Alice in Stretch & SqueezeLand: 18 Sphere Maps

August 12, 2012

Chapter Abstract

Periodic orbits are rigidly organized in *any* three-dimensional manifold $\mathcal{M}^3$ by transversality.

If $\mathcal{M}^3 \subset R^3$, Gauss Linking numbers are useful:
$\mathcal{M}^3 = D^3; D^2 \times S^1; (I \times S^1) \times S^1$.

If $\mathcal{M}^3$ can't be squeezed into R^3, new tools are needed:
$\mathcal{M}^3 = (S^1 \times S^1) \times S^1, S^2 \times S^1, f(w,x,y,z) = 0, \cdots$.

If a flow is suspended on S^2 what can we expect?

We use the circle map as a model for the sphere map $S^2 \to S^2$.

All the usual culprits are here.

Constraints

Often constraints exist on a dynamical system

$$\frac{d\mathbf{S}}{dt} = f(\mathbf{S}) \qquad e.g. \qquad \mathbf{S}\cdot f(\mathbf{S}) = 0$$

Constraints lower the dimension of the phase space.
One class of physical systems includes spins (NMR, spintronics, ...). In such cases the spin length is a conserved quantity and the phase space is the sphere surface $S^2 \subset R^3$.

Driven Constrained Systems

Under periodic driving the dynamical equations are

$$\frac{d\mathbf{S}_i}{dt} = f_i(\mathbf{S}) + A_i \cos(\omega t)$$

The phase space is enlarged to $S^2 \times S^1$.
We need to develop new methods to determine the rigid organization of unstable periodic orbits.
Stroboscopic recordings at $T = 2\pi/\omega$ map the sphere surface into itself, so we can study mappings

$$S^2 \rightarrow S^2$$

?? Sphere Maps ??

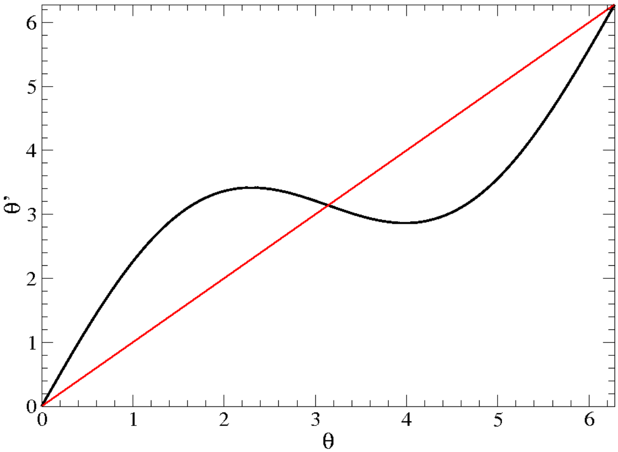
We can use *circle* maps as a guide to the study of *sphere* maps. The classical circle map (Arnold) is

$$\theta' = \omega_0 + \theta + k\sin(\theta)$$

1. Rigid rotation: ω_0
2. Linear term: θ
3. Nonlinear folding term: $k\sin(\theta)$

This simple map exhibits lots of fun properties.

Circle Map



$\omega_0 = 0 \qquad k = 1.5$

?? Sphere Maps ??

Another way to construct circle maps involves a simple algorithm.

1. Start with a curve $s(\theta)$.
2. Compute the normal to the curve at θ
3. Determine the angle of the normal, θ'.

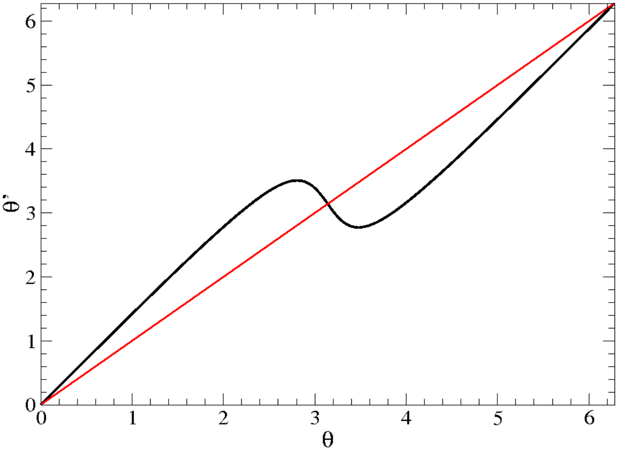
Gauss Map:

$$\theta \to \theta'$$

Implement ther Gauss Map on the classical curve called the limaçon:

$$r(\theta) = 2 + a\cos(\theta)$$

Circle Map via Limaçon



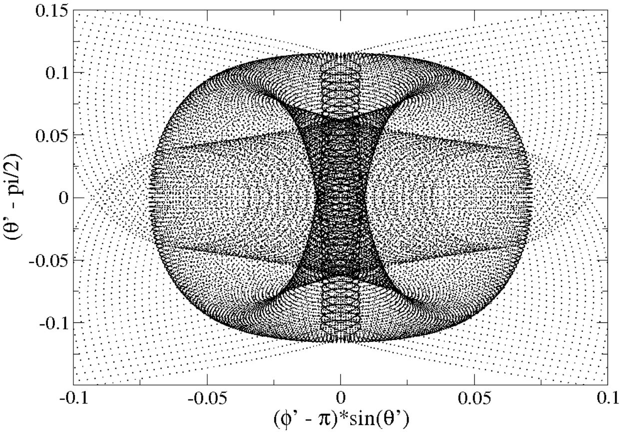
?? Sphere Maps ??

Now apply this algorithm to a sphere. Push your finger into the surface and deform to a spherical limacon

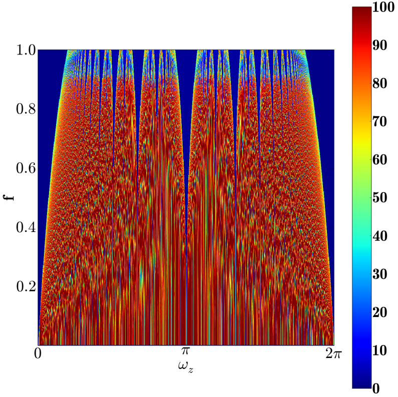
$$r(\theta_1, \theta_2, \cdots) = 2 + \sum_{j=1}^{d} a_j \cos(\theta_j)$$

This produces the map $(\theta, \phi) \to (\theta', \phi')$.
The map is invertible for $|a_1| + |a_2| < 1$ and is not invertible otherwise. Noninvertible maps have exciting bifurcation structures. These maps depend on 3 rotation parameters and one nonlinear parameter. Several combinations are shown in the following color plots.

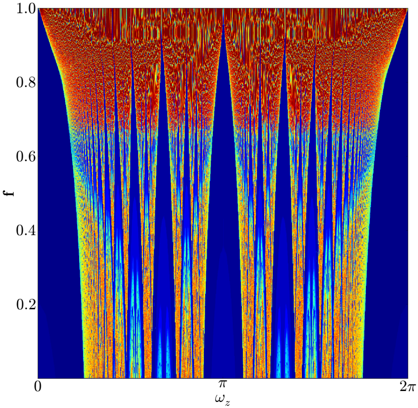
Gauss Map of Spherical Limacon



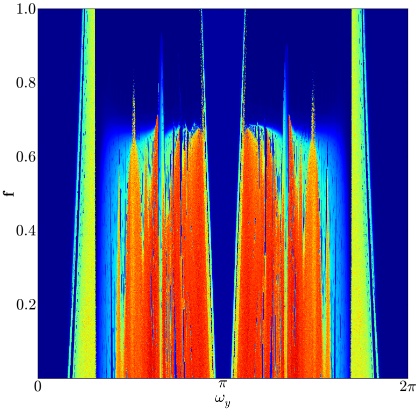
Rotation around z axis



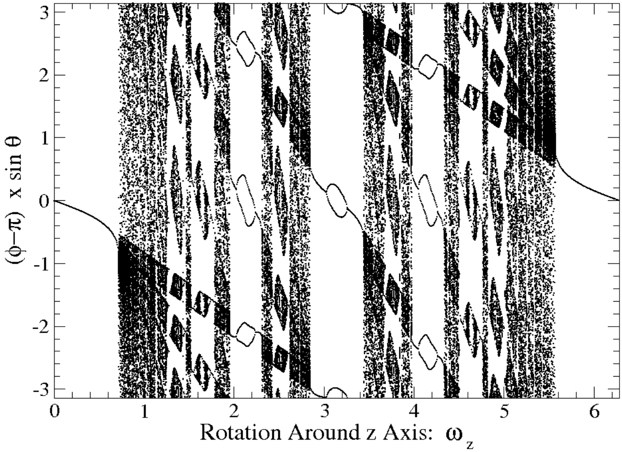
Rotation around z axis



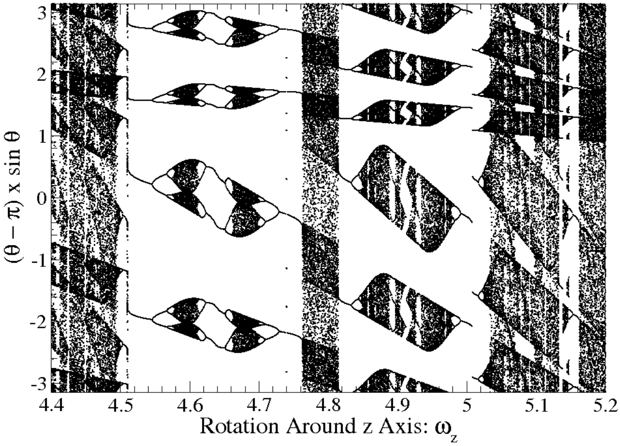
Rotation around y axis



Bifurcation Diagram



Another Bifurcation Diagram



A Mode-Locked Trajectory