UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

Remark: "One of the few theorems useful to experimentalists."

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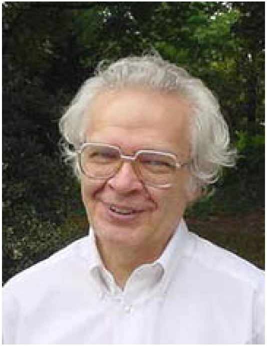
Two Standard Strange Attractors



Otto Rössler

Edward Lorenz

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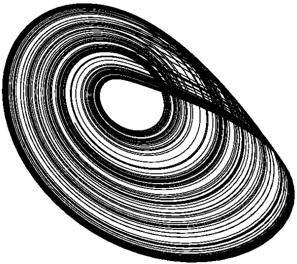
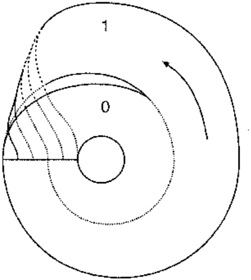
Rössler:

Attractor



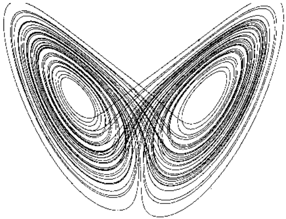
Branched Manifold

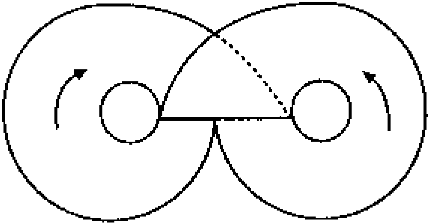


A Mechanism with Symmetry

Lorenz:

Attractor **Branched Manifold**





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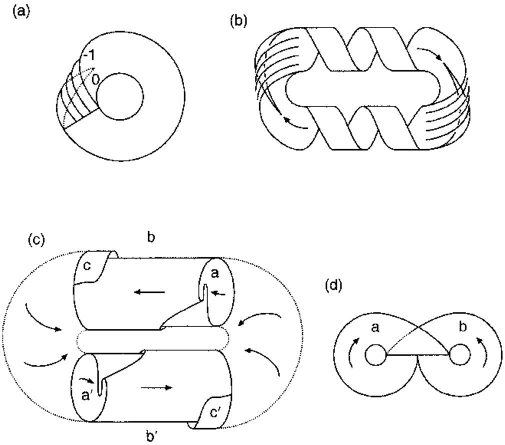
Inequivalent Branched Manifolds



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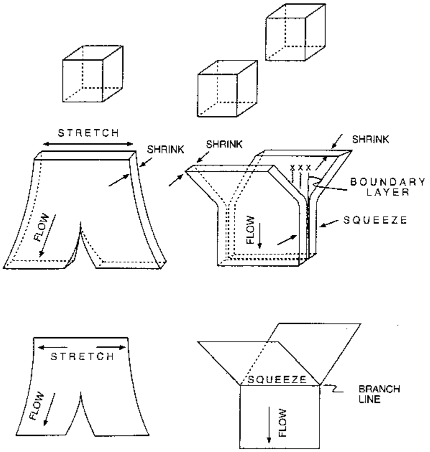
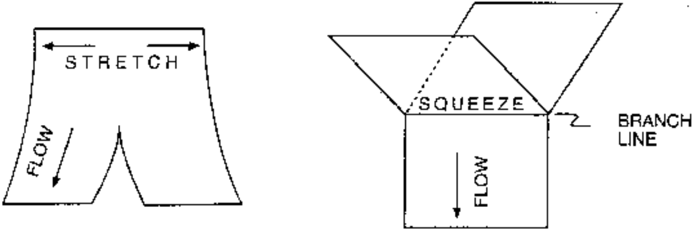
Stretching — Squeezing



Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units



subject to the conditions:

- Outputs to Inputs
- No Free Ends

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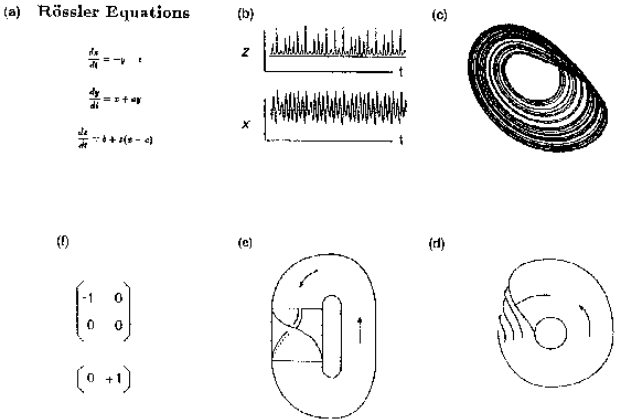
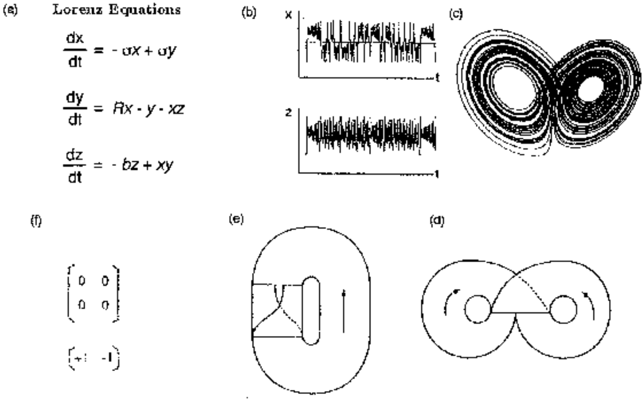
Rössler System



Lorenz System



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Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

We Like to be Organized

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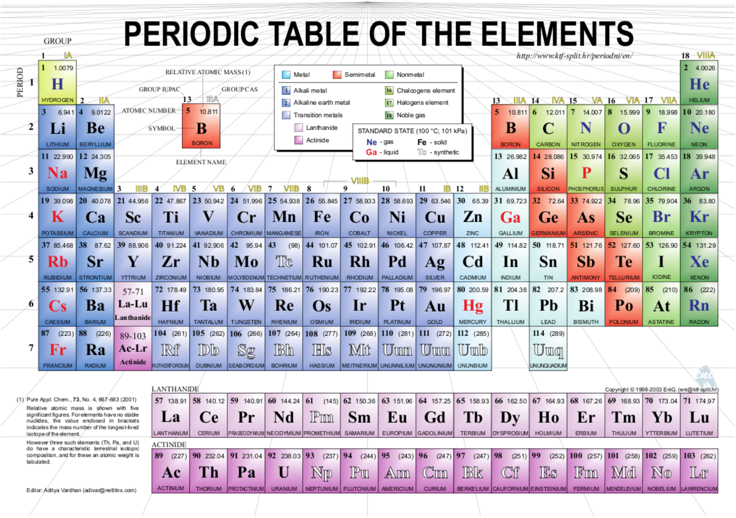
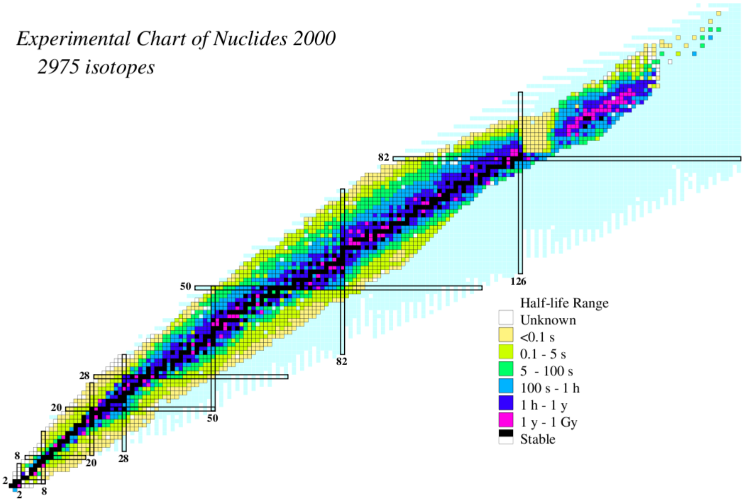


We Like to be Organized



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The Topological Team

Hernan G. Solari
(disguised as Cristal)

Gabriel B. Mindlin

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Topological Analysis Program

- Locate Periodic Orbits
- Create an Embedding
- Determine Topological Invariants (LN)
- Identify a Branched Manifold
- Verify the Branched Manifold

Additional Steps

- Model the Dynamics
- Validate the Model

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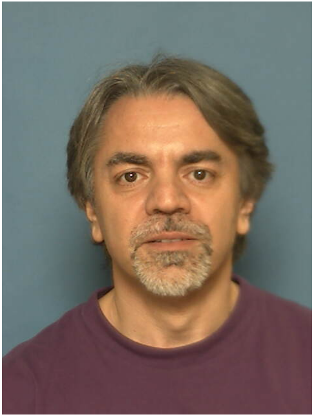
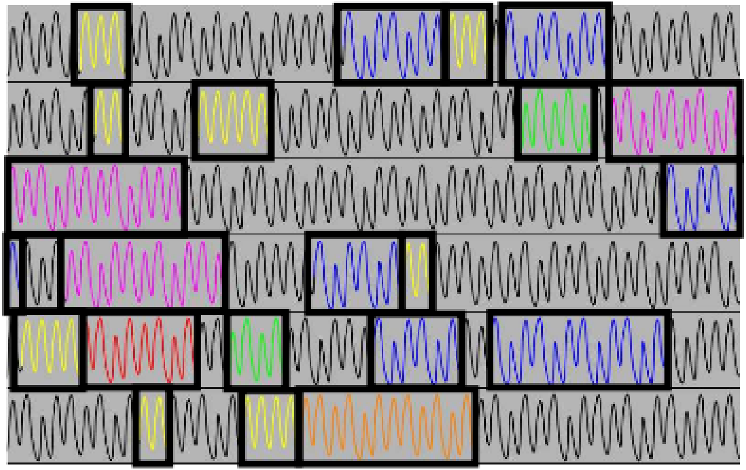
The Topological Team

Nicholas B. Tufillaro



Mario A. Natiello

Method of Close Returns



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Embeddings (= Black Magic)

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

Locate UPOs

An Embedding and Periodic Orbits

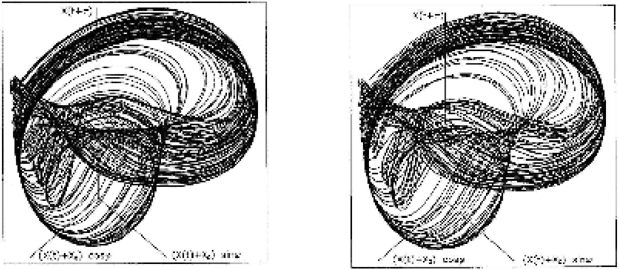

Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

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An Embedding and Periodic Orbits

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Linking Number of Orbit Pairs

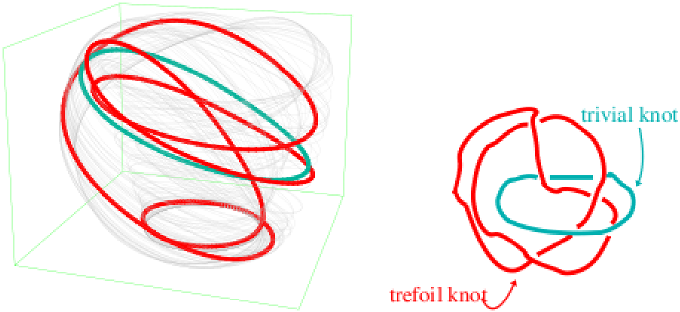

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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Compute Table of Experimental LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1s$	$1f$	2_1	$3f$	$3s$	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Lefranc - Cargese

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Guess Branched Manifold

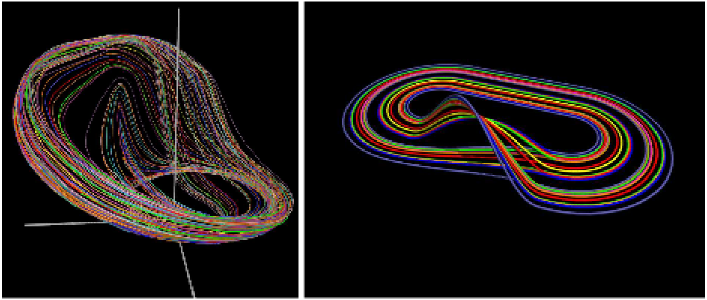

Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

A Chaotic Walk with Friends

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Identification & ‘Confirmation’

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Fail to Reject H_0)

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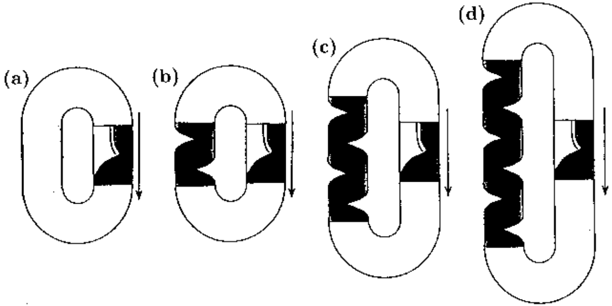
What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



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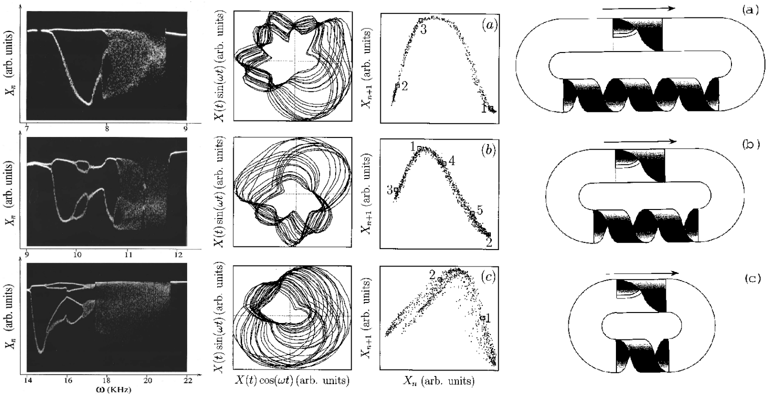
Evolution Under Parameter Change



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Evolution Under Parameter Change

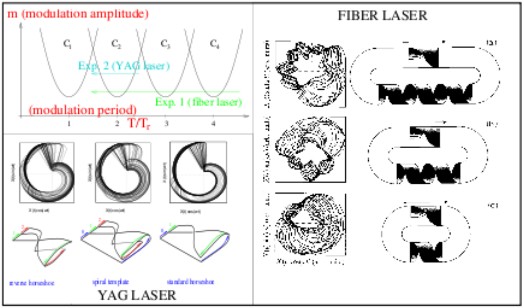

Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

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Analysis of Nonstationary Data

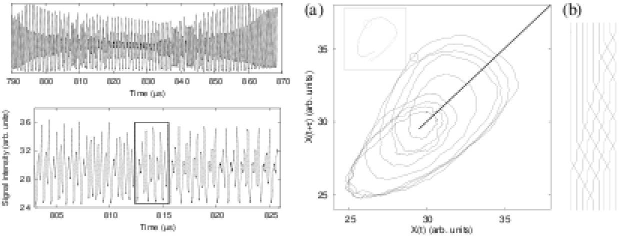

Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

A Prediction Realized

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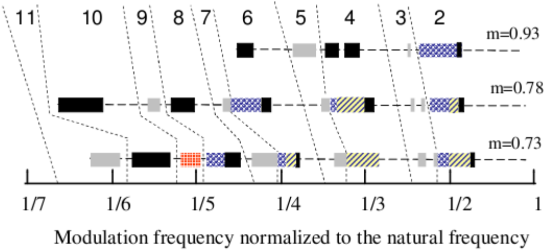
TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						



Used and Martin (2010)

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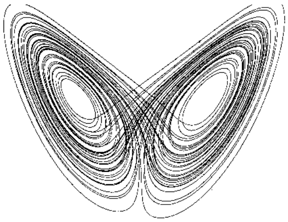
Compare with Original Objectives

Construct a simple, algorithmic procedure for:

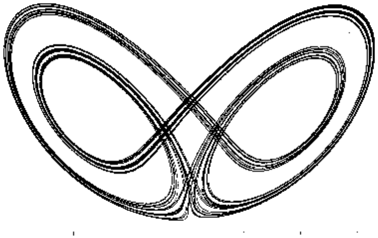
- Classifying strange attractors
- Extracting classification information

from experimental signals.

There Are Some Missing Orbits



Lorenz



Shimizu-Morioka

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There Are Some Missing Orbits

Francisco Papoff

Arimondo et al.

- The branched manifold remains unchanged,
- the spectrum of orbits on it changes.

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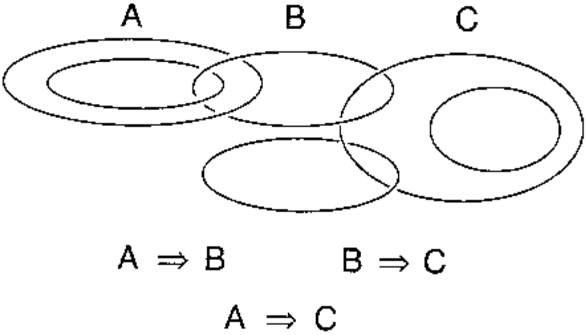
Topological Orbit Forcing



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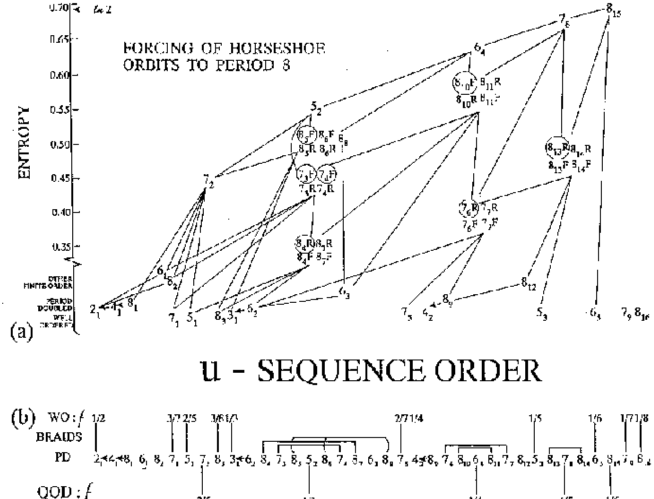
Forcing Diagram - Horseshoe



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Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

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Groups and Strange Attractors

Christophe Letellier

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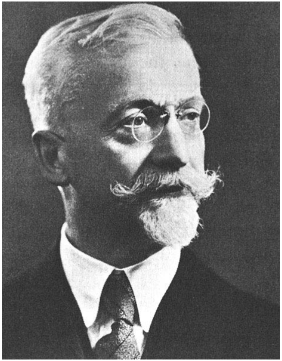
Ask the Masters: 9

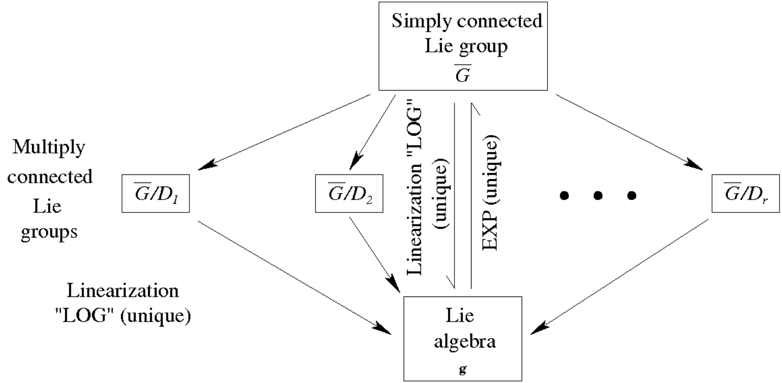
What is the relation between symmetry groups & strange attractors?



Elie Cartan

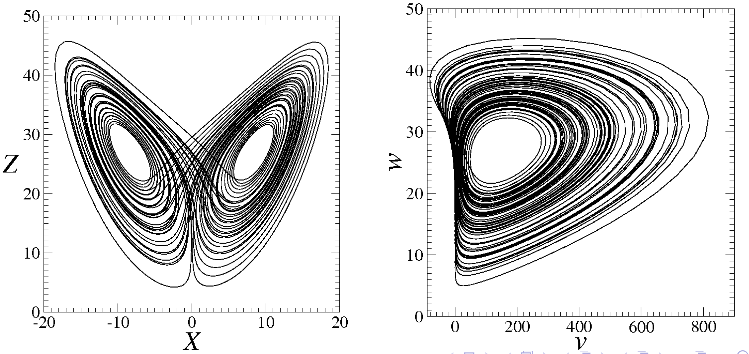
Universal Covering Group

Cartan's Theorem for Lie Groups

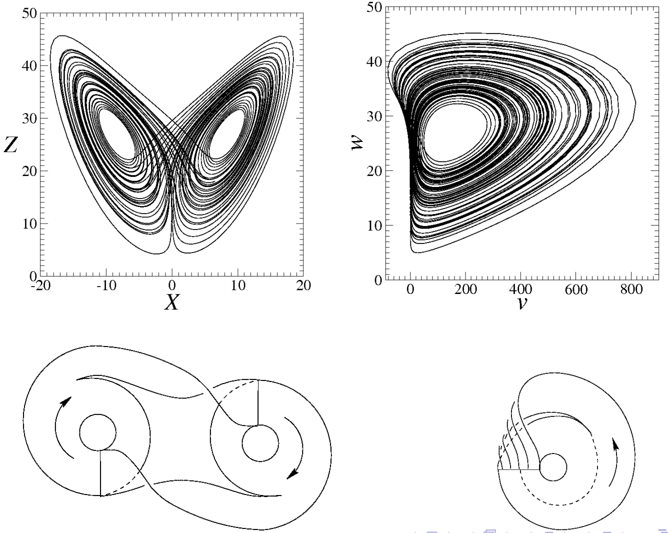


Modding Out a Rotation Symmetry

$$\left(\begin{array}{c} X \\ Y \\ Z \end{array}\right) \rightarrow \left(\begin{array}{c} u \\ v \\ w \end{array}\right) = \left(\begin{array}{c} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{array}\right)$$

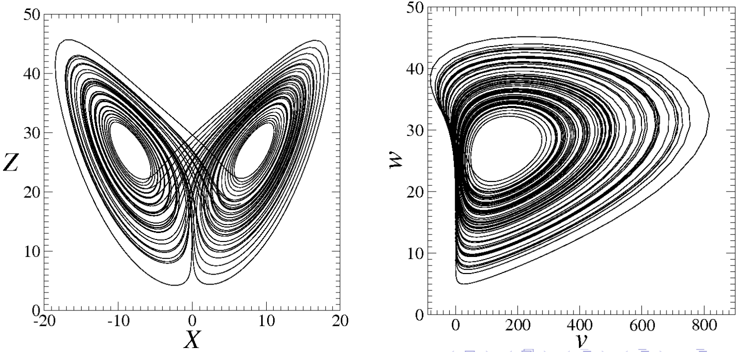


Lorenz Attractor and Its Image



Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$



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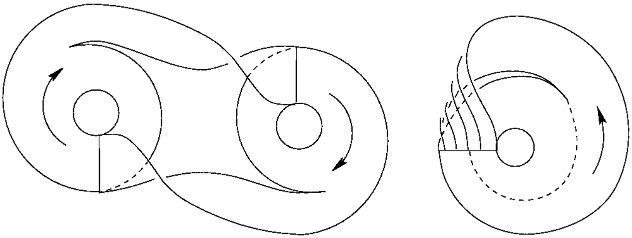
Cover-Image Branched Manifolds



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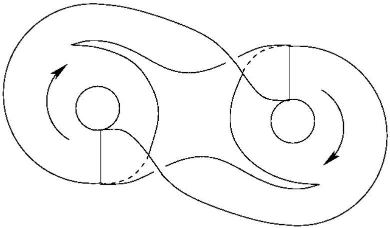
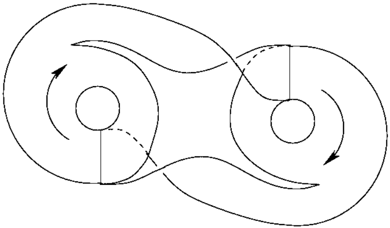
Two Two-fold Lifts Different Symmetry



Rotation Symmetry



Inversion Symmetry

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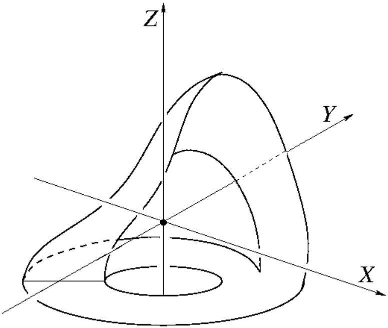
Topological Index: Choose Group
Choose Rotation Axis (Singular Set)



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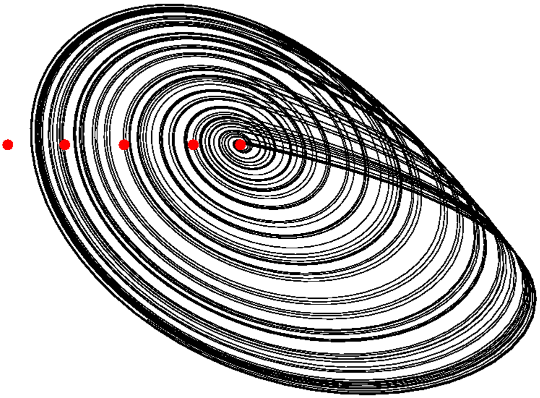
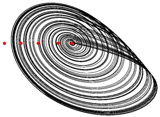
Different Rotation Axes Produce Different (Nonisotopic) Lifts



Nonisotopic Locally Diffeomorphic Lifts

(a) $\mu = 0.0$ (c) $\mu = -2.083$ (e) $\mu = -4.166$

(b) $\mu = -0.84548$ (d) $\mu = -3.14674$



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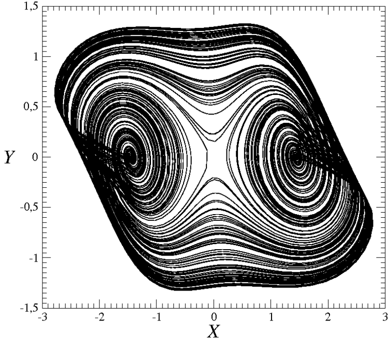
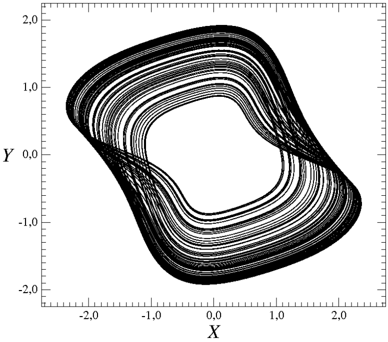
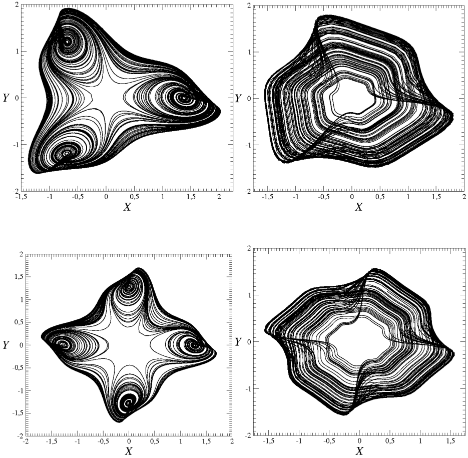
Two Two-fold Covers Same Symmetry

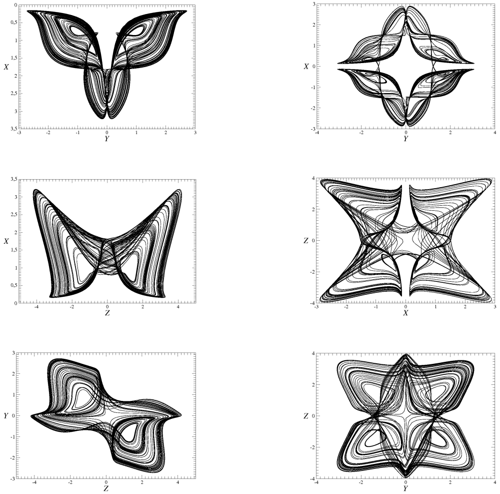




Three-fold, Four-fold Covers



Two Inequivalent Lifts with V_4 Symmetry



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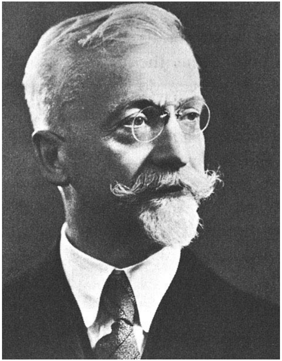
Ask the Masters: 9



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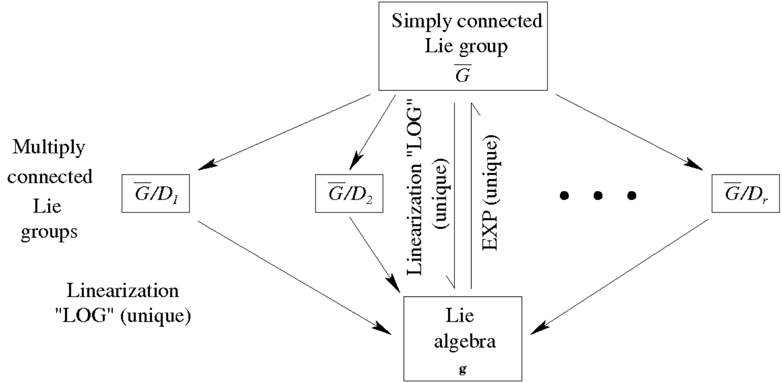
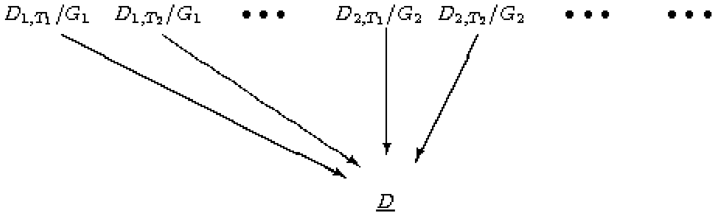
Cartan's Theorem for Lie Groups



Universal Image Dynamical System

Locally Diffeomorphic Covers of $\underline{D}$



$\underline{D}$: Universal Image Dynamical System

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Constraints on Branched Manifolds

"Inflate" a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

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Ask the Masters: 10

How do we characterize surfaces?

Leonard Euler
Count holes: $\chi(\partial\mathcal{M})$

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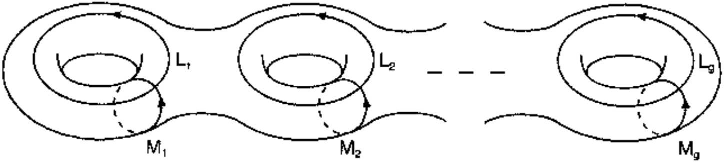
Torus, Longitudes, Meridians



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Usual Culprits: 10

What kind of "dressed tori" enclose strange attractors?

Tsvetelin D. Tsankov
Markov Matrices and Symmetric Cycles

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Surface Singularities

Flow field: three eigenvalues: +, 0, −

Vector field "perpendicular" to surface

Eigenvalues on surface at fixed point: +, −

All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

 # fixed points on surface = index = 2g - 2

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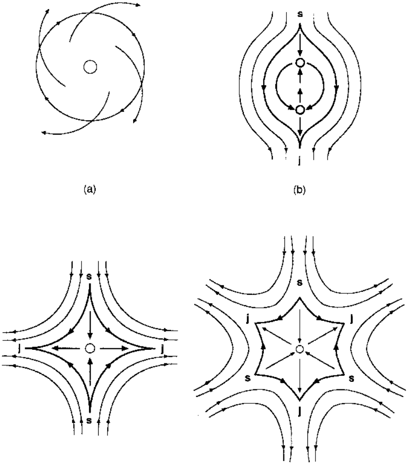
Flow Near a Singularity



(a) (b) (c) (d)

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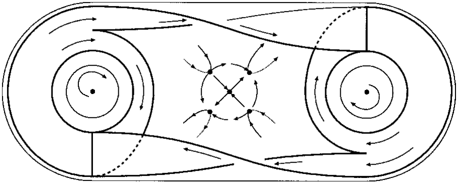
Torus Bounding Lorenz-like Flows



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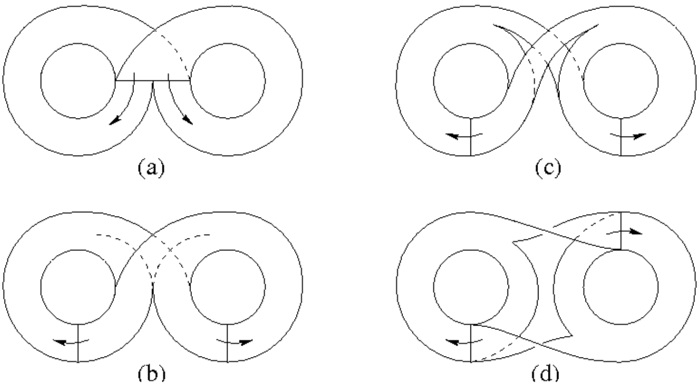
Twisting the Lorenz Attractor



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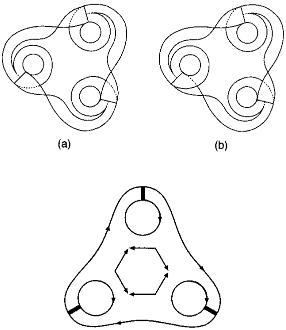
Two possible branched manifolds in the torus with g=4.



(c)

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Bounding Tori contain all known Strange Attractors

Tab.1. All known strange attractors of dimension $d_L < 3$ are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g-1$ Orbit
Rossler, Duffing, Burke and Shaw	A_1	1
Various Lasers, Gateau Roule	A_1	1
Neuron with Subthreshold Oscillations	A_1	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	A_2	$(12)^2$
Multispiral attractors	A_n	$(12^{n-1})^2$
C_n Covers of Rossler	C_n	1^n
C_2 Cover of Lorenz[(a)]	C_4	1^4
C_2 Cover of Lorenz[(b)]	A_3	$(122)^2$
C_n Cover of Lorenz[(a)]	C_{2n}	1^{2n}
C_n Cover of Lorenz[(b)]	P_{n+1}	$(1n)^n$
$2 \to 1$ Image of Fig. 8 Branched Manifold	A_3	$(122)^2$
Fig. 8 Branched Manifold	P_5	$(14)^4$

[(a)] Rotation axis through origin.
[(b)] Rotation axis through one focus.

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Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 $\times$ g-1 matrices

One is cyclic g-1 $\times$ g-1 matrix

Other represents union of cycles

Labeling via (permutation) group theory

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Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

g	m	$(p_1, p_2, \ldots p_m)$	$n_1 n_2 \ldots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

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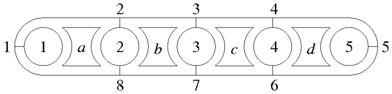
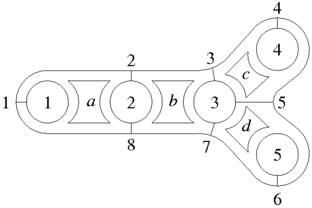
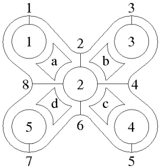
Some Genus-9 Bounding Tori

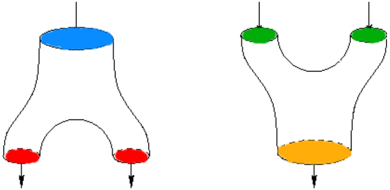






Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

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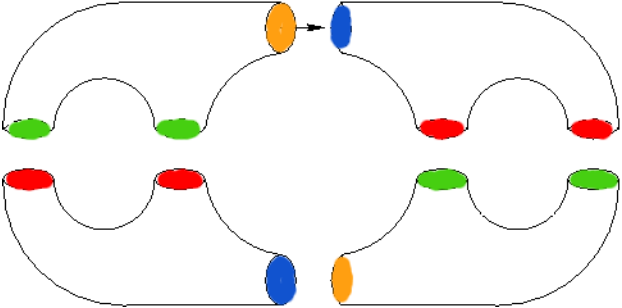
Application: Lorenz Dynamics, g=3



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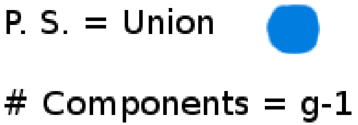
Construction of Poincaré Section



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The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus, g.

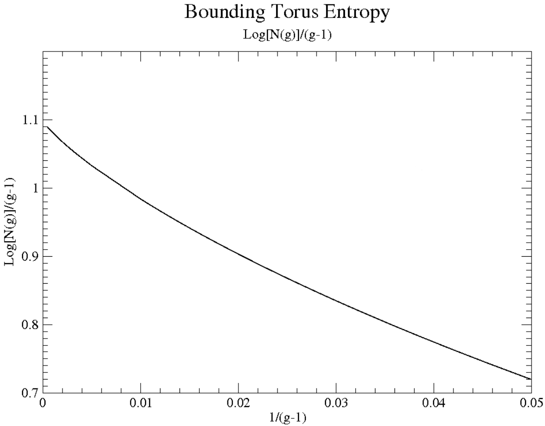
g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

Usual Culprits: 11

How quickly does the number of bounding tori increase with g?

Jacob Katriel
Magician with permutation group cycles.

The Growth is Exponential
The Entropy is log 3



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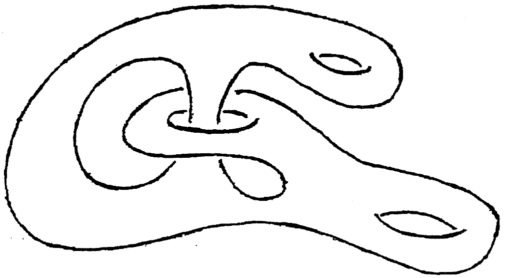
Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

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Ask the Masters: 11

Could there be representation theory for strange attractors?

Eugene Wigner

Of course.

There is a representation theory for everything!

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Embeddings

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

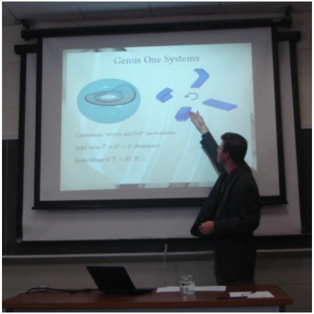
Irreducible is by Dimension

Usual Culprits: 12

Is there a representation theory for strange attractors?

What is it? How does it work?

Daniel J. Cross
Hard at work (pretending)!



Daniel J. Cross
Instructing us.

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Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

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Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

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Equivalences by Injection

Obstructions to Isotopy

R^3	$\longrightarrow$	R^4	$\longrightarrow$	R^5
Global Torsion		Global Torsion		
Parity				
Knot Type				

There is one *Universal* reducible representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

Usual Culprits: 13 & 14

After fixed points — Organizing curves?

What? How?

Tim Jones

Jean-Marc Ginoux

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Rotating the Attractor

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X,Y) \\ F_2(X,Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix}$$

$\Omega = n\ \omega_d$ | $q\ \Omega = p\ \omega_d$

Global Diffeomorphisms | Local Diffeomorphisms (p-fold covers)

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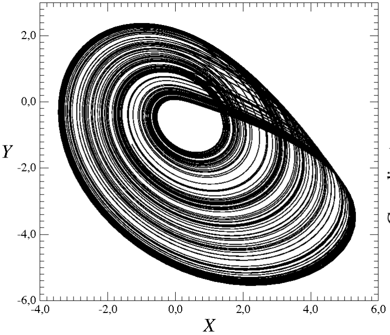
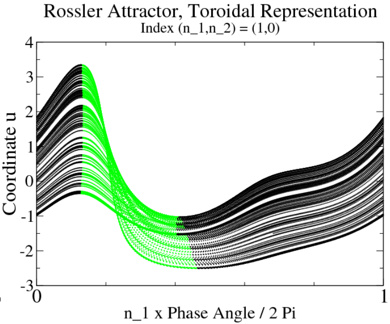
Rossler Attractor: Two Representations

R^3 $\qquad$ $D^2 \times S^1$





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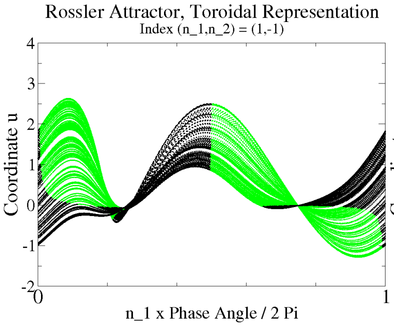
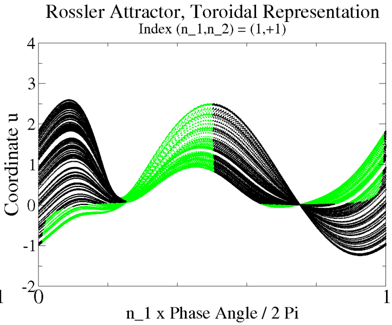
Rossler Attractor:

Two More Representations with $n = \pm 1$





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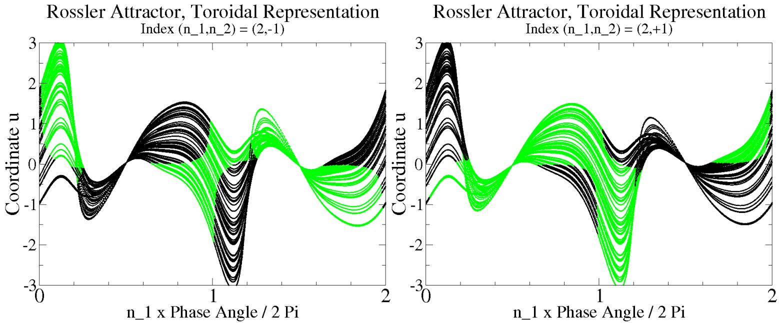
Rossler Attractor:

Two Two-Fold Covers with $p/q = \pm 1/2$



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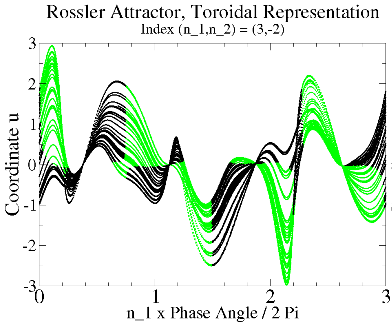
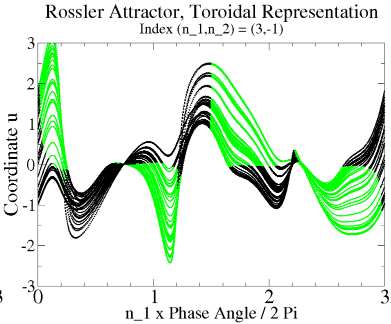
Rossler Attractor:

Two Three-Fold Covers with $p/q = -2/3, -1/3$





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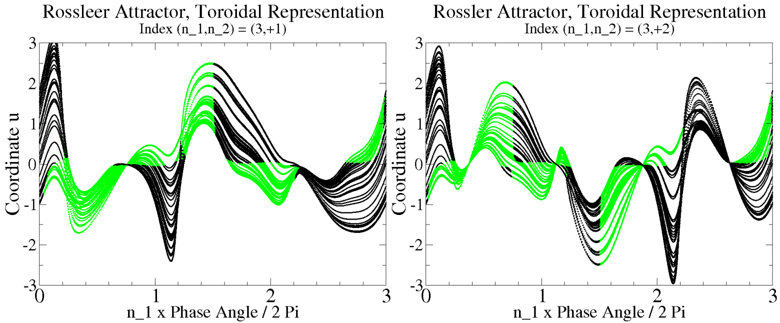
Rossler Attractor:

And Even More Covers (with $p/q = +1/3, +2/3$)



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Angular Momentum and Energy

$$L(0) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} XdY - YdX \qquad K(0) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} \frac{1}{2}(\dot{X}^2 + \dot{Y}^2)dt$$

$$L(\Omega) = \langle u\dot{v} - v\dot{u}\rangle \qquad K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2)\rangle$$

$$= L(0) + \Omega\langle R^2\rangle \qquad = K(0) + \Omega L(0) + \frac{1}{2}\Omega^2\langle R^2\rangle$$

$$\langle R^2\rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} (X^2 + Y^2)dt = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} (u^2 + v^2)dt$$

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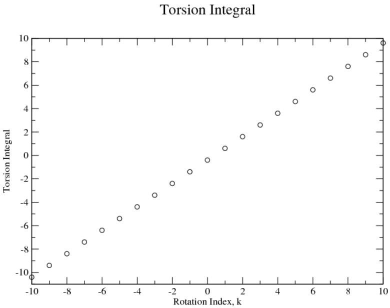
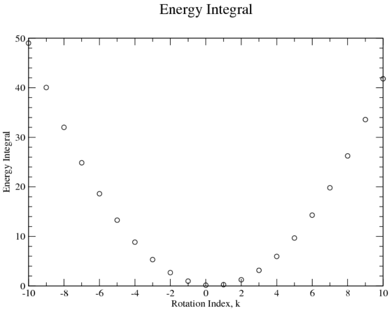
Energy and Angular Momentum

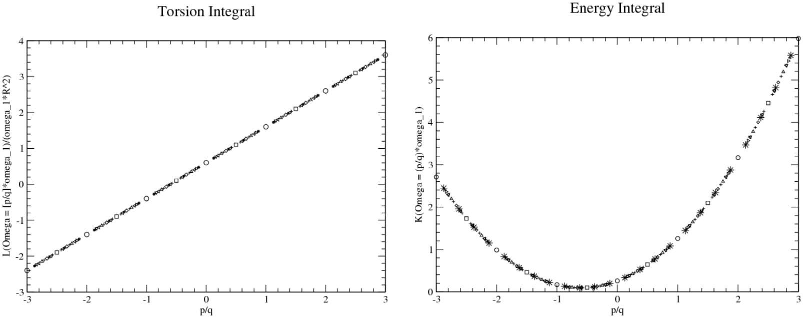
Diffeomorphic, Quantum Number n





Energy and Angular Momentum

Subharmonics, Quantum Numbers p/q



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Summary

1 Question Answered $\Rightarrow$

2 Questions Raised

We must be on the right track !

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Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

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Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

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The Classification Theory has 4 Levels of Structure

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The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits

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The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds

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The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

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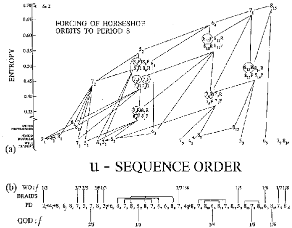
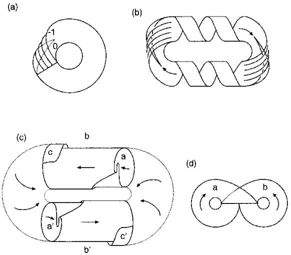
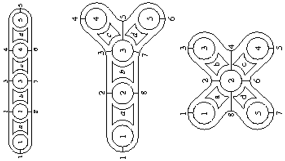
The Classification Theory has 4 Levels of Structure

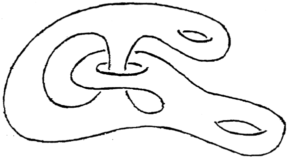
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









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Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

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Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g-1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors

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We hope to find:

- Robust topological invariants for R^N, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \to D^2$, $n \times D^2 \to n \times D^2$ (e.g., Lorenz), $D^N \to D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy