Alice in Stretch & SqueezeLand: 14 Representations

August 12, 2012

Chapter Abstract

Data can be described (embedded) in many different ways.

Each faithful (1 to 1) description is a *representation*.

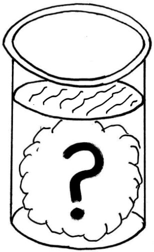
When are two representations equivalent or not?

What *representation labels* are needed to distinguish among *inequivalent representations*?

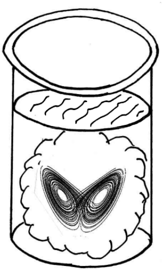
Insights from Group Theory provide guidance.

What do we do with Data?

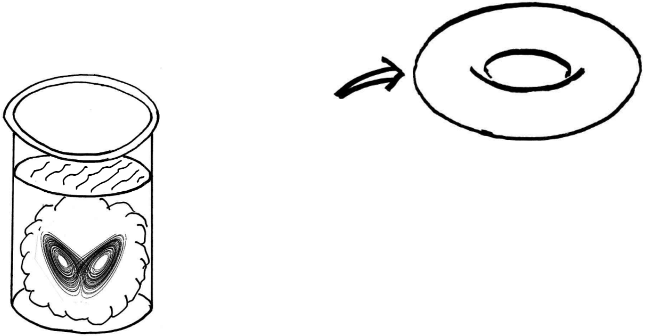
What do we do with Data?



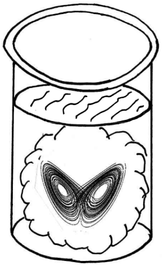
What do we do with Data?

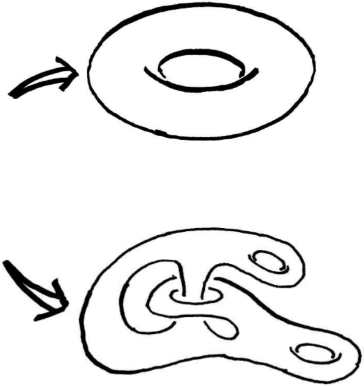


What do we do with Data?



What do we do with Data?





Outline

What to do with Data

Step 1: Data $\rightarrow$ Embedding

Step 2: Analyze Reconstructed Attractor

Step 3: What do you learn about:

The Data
The Embedding

????????

Important Theorems

Whitney (1936): $\mathcal{M}^n \to R^N$:
N generic functions -
Embedding if $N \geq 2n+1$.

Takens (1981) : $(\mathcal{M}^n, \dot{X} = F(X)) \to (R^N, Flow)$:
One generic function at N measurement intervals.
Embedding if $N \geq 2n+1$.

Wu (1958): All embeddings $\mathcal{M}^n \to R^N$ are isotopic for $N \geq 2n+1$ and $n > 1$.

Embeddings and Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

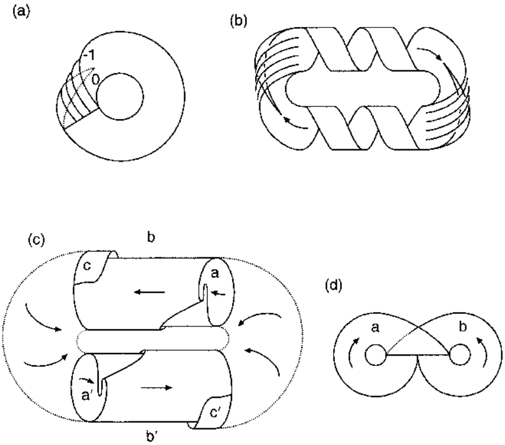
Preference is for embeddings of lowest possible dimension.

Possible Inequivalence for $n \leq N \leq 2n$.

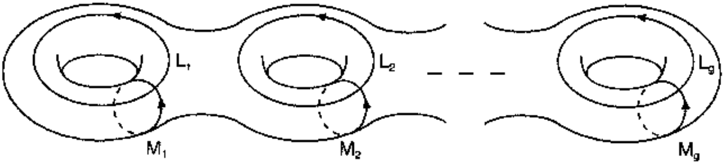
What do you want to learn?

- Geometry (Fractals, ...): "Independent" of Embedding
- Dynamics (Lyapunovs, ...) "Independent" of Embedding, but beware of spurious LEs
- Topology: some indices depend on embedding, others (*mechanism*) do not.

Revealed by Branched Manifolds



Classification of 3D Attractors



Program: $\mathcal{M}^3 \rightarrow R^3, R^4, R^5, R^6$

Inequivalent Representations in R^3: $g = 1$

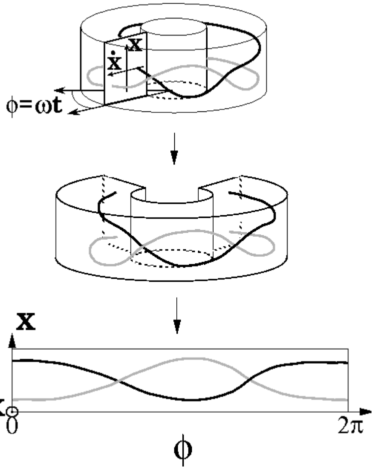
Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

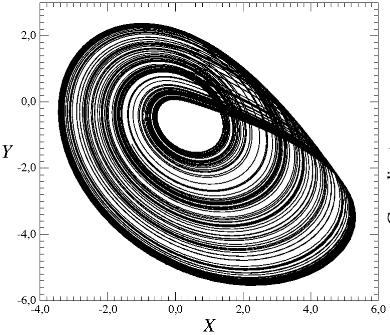
Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

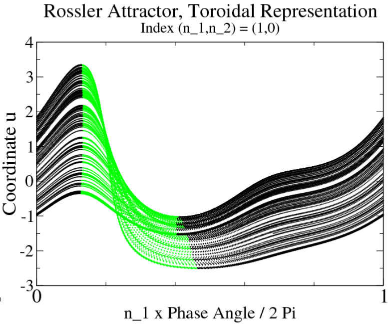
Cutting Open a Torus



Rossler Attractor: Two Representations

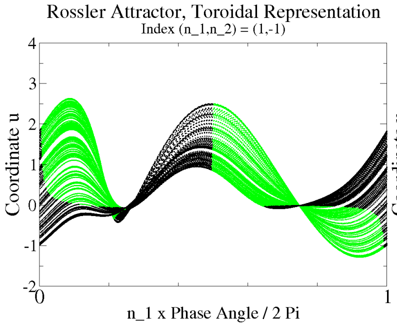
R^3 $\qquad$ $D^2 \times S^1$

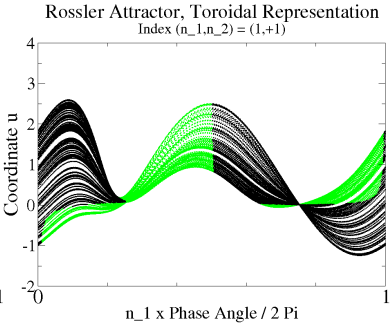




Rossler Attractor:

Two More Representations with $n = \pm 1$





Knot Representations

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

$$\text{Repere Mobile: } \mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)$$

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

$$(X(t), Y(t)) \rightarrow \mathbf{X}(t) = \mathbf{K}(\theta) + X(t)\mathbf{n}(\theta) + Y(t)\mathbf{b}(\theta)$$

$$\frac{\theta}{2\pi} = \frac{t}{T}$$

Equivalent Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension $= 3$.

Are these obstructions removed by injections into higher dimensions: R^4, R^5, R^6 ?

Necessary Labels

	Parity	Knot Type	Global Torsion
R^3	Y, $\pm$	Y	Y, N
R^4	-	-	Y, Z_2
R^5	-	-	-

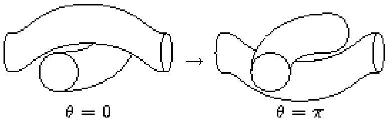
There is one *Universal* representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

$R^3 \to R^4$

Parity Isotopy in R^4

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{\text{Inject}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{pmatrix} \xrightarrow{\text{Isotopy}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \cos\theta \\ x^3 \sin\theta \end{pmatrix} \xrightarrow[\theta=\pi]{\text{Project}} \begin{pmatrix} x^1 \\ x^2 \\ -x^3 \end{pmatrix}.$$

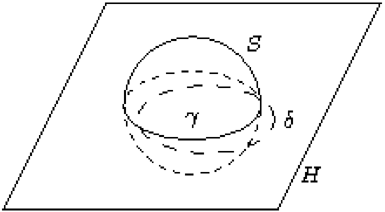
Knot Type Isotopy in R^4



Global Torsion Isotopy in R^5

$$\begin{bmatrix} s \\ re^{i\phi} \end{bmatrix} \mapsto \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix} \to \left[\begin{array}{c|cc} 1 & \multicolumn{2}{c}{0} \\ \hline 0 & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{array}\right] \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix}$$

Continued Inequivalence in R^4



The General Program

- $\mathcal{M}^n \to R^n$
- Identify all representation labels
- $R^n \to R^{n+1}$: Which labels drop away?
- $\to n+2, n+3, \ldots, 2n$: Which labels drop away?

- Group Theory: Complete set of Reps separate points.
- Dynamical Systems: Complete set of Reps separate diffeomorphisms.

Representation Theory for $g > 1$

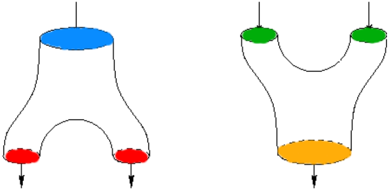
Can we extend the representation theory for strange attractors "with a hole in the middle" (i.e., genus $= 1$) to higher-genus attractors?

Yes. The results are similar.

Begin as follows:

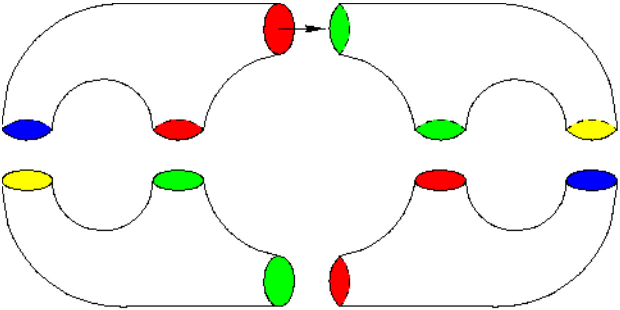
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



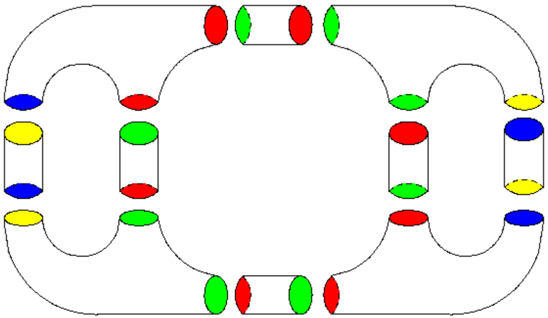
- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Application: Lorenz Dynamics, g=3



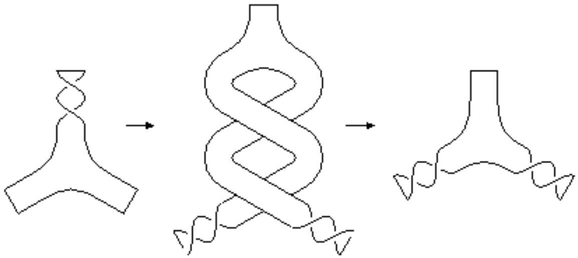
$g - 1$ Pairs of “trinions”

Insert A Flow Tube at Each Input

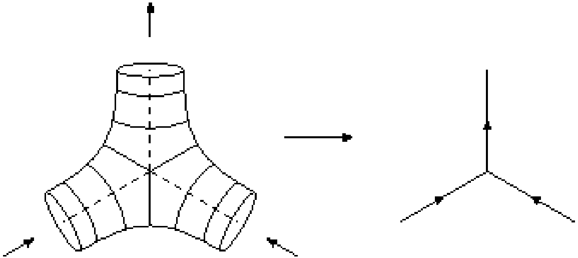


$3 \times (g-1)$ Local Torsion integers:	Isotope in R^5
Parity:	Isotope in R^4
Knot Type:	Isotope in R^4

Embeddings

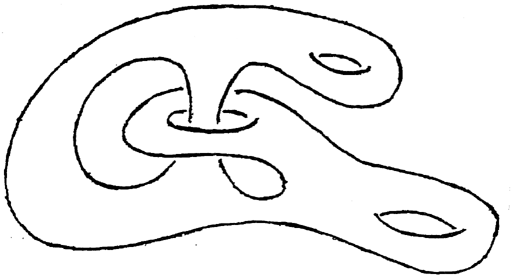


Reduction to Networks



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Intrinsic Tori



A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

Equivalences by Injection

Obstructions to Isotopy

Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus-g strange attractor become equivalent under isotopy.

zzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz

Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.

Inequivalent Irreducible Representations

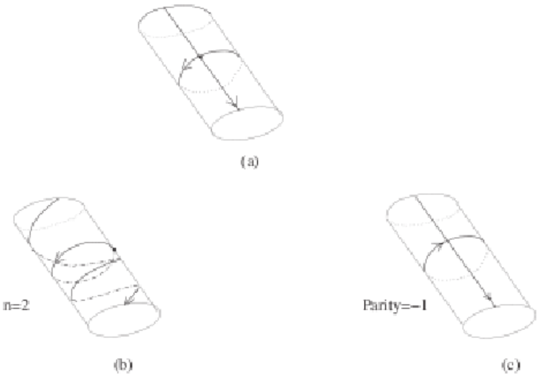
Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

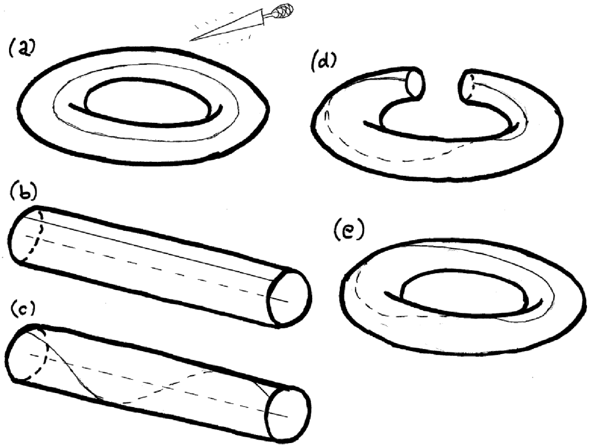
$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Global Torsion & Parity



Inequivalence in R^3



Equivalent Reducible Representations

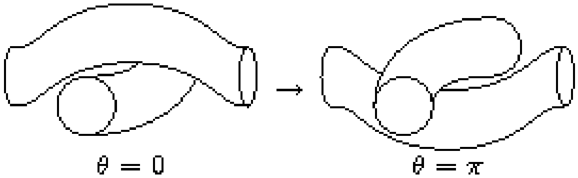
Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

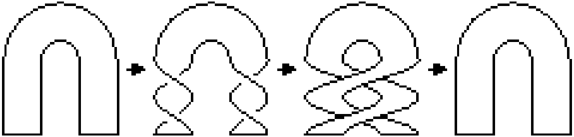
Equivalences

Crossing Exchange in R^4



Parity reversal is also possible in R^4 by isotopy.

2 Twists = 1 Writhe = Identity



$$Z \longrightarrow Z_2$$

$$\text{Global Torsion} \longrightarrow \text{Binary Op}$$

Equivalences by Injection

Obstructions to Isotopy

R^3	$\rightarrow$	R^4	$\rightarrow$	R^5
Global Torsion		Global Torsion		
Parity				
Knot Type				

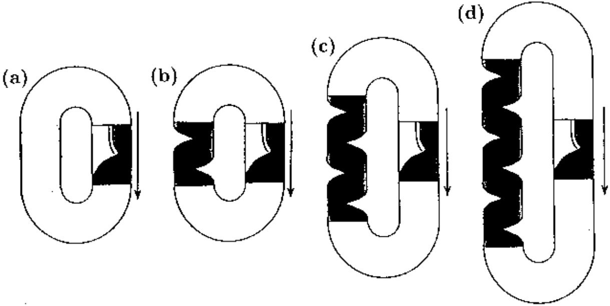
There is one *Universal* reducible representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

La Fin

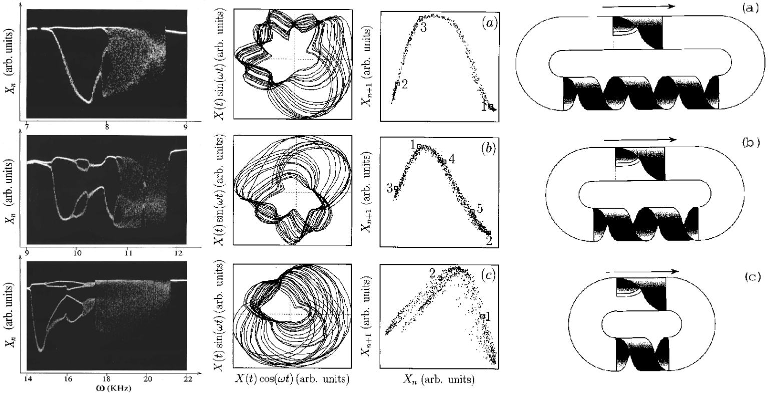
Merci Bien pour votre attention.

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change



Lefranc - Cargese