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Periodic orbits are rigidly organized in any three-dimensional manifold \( M^3 \) by transversality.

If \( M^3 \subset R^3 \), Gauss Linking numbers are useful:
\[ M^3 = D^3; D^2 \times S^1; (I \times S^1) \times S^1. \]

If \( M^3 \) can’t be squeezed into \( R^3 \), new tools are needed:
\[ M^3 = (S^1 \times S^1) \times S^1, S^2 \times S^1, f(w, x, y, z) = 0, \ldots. \]

If a flow is suspended on \( S^2 \) what can we expect?

We use the circle map as a model for the sphere map \( S^2 \to S^2 \).

All the usual culprits are here.
Constraints

Often constraints exist on a dynamical system

$$\frac{dS}{dt} = f(S) \quad e.g. \quad S \cdot f(S) = 0$$

Constraints lower the dimension of the phase space. One class of physical systems includes spins (NMR, spintronics, ...). In such cases the spin length is a conserved quantity and the phase space is the sphere surface $S^2 \subset R^3$. 
Driven Constrained Systems

Under periodic driving the dynamical equations are

\[ \frac{dS_i}{dt} = f_i(S) + A_i \cos(\omega t) \]

The phase space is enlarged to \( S^2 \times S^1 \).

We need to develop new methods to determine the rigid organization of unstable periodic orbits.

Stroboscopic recordings at \( T = 2\pi/\omega \) map the sphere surface into itself, so we can study mappings

\[ S^2 \rightarrow S^2 \]
Sphere Maps

?? Sphere Maps ??

We can use circle maps as a guide to the study of sphere maps. The classical circle map (Arnold) is

\[ \theta' = \omega_0 + \theta + k \sin(\theta) \]

1. Rigid rotation: \( \omega_0 \)
2. Linear term: \( \theta \)
3. Nonlinear folding term: \( k \sin(\theta) \)

This simple map exhibits lots of fun properties.
Circle Map

\[ \omega_0 = 0 \quad k = 1.5 \]
Another way to construct circle maps involves a simple algorithm.

1. Start with a curve \( s(\theta) \).
2. Compute the normal to the curve at \( \theta \).
3. Determine the angle of the normal, \( \theta' \).

Gauss Map:

\[
\theta \rightarrow \theta'
\]

Implement the Gauss Map on the classical curve called the limaçon:

\[
r(\theta) = 2 + a \cos(\theta)
\]
Sphere Maps

Circle Map via Limaçon

\[ \theta' = \theta \]
Now apply this algorithm to a sphere. Push your finger into the surface and deform to a spherical limacon

\[ r(\theta_1, \theta_2, \cdots) = 2 + \sum_{j=1}^{d} a_j \cos(\theta_j) \]

This produces the map \((\theta, \phi) \rightarrow (\theta', \phi')\).

The map is invertible for \(|a_1| + |a_2| < 1\) and is not invertible otherwise. Noninvertible maps have exciting bifurcation structures. These maps depend on 3 rotation parameters and one nonlinear parameter. Several combinations are shown in the following color plots.
Sphere Maps

Gauss Map of Spherical Limacon
Rotation around $z$ axis
Rotation around $z$ axis
Rotation around $y$ axis
Sphere Maps

Bifurcation Diagram

Rotation Around z Axis: \( \omega_z \)
Sphere Maps

Another Bifurcation Diagram
Sphere Maps

A Mode-Locked Trajectory