From Nonlinear Dynamics to BioMedicine

The Topology of Chaos

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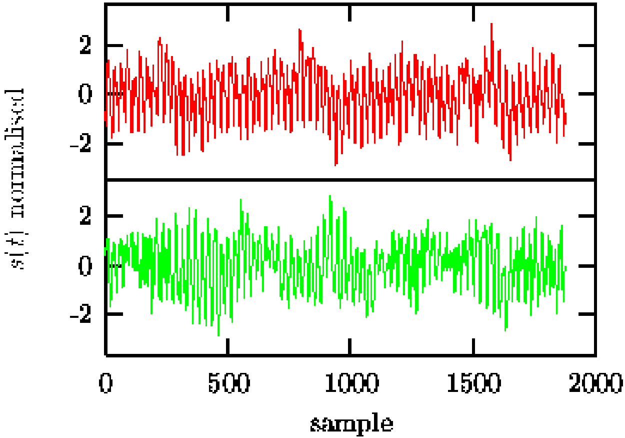
Is This Predictable or Not?



Fundamental Question

If you believe there is no relation between cause and effect, go home now.
If you believe there is, you must work with dynamical systems (Newtonian) equations.
These have the form:

$$\frac{dx_i}{dt} = f(x_i; c_k)$$

x_i	c_k
State Variables	Control Parameters
Chemical Concentrations	Temperature
Potential Differences	Magnetic Fields

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Outline

Nature of Nonlinear Systems

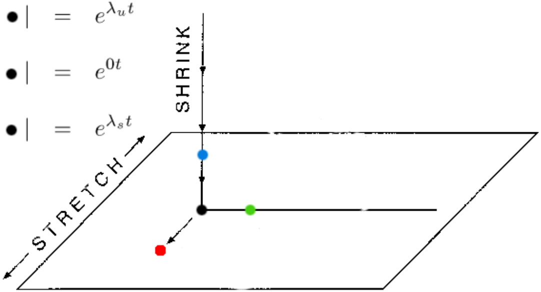
Linear	NonLinear
$\frac{d}{dt}\left[\ x\ \right] = \left[\ M\ \right]\left[\ x\ \right]$	$\frac{d}{dt}\left[\ x\ \right] = \left[\ M\ \right]\left[\ x\ \right] + \begin{array}{c}\text{More}\\ \text{Stuff}\end{array}$
Solution :	Solution :
$\left[\ x(t)\ \right] = e^{\left[\ M\ \right]t}\left[\ x(0)\ \right]$	???
$x \rightarrow 0 \text{ if } Re(\lambda_i) < 0, \text{ all } i$	

Typical Solutions for Linear Equations

$|\text{red} - \text{black}| = e^{\lambda_u t}$

$|\text{green} - \text{black}| = e^{0t}$

$|\text{blue} - \text{black}| = e^{\lambda_s t}$



Two of Our Heroes

J. B. Fourier

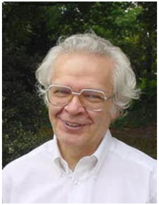
H. Poincare

Why They Are Our Heroes

- Fourier: taught us how to use periodic orbits to describe linear systems.
- Poincaré: taught us to use periodic orbits to understand nonlinear systems.

Two Benchmark Nonlinear Systems

Rossler



$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b - cz \quad + zx \end{aligned}$$

Lorenz

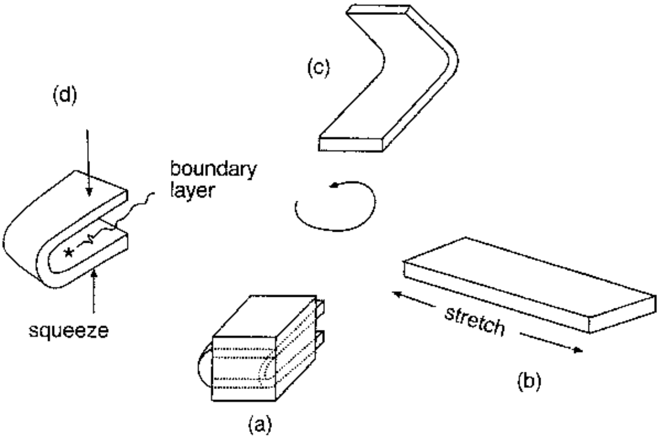
$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= Rx - y \quad -xz \\ \dot{z} &= -bz \quad +xy \end{aligned}$$

Definition of Chaos

Motion is chaotic if it is

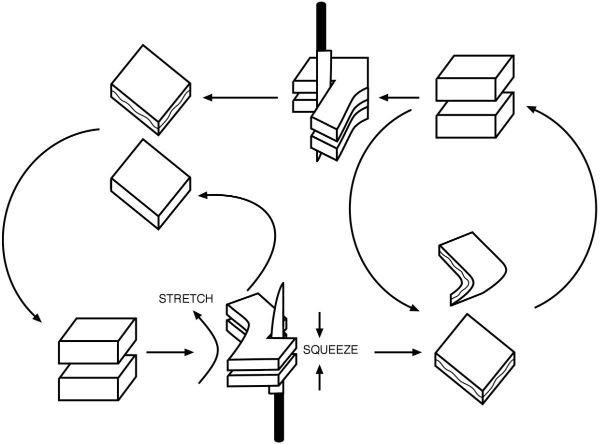
1. Deterministic
2. Bounded
3. Recurrent
4. Sensitive to Initial Conditions

Stretching and Folding



Mechanisms for Generating Chaos

Tearing and Squeezing



Production of Chaos

Chaos (chaotic motion, any strange attractor) is generated by the continuous repetition of two processes:

1. Stretching
2. Squeezing

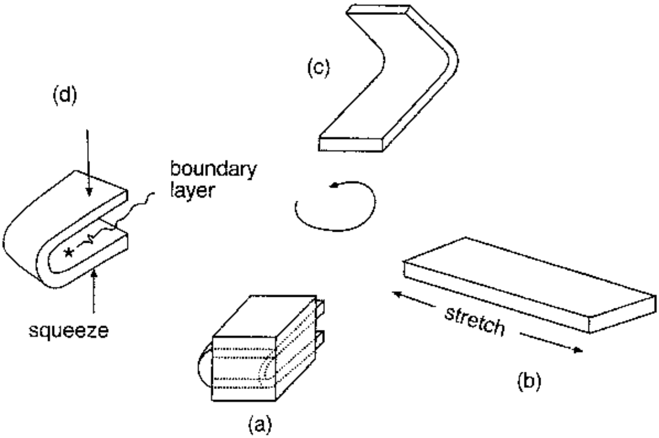
Consequences of Stretching / Squeezing

1. "Stretching" causes nearby points to separate
2. Stretching grows exponentially with time (short times)

$$\delta x(t) \simeq e^{\lambda t}\delta x(0) \qquad \lambda > 0$$

3. λ is a Lyapunov exponent
4. *Almost* all points move apart
5. A measure zero set does not
6. Lyapunov exponents can be tricky to estimate

Stretching and Folding



How to “Measure” Chaos?

Method	Measure	Data Rqmts.
Dynamical	Lyapunov Exponents	L, C, -
Geometrical	Fractal Dimensions	LL, CC, SS
Topological	Linking Numbers	M, M, S

L = Long, C = Clean, S = Stationary

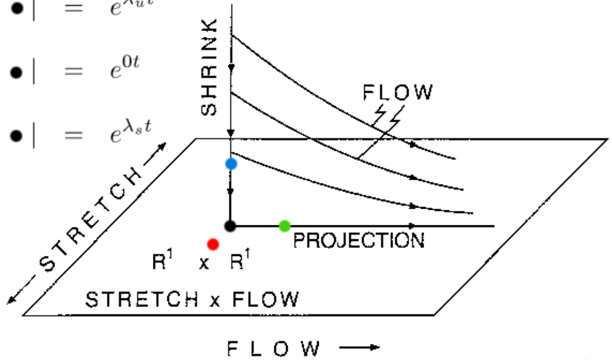
Problems Estimating Lyapunov Exponents

Lyapunov exponents measure how points move with respect to each other.

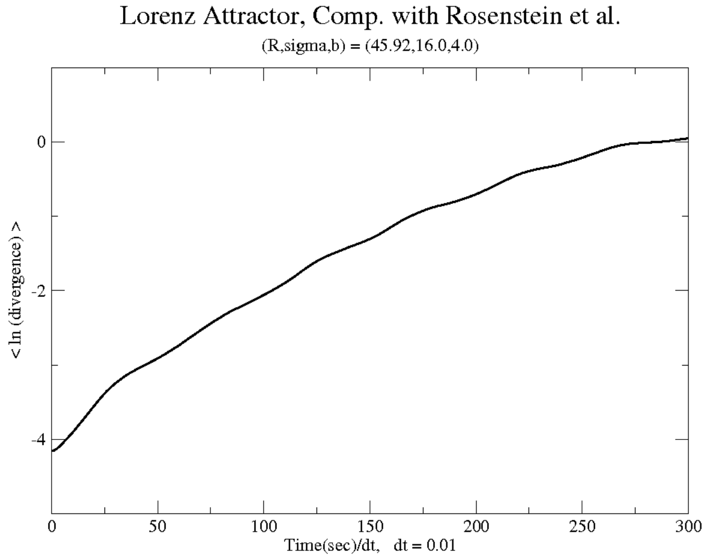
$$|\bullet - \bullet| = e^{\lambda_u t}$$

$$|\bullet - \bullet| = e^{0t}$$

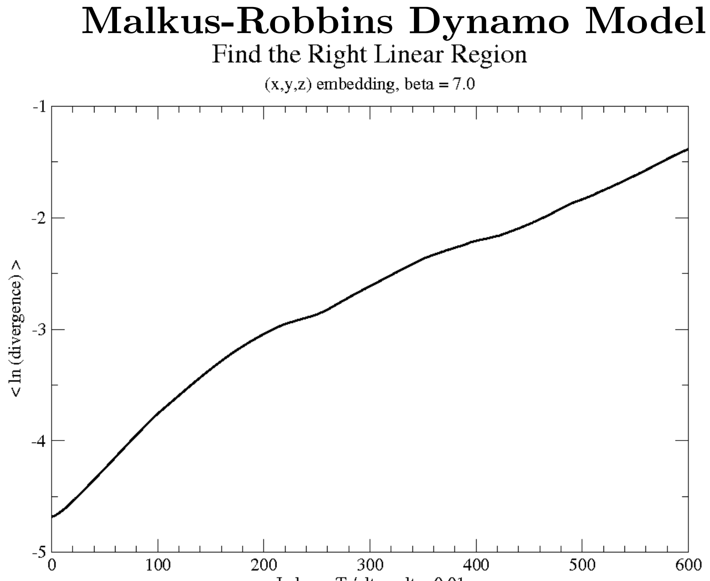
$$|\bullet - \bullet| = e^{\lambda_s t}$$



Problems Estimating Lyapunov Exponents

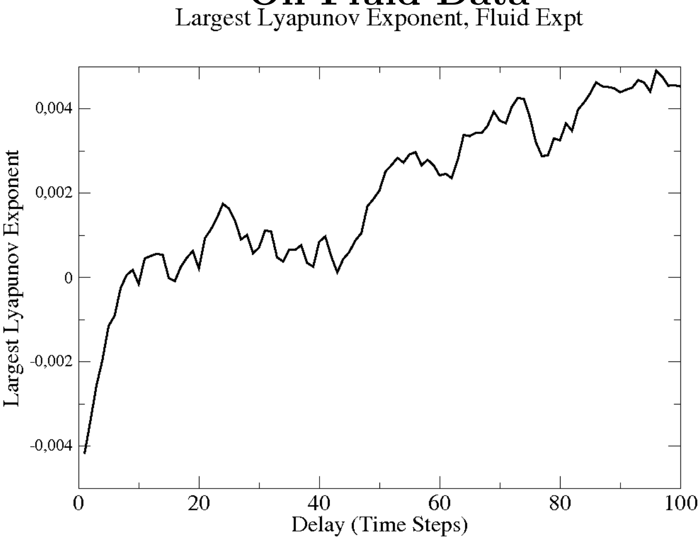


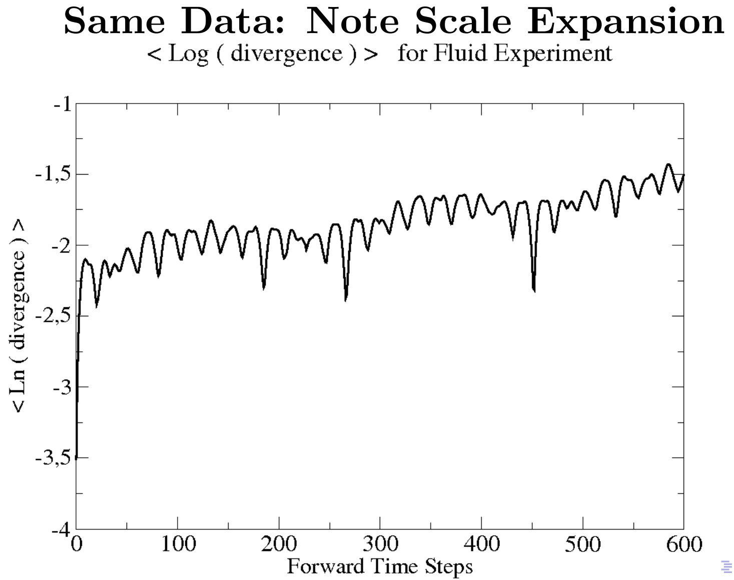
Problems Estimating Lyapunov Exponents



Problems Estimating Lyapunov Exponents

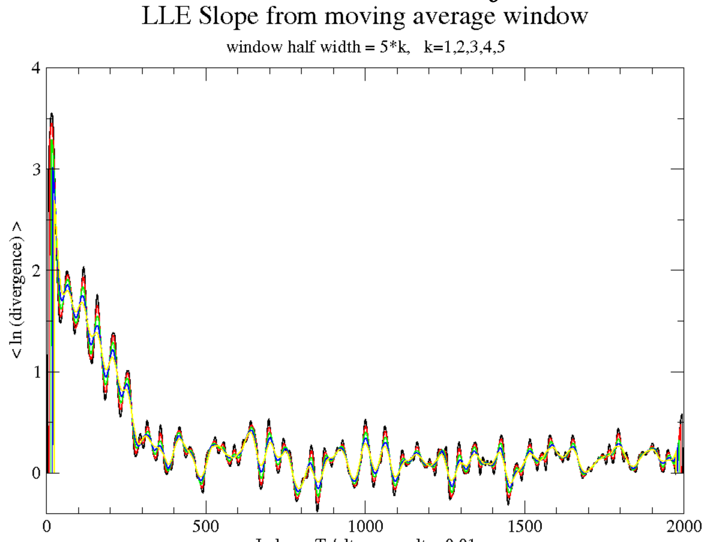
On Fluid Data



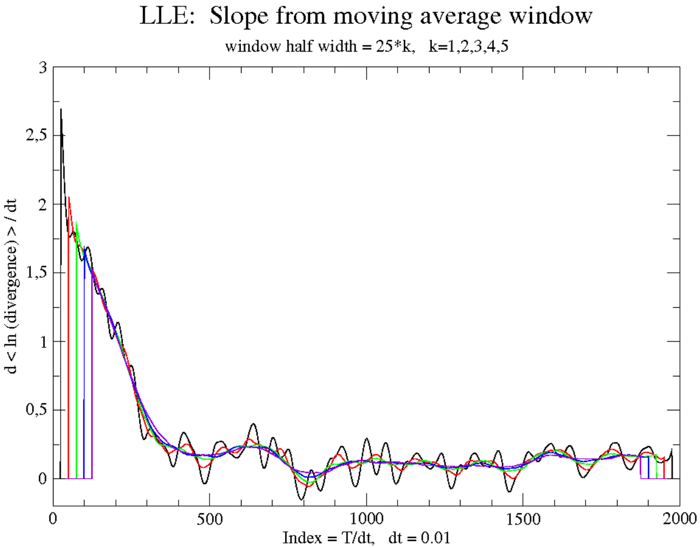

Same Data: Note Scale Expansion
< Log (divergence) > for Fluid Experiment
-1
-1,5
-2
-2,5
-3
-3,5
-4
0
100
200
300
400
500
600
Forward Time Steps
< Ln (divergence) >

Problems Estimating Lyapunov Exponents

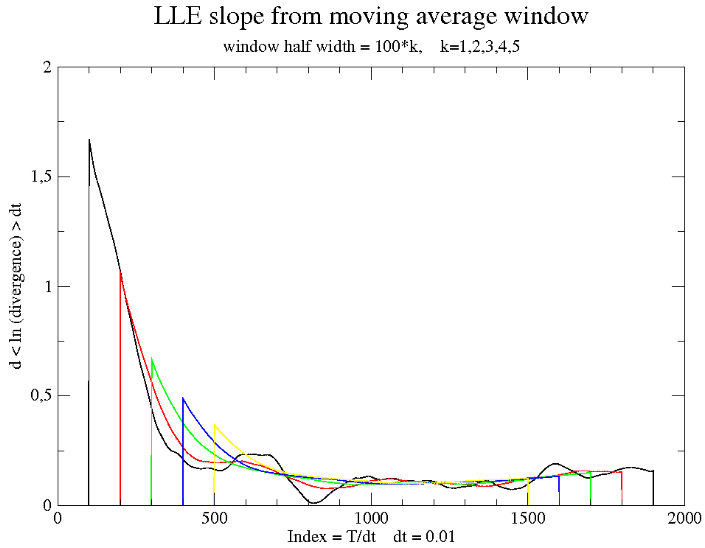
What Value would you Like?



Problems Estimating Lyapunov Exponents



Problems Estimating Lyapunov Exponents



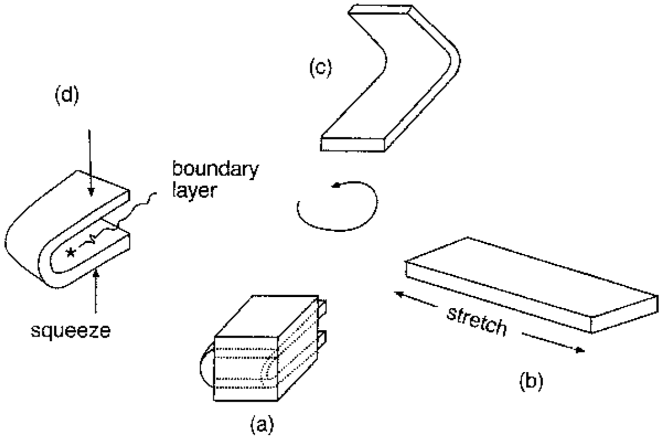
Consequences of Stretching / Squeezing

1. Recurrence causes "Squeezing".
2. Squeezing causes nearby points to move closer

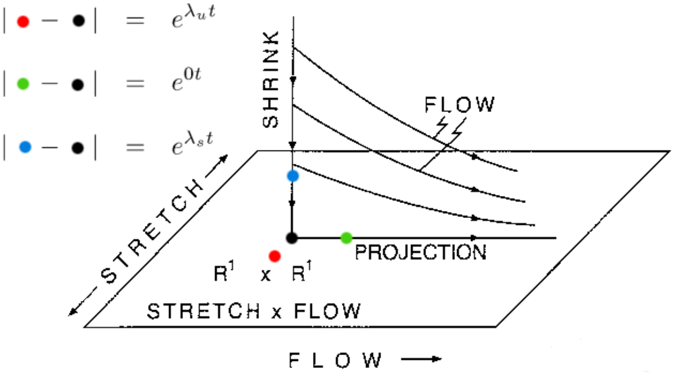
$$\delta x(t) \simeq e^{\lambda t} \delta x(0) \qquad \lambda < 0$$

3. λ is another Lyapunov exponent.
4. Repetition of Squeezing builds up a flakey, milleffeuil like structure
5. This is called a Fractal
6. The number of points $|x - y| < \epsilon$ scales like $N(\epsilon) \simeq \epsilon^D$
7. Fractal dimensions (e.g., D) are also tricky to estimate.

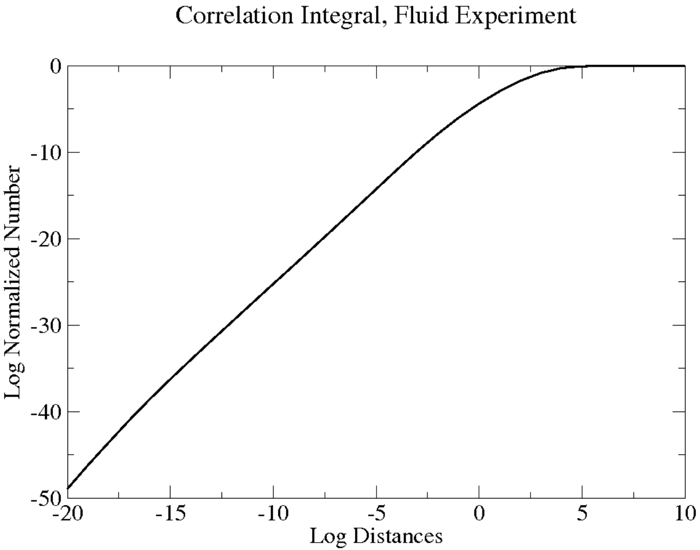
One Mechanism for Generating Chaos

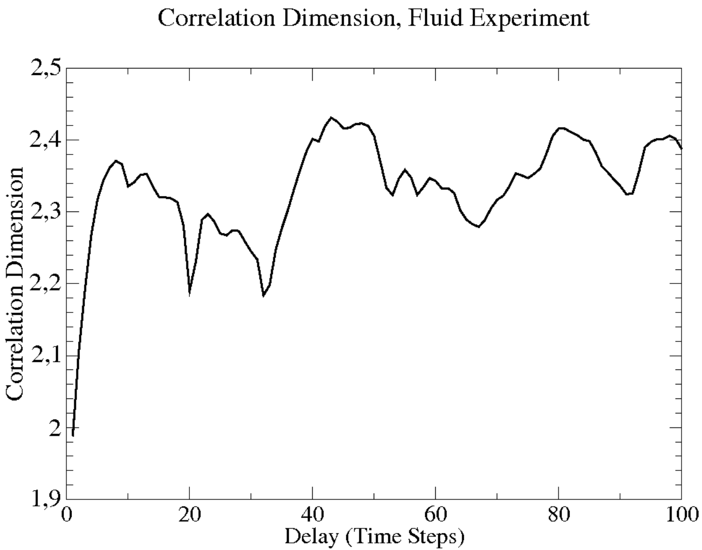


Stretching and Folding



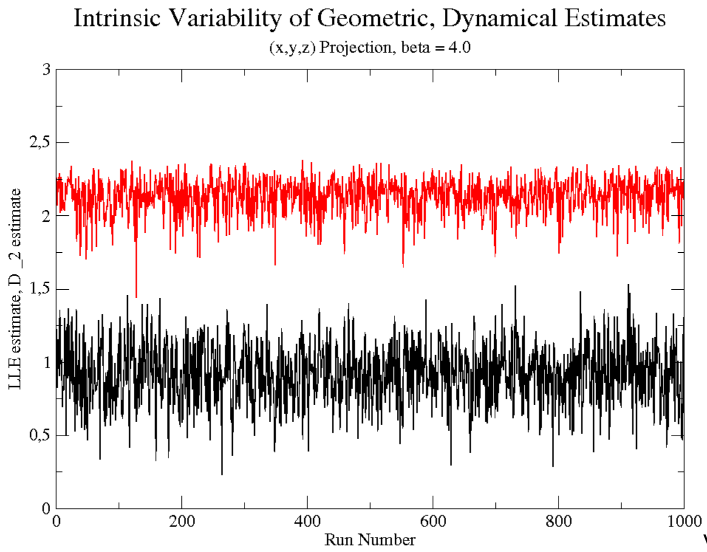
Problems Estimating Fractal Dimensions



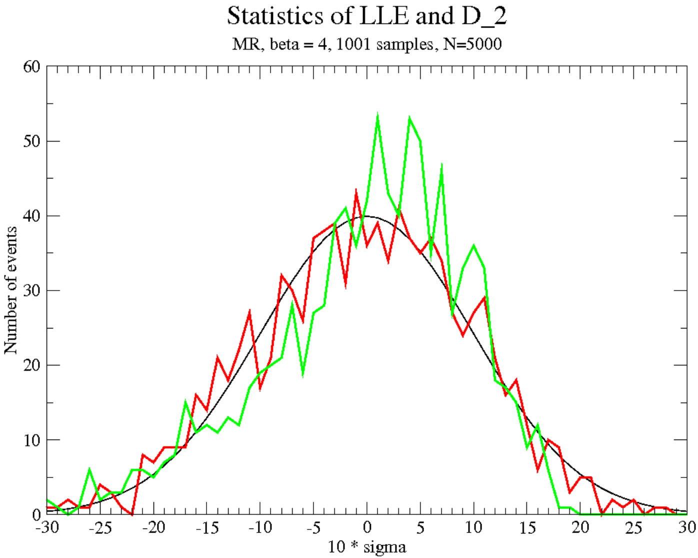

Correlation Dimension, Fluid Experiment
Correlation Dimension
2,5
2,4
2,3
2,2
2,1
2
1,9
0
20
40
60
80
100
Delay (Time Steps)

D2-

fluid.png



variability


Statistics of LLE and D_2
MR, beta = 4, 1001 samples, N=5000
Number of events
10 * sigma
60
50
40
30
20
10
0
-30
-25
-20
-15
-10
-5
0
5
10
15
20
25
30

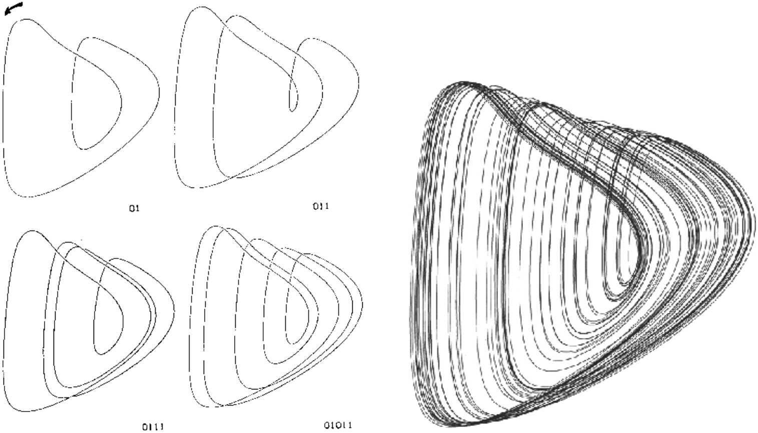
statistics.

Periodic Orbits

A strange attractor is the ‘Ω’ limit set of the flow.

- There are unstable periodic orbits “in” the strange attractor.
- Many.
- They outline the strange attractor.
- They provide a skeleton for the strange attractor.
- They can be extracted from the attractor.
- Their organization can be determined (in R^3).
- This analysis method applies in R^3 *only*, for now.

UPOs Outline Strange attractors



BZ reaction

UPOs Outline Strange attractors

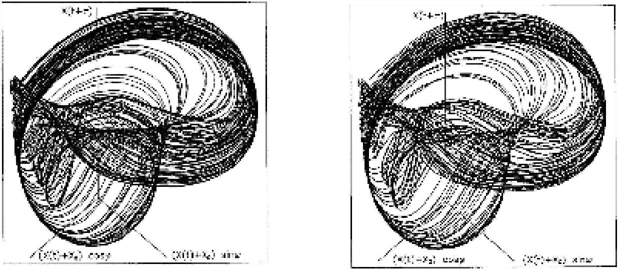


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

Organization of UPOs in R^3:

Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

Linking Numbers

Linking Number of Two UPOs

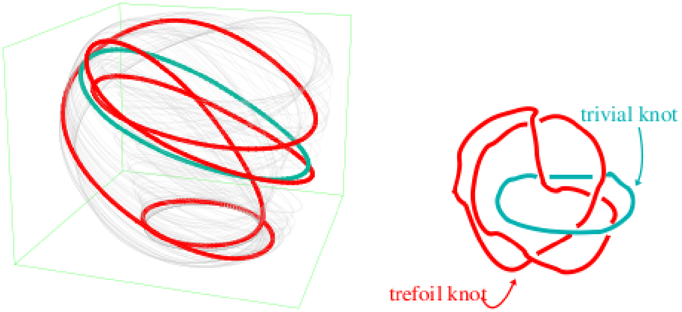


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

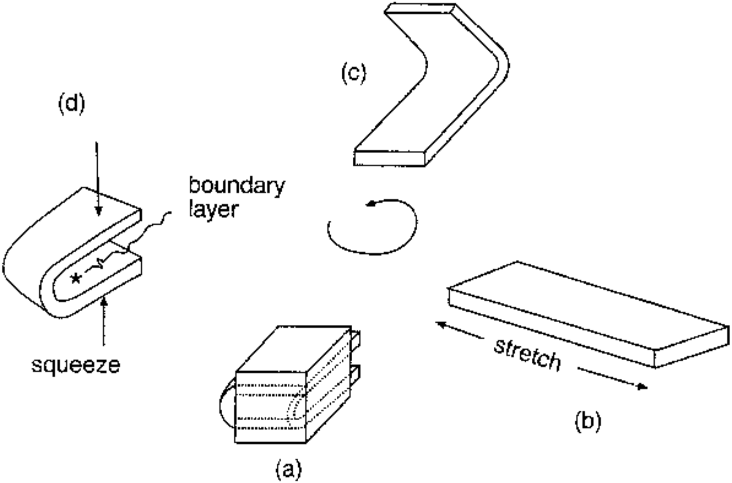
Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

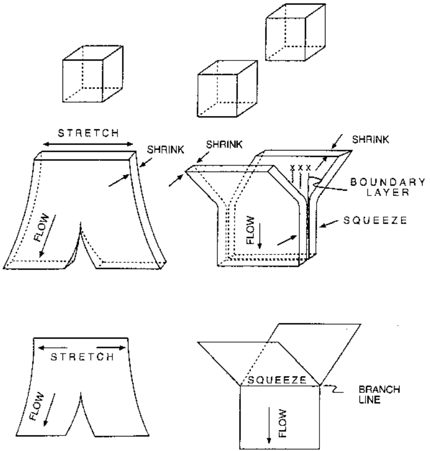
[a] All indices are negative.

One Stretch-&-Squeeze Mechanism



Motion of Blobs in Phase Space (Poincaré)

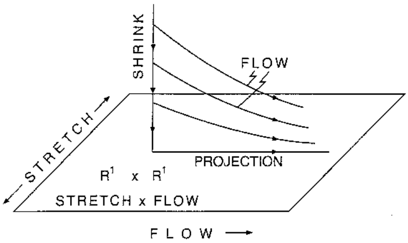
Stretching — Squeezing



Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



This projects the flow donw onto the unstable manifold.

Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: Certain Assumptions

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$
- **Generates a** **<u>hyperbolic</u>** **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

- The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.

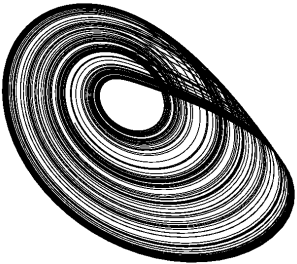
- UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

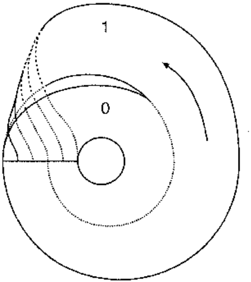
Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

Rössler:

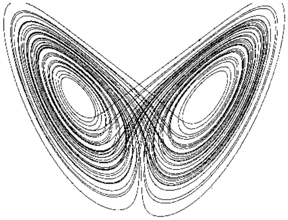
Attractor **Branched Manifold**

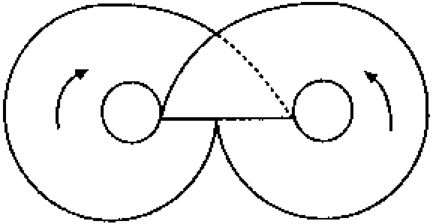




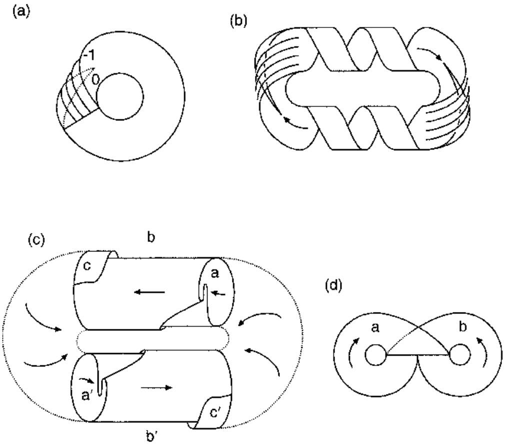
Lorenz:

Attractor **Branched Manifold**



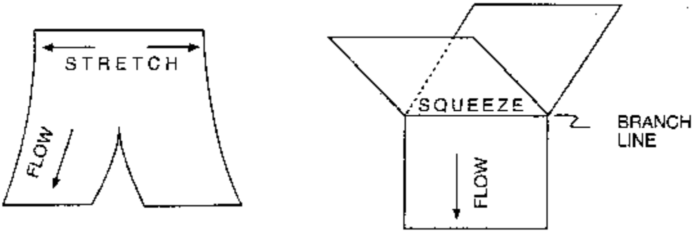


Inequivalent Branched Manifolds



Aufbau Princip for Branched Manifolds

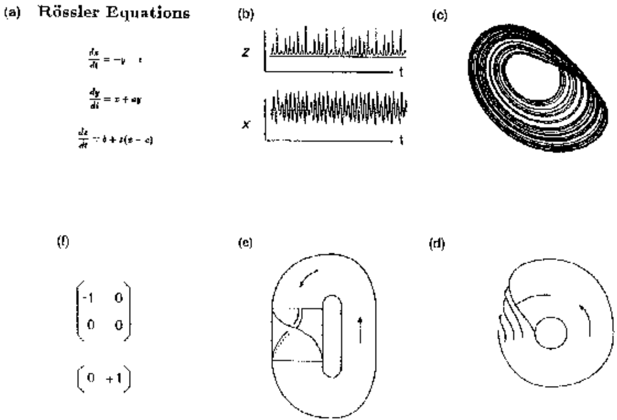
Any branched manifold can be built up from stretching and squeezing units



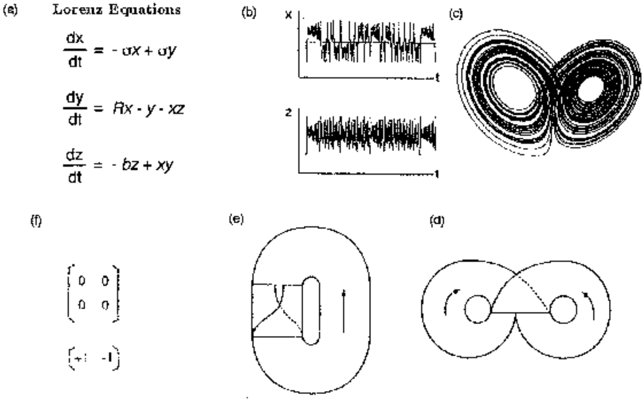
subject to the conditions:

- Outputs to Inputs
- No Free Ends

Rossler System



Lorenz System

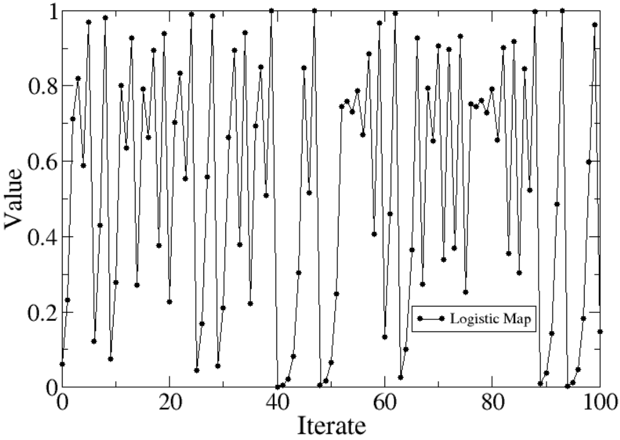


Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

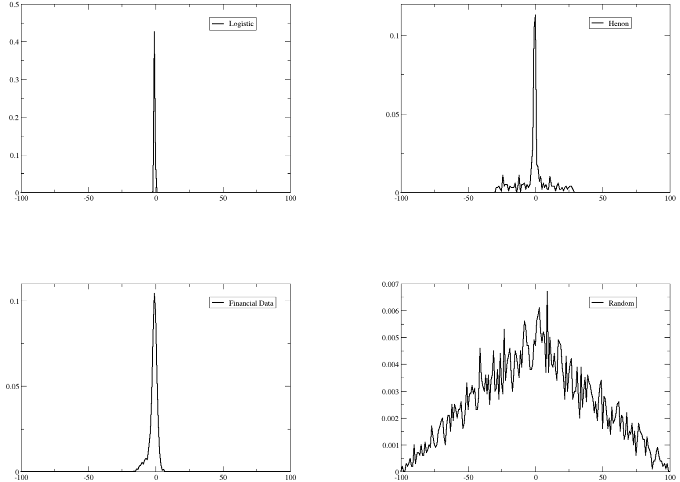
Determinism

How to predict the future from the past



Some Prediction Results

Tightly binned predictions suggest determinism



Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

Identify a Branched Manifold

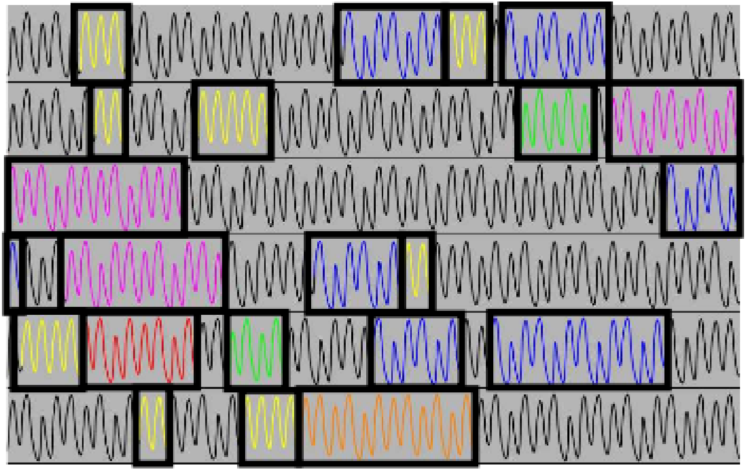
Verify the Branched Manifold

Model the Dynamics

Validate the Model

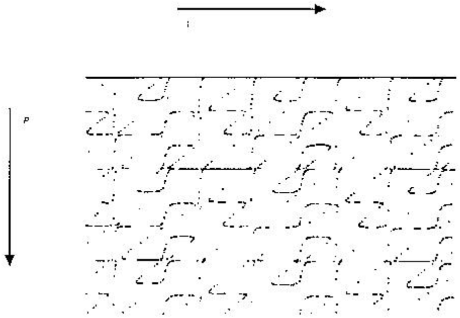
Locate UPOs

Method of Close Returns



Method of Close Returns

$$|x_i - x_{i+p}| < \epsilon, \quad \text{pixel} \rightarrow \text{black}$$



Embeddings

This is a tricky business. There are many problems ...

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

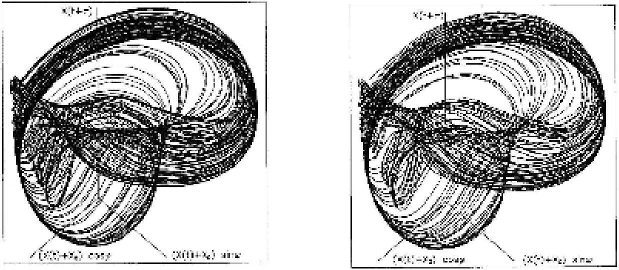
Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

Locate UPOs

Periodic Orbits Outline the Attractor



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Lefranc - Cargese

Linking Number of Orbit Pairs

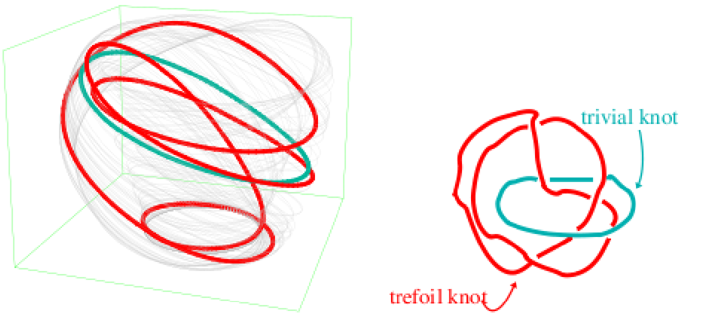

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

Compute Table of Expt'l LN

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2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	1s	1f	2_1	3f	3s	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Guess Branched Manifold

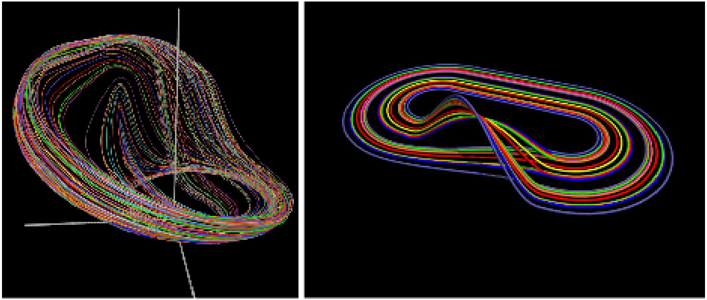


Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

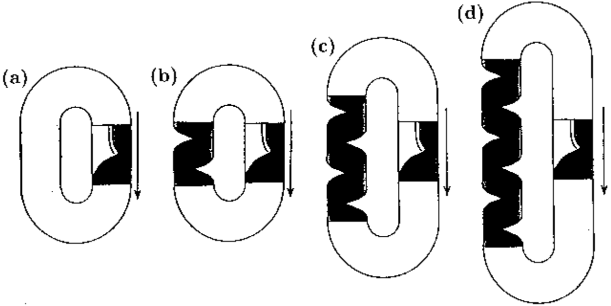
Lefranc - Cargese

Identification & 'Confirmation'

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change

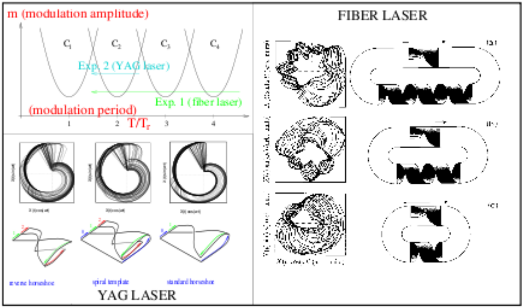
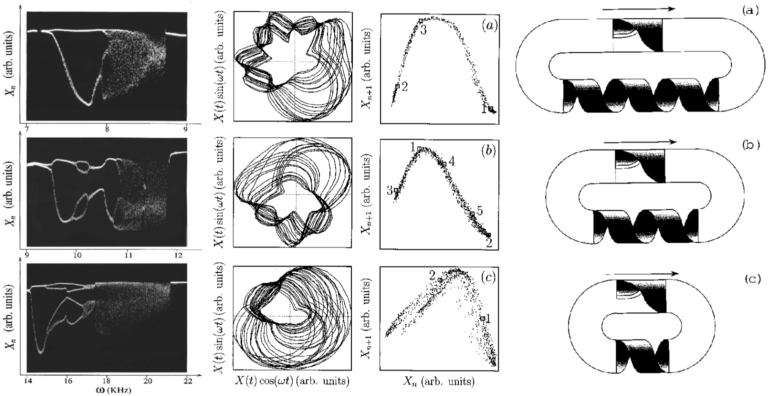


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Lefranc - Cargese

Evolution Under Parameter Change



Lefranc - Cargese

Analysis of Nonstationary Data

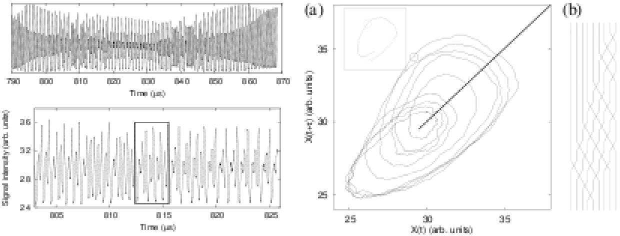


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

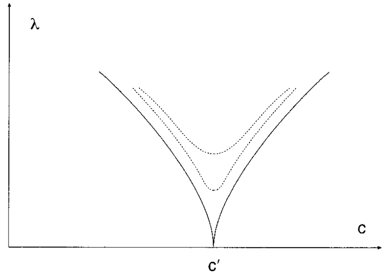
Lefranc - Cargese

Model the Dynamics

A hodgepodge of methods exist: # Methods $\simeq$ # Physicists

Validate the Model

Needed: Nonlinear analog of χ^2 test. OPPORTUNITY: Tests that depend on entrainment/synchronization.



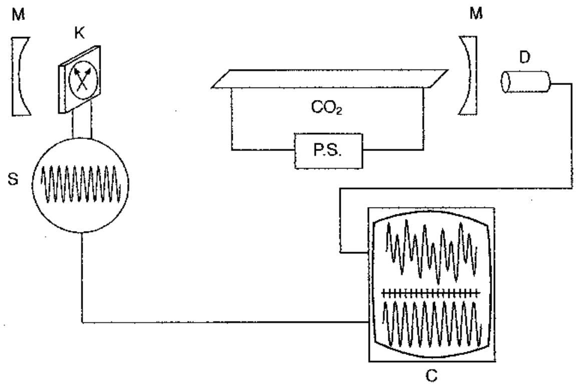
Compare with Original Objectives

Construct a simple, algorithmic procedure for:

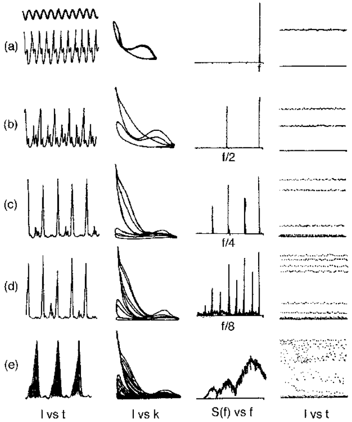
- Classifying strange attractors
- Extracting classification information

from experimental signals.

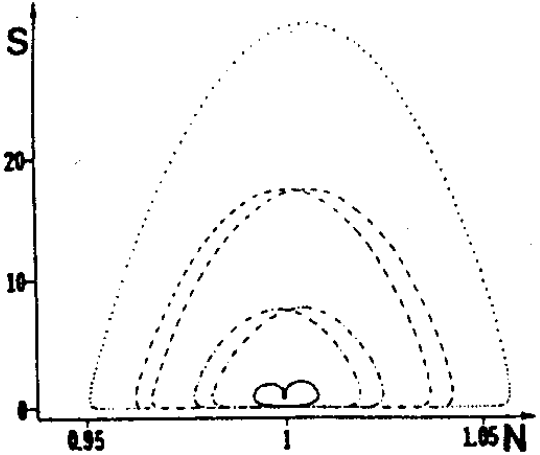
Laser Experimental Arrangement



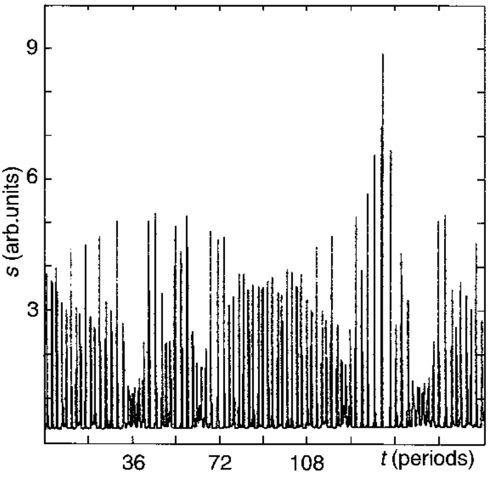
Oscilloscope Traces



Coexisting Basins of Attraction

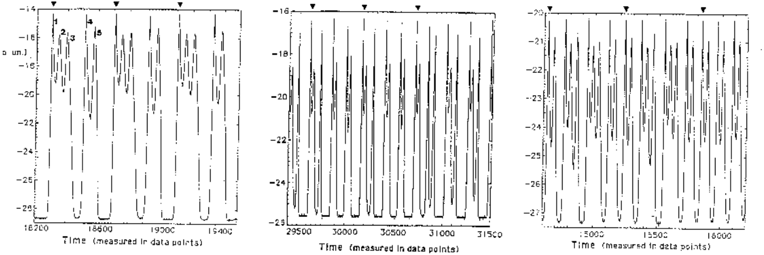


Note the Spikiness

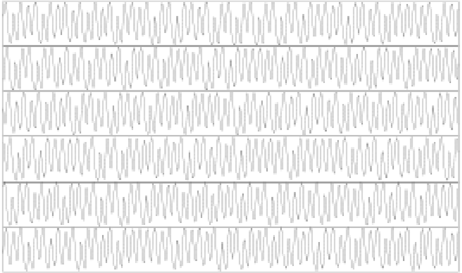


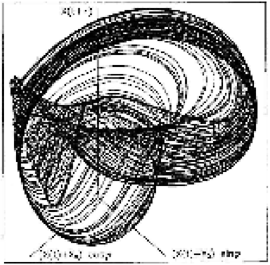
Time Series Data

A Short Part of the Time Series



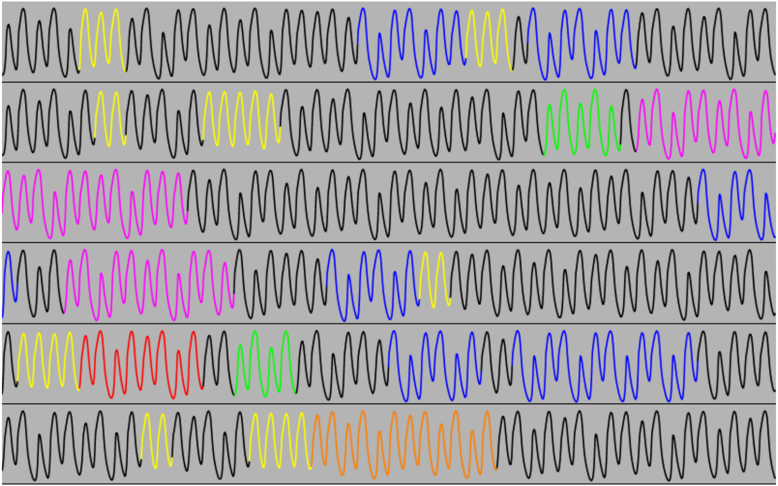
Experimental Data: LSA



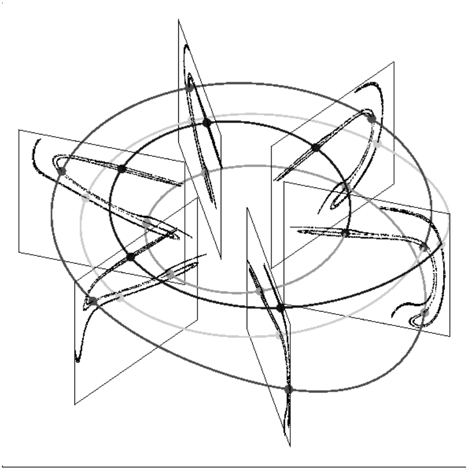


Lefranc - Cargese

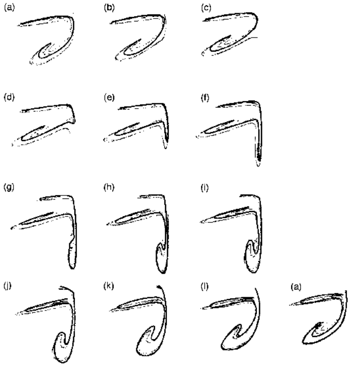
Experimental Data: LSA



Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Rotating the Poincaré Section around the axis of the torus

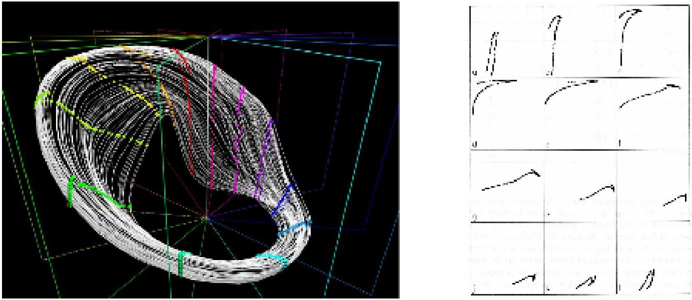
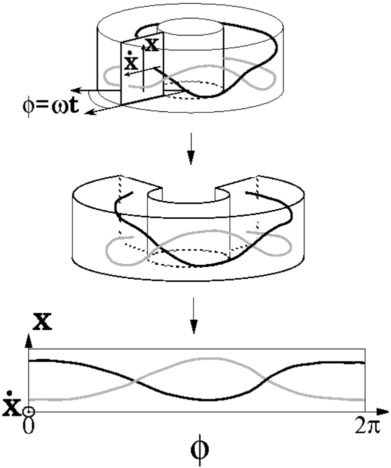


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

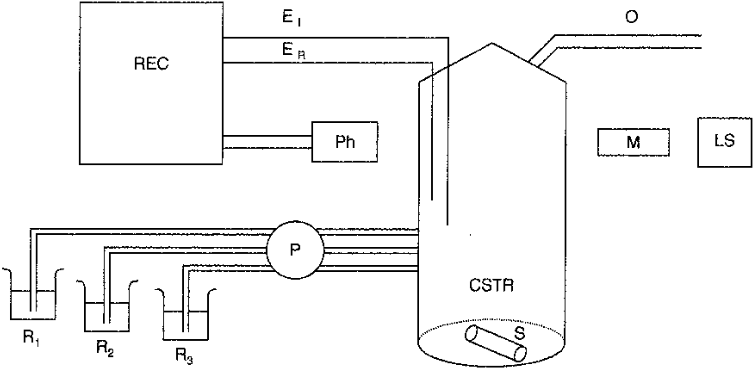
Lefranc - Cargese

Another Visualization

Cutting Open a Torus

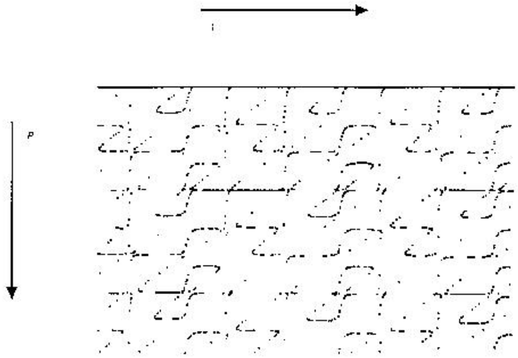


Belousov-Zhabotinskii Experimental Configuration

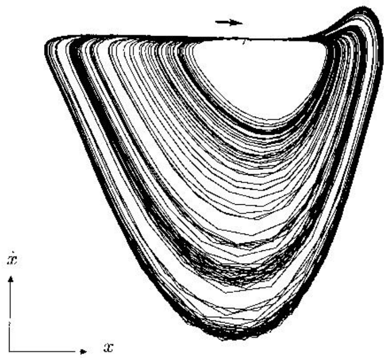


Close Returns Plot

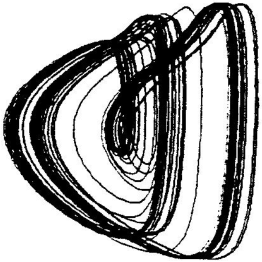
$$|x_i - x_{i+p}| < \epsilon \qquad \text{pixel} \rightarrow \text{black}$$



First Embedding Attempt: $x, \dot{x}, \ddot{x}$

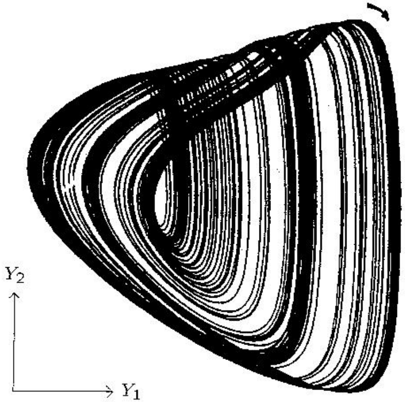


Second Embedding Attempt: $\int x, x, \dot{x}$



Nonstationary!

Third embedding attempt: $\int xe^{-t'/\tau}, x, \dot{x}$



Once you have an embedding:

- Find a Poincaré Section
- Construct a First Return Map on the Section
- Introduce a Symbolic Encoding
- Encode all Unstable Periodic Orbits
- Find their Linking Numbers

From Nonlinear Dynamics to BioMedicine

Robert Gilmore

Motivation-01
Motivation-02
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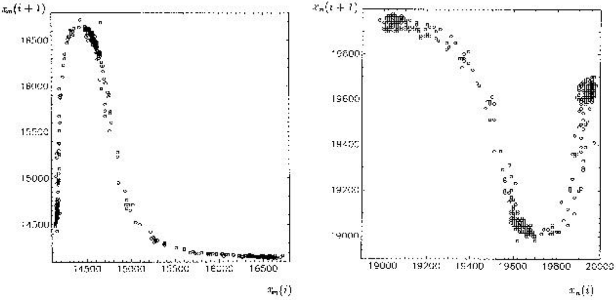
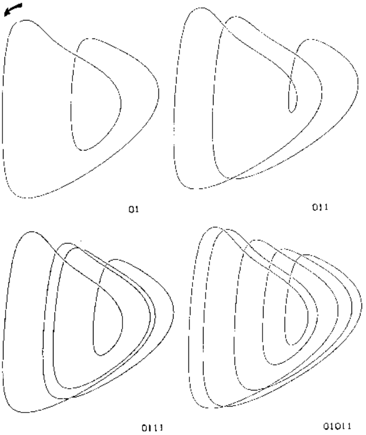
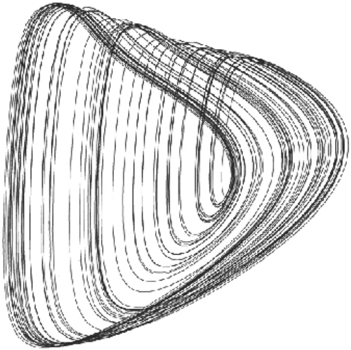
Nonlinear-02
Nonlinear-03
Nonlinear-04
Nonlinear-05
Chaos-01
Chaos-02
Chaos-03
Chaos-04
Chaos-05

Two Symbols Suffice! 0 and 1



Some Named Low-Period Orbits





Some Extracted and Reconstructed Periodic Orbits

Orbit	Name	Symbolics	Local Torsion	Self-Linking
1	1_1	1	1	0
2	2_1	01	1	1
3	3_1	011	2	2
4	4_1	0111	3	5
5	5_1	01 011	3	8
6	6_2	011 0M1	3	9
7	7_2	$(01)^2 011$	4	16
8a	8_1	$(01)^2 0111$	5	23
8b	8_3	$01(011)^2$	5	21
9	9_3	$(01)^3 011$	5	28
10a	10_6	$(011)^2 0101$	6	33
10b	10_6	$(011)^2 0111$	7	33
11	11_9	$01(011)^3$	7	40
13a		$(01)^2 011\ 01\ 0111$	8	62
13b		$(01)^3 011\ 0111$	8	60

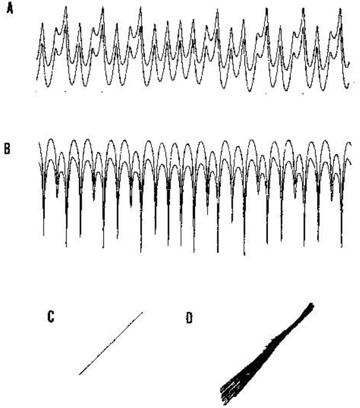
Table of Experimental Linking Numbers

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

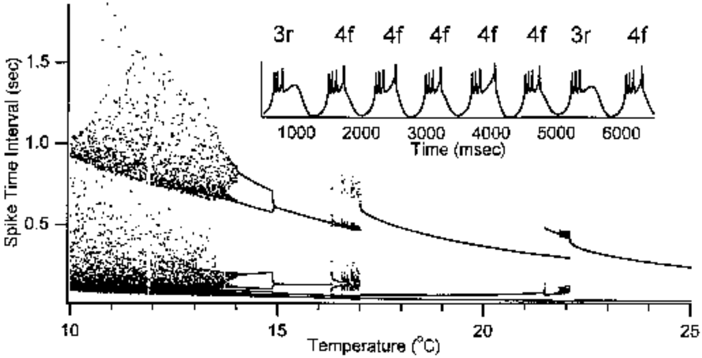
[a] All indices are negative.

Testing the Result

(a), (c) y_1^m compared with y_1^d. (b), (d) y_3^m compared with y_3^d.

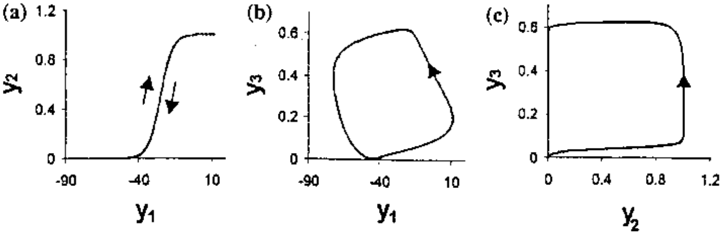


Is This Chaotic or Not?

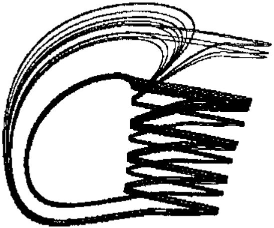


Reducing Dimension

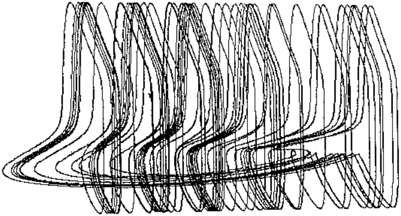
Correlations Imply Fewer Independent Variables



Two Different Planar Projections

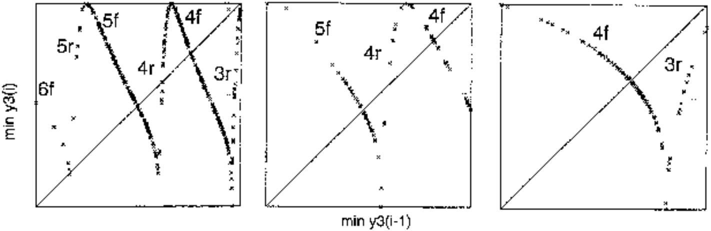


y_4-y_5 Plane



$\dot{y}_4$-y_5 Plane

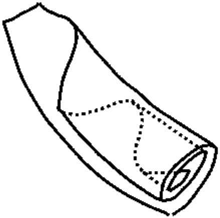
Return Map Migrates with Temperature



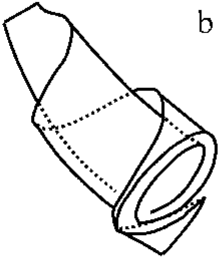
12^o C 13.5^o C 16.6^o C

Scroll Templates

a



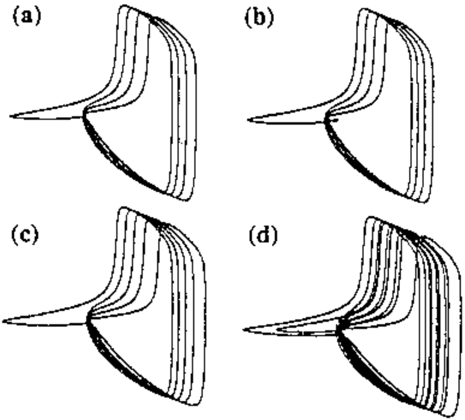
b



Branch	Array	0	1	2	3	4	5	6	7	8	9
0	+N-0	0	0	0	0	0	0	0	0	0	0
1	-N+0	0	1	2	2	2	2	2	2	2	2
2	+N-1	0	2	2	2	2	2	2	2	2	2
3	-N+1	0	2	2	3	4	4	4	4	4	4
4	+N-2	0	2	2	4	4	4	4	4	4	4
5	-N−2	0	2	2	4	4	5	6	6	6	6
6	-N-3	0	2	2	4	4	6	6	6	6	6
7	-N−3	0	2	2	4	4	6	6	7	8	8
8	−N-4	0	2	2	4	4	6	6	8	8	8
9	-N+4	0	2	2	4	4	6	6	8	8	9

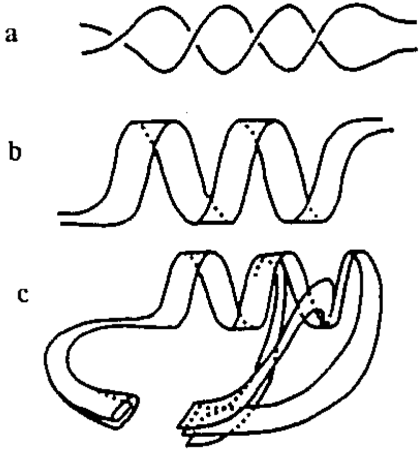
Branch	Array	0	1	2	3	4	5	6	7	8	9
0	0	0	0	2	2	4	4	6	6	8	8
1	-1	0	1	2	2	4	4	6	6	8	8
2	+1	2	2	2	2	4	4	6	6	8	8
3	-2	2	2	2	3	4	4	6	6	8	8
4	+2	4	4	4	4	4	4	6	6	8	8
5	-3	4	4	4	4	4	5	6	6	8	8
6	+3	6	6	6	6	6	6	6	6	8	8
7	-4	6	6	6	6	6	6	6	7	8	8
8	+4	8	8	8	8	8	8	8	8	8	8
9	-5	8	8	8	8	8	8	8	8	8	9

Periodic Orbits



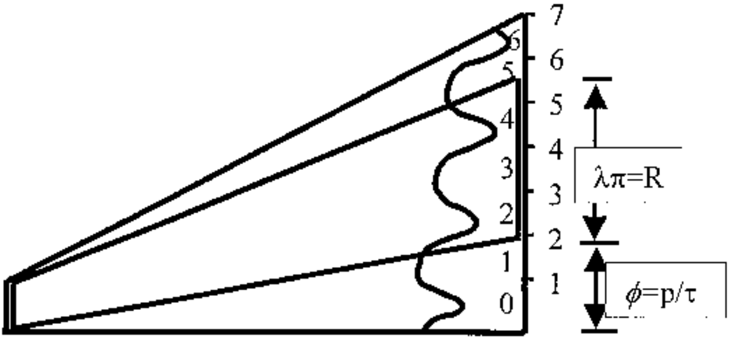
(a) $4f$ (b) $4r$ (c) $5f$ (d) $(4f, 5f)$

Creating the Branched Manifold



(a) Twist (b) Relax (c) Rejoin

Simple Two-Parameter Model



ϕ = Drift $\qquad$ λ = Stretch (Lyapunov Exp)

The Setup

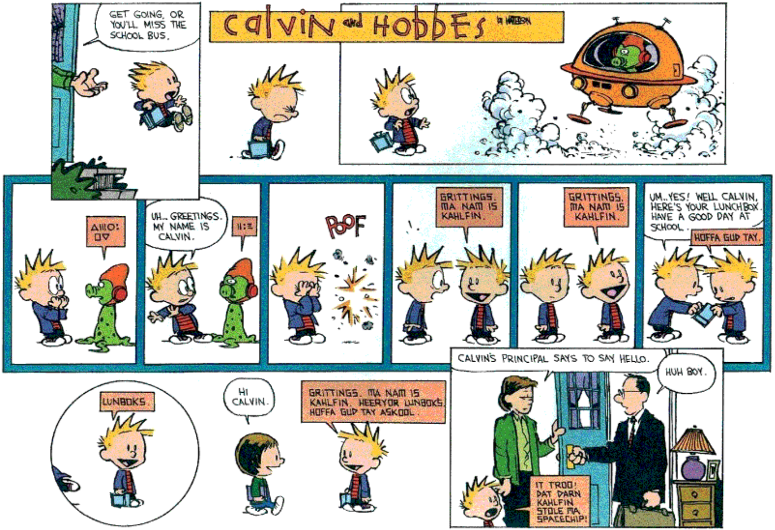
- Suppose you want real-time data from a certain subject.
- And the subject is:

The Setup

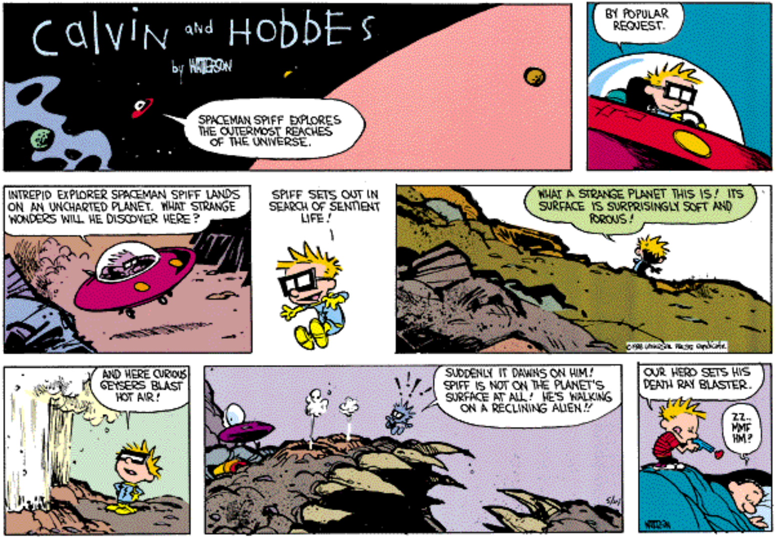
- Suppose you want real-time data from a certain subject.
- And the subject is:

CALVIN

Brain Rampant

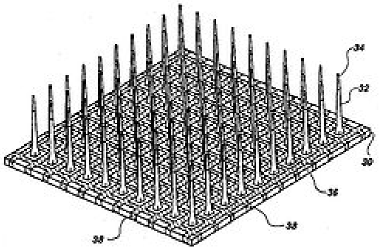


Dad's of the World – Watch Out !



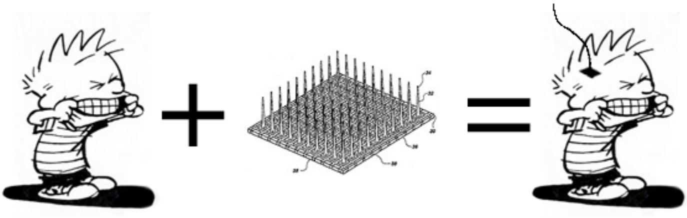
Advance in Technology

A normal implant would provide one time series.
This guy has a behavioral time scale approx. 10^{-1} sec.
Use an electrode array implant to record lots of time series simultaneously.

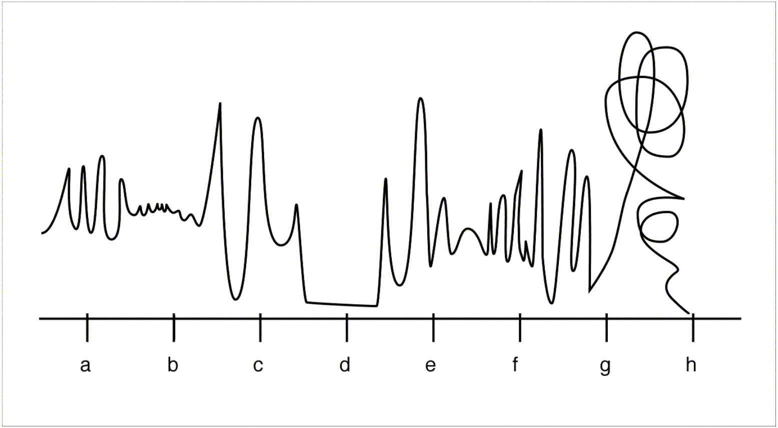


Record from Lots of Spots

Now Calvin is Wired



Highly Effervescent Time Series



a: PB & J Sandwich
b: Skool
c: Toboggan on Hobbes
d: Dad Explains Something
e: Drives Spaceship
f: Speaks with Martian
g: **A GIRL !!!**

Biological Algorithm

Use the Ergodic Theorem (Hypothesis, Guess, Hope, Desperation Wish) to assume lots of short snippets from the $10^{2.5\pm0.5}$ electrodes can be reconstructed into a single long times series, one for each behavioral mode.

The reconstruction can be carried out via a "biological algorithm".

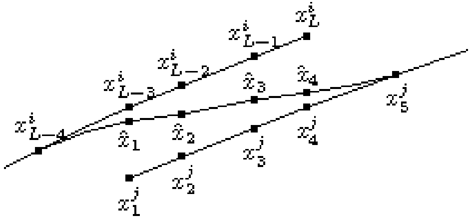
Biological Algorithm for Data Annealing

- Time-tag time series from each electrode
- Cut out short snippets w. same time-tag from each electrode record
- Use DNA type comparison to join them

$$\begin{array}{l} ...G\ A\ C\ T\ C\quad T\ A\ G\ C \\ \qquad\qquad\qquad\quad A\ T\ C\ G\quad T\ A\ T\ T... \end{array}$$

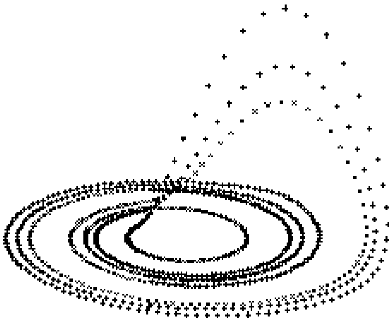
- Study each longer time series to determine behavior fingerprint

Interpolating Connection Between Snippets



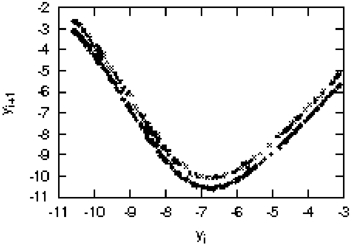
Rossler Attractor

This period-7 orbit was reconstructed from 4 short snippets.



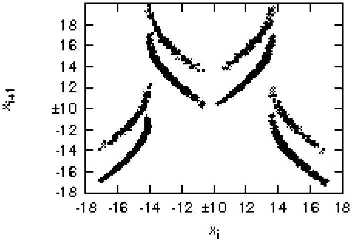
Rossler Attractor

Return maps for the original attractor and the attractor reconstructed from many short snippets.

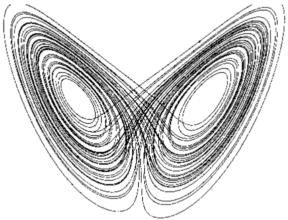


Lorenz Attractor

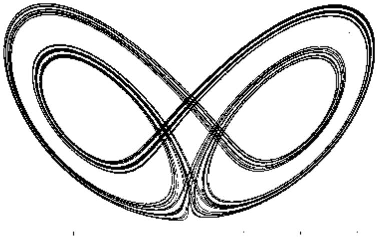
Return maps for the original attractor and the attractor reconstructed from many short snippets.



There Are Some Missing Orbits

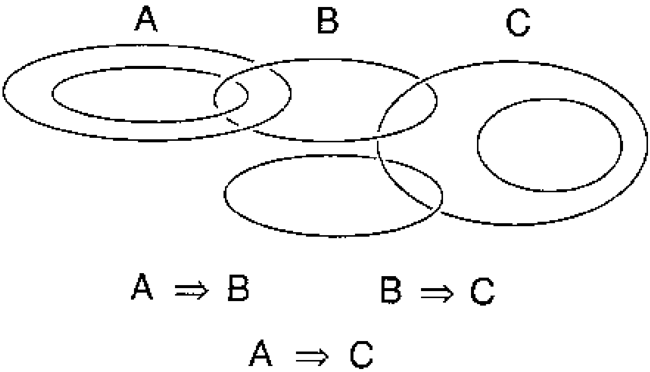


Lorenz

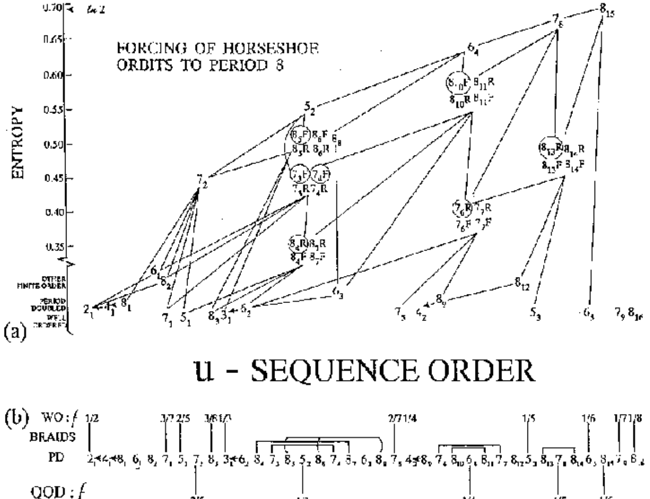


Shimizu-Morioka

Orbit Forcing



Forcing Diagram - Horseshoe



Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

Constraints on Branched Manifolds

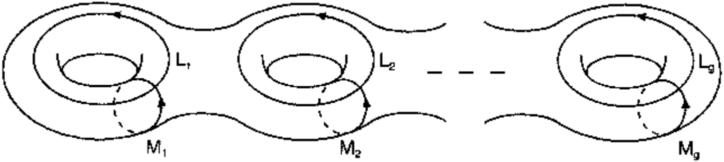
“Inflate” a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Surface Singularities

Flow field: three eigenvalues: +, 0, –

Vector field "perpendicular" to surface

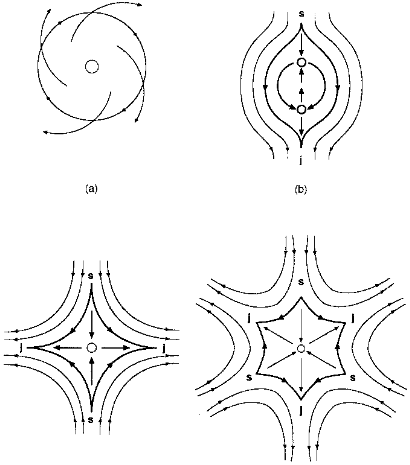
Eigenvalues on surface at fixed point: +, –

All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

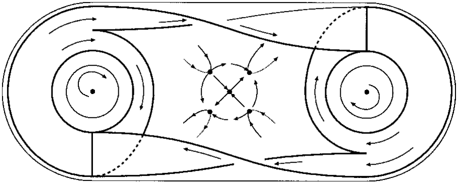
 # fixed points on surface = index = 2g - 2

Flow Near a Singularity

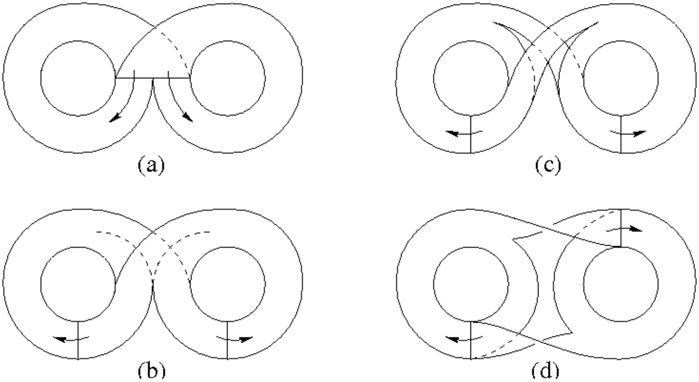


(a) (b) (c) (d)

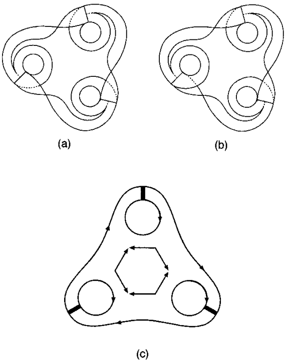
Torus Bounding Lorenz-like Flows



Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.



Bounding Tori contain all known Strange Attractors

Tab.1. All known strange attractors of dimension $d_L < 3$ are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g-1$ Orbit
Rossler, Duffing, Burke and Shaw	A_1	1
Various Lasers, Gateau Roule	A_1	1
Neuron with Subthreshold Oscillations	A_1	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	A_2	$(12)^2$
Multispiral attractors	A_n	$(12^{n-1})^2$
C_n Covers of Rossler	C_n	1^n
C_2 Cover of Lorenz$^{(a)}$	C_4	1^4
C_2 Cover of Lorenz$^{(b)}$	A_3	$(122)^2$
C_n Cover of Lorenz$^{(a)}$	C_{2n}	1^{2n}
C_n Cover of Lorenz$^{(b)}$	P_{n+1}	$(1n)^n$
$2 \to 1$ Image of Fig. 8 Branched Manifold	A_3	$(122)^2$
Fig. 8 Branched Manifold	P_5	$(14)^4$

$^{(a)}$ Rotation axis through origin.
$^{(b)}$ Rotation axis through one focus.

Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 $\times$ g-1 matrices

One is cyclic g-1 $\times$ g-1 matrix

Other represents union of cycles

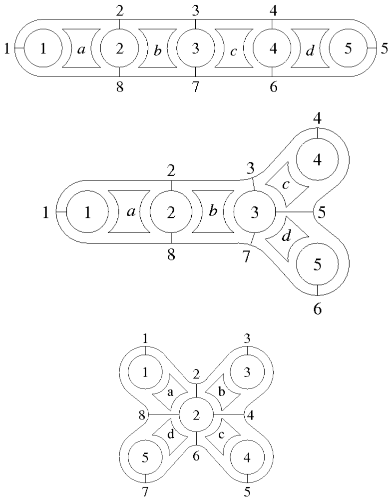
Labeling via (permutation) group theory

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

g	m	$(p_1, p_2, \ldots p_m)$	$n_1 n_2 \ldots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

Some Genus-9 Bounding Tori



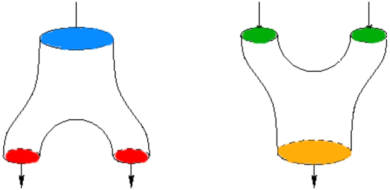
Labels for Bounding Tori

Each is described by a Transition Matrix. This is the sum of a general cyclic rotation and another Permutation Group Element. For second case, previous:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (12345678) + (1)(28)(357)(4)(6)$$

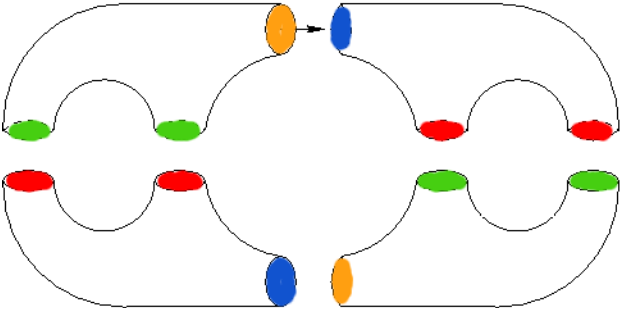
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Application: Lorenz Dynamics, g=3



From Nonlinear Dynamics to BioMedicine

Robert Gilmore

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Construction of Poincaré Section

P. S. = Union 

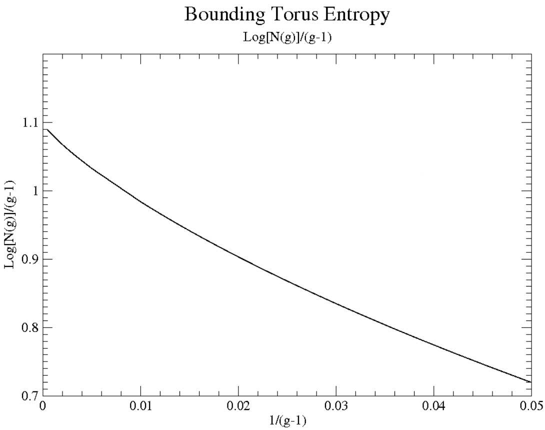
Components = g-1

The Growth is Exponential

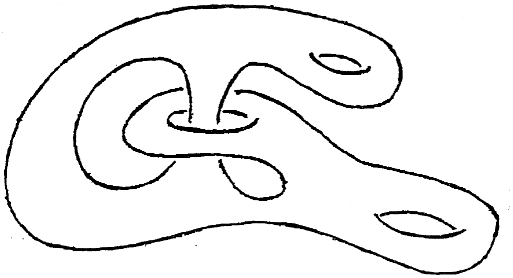
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

The Growth is Exponential
The Entropy is log 3



Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.