Alice in Stretch & SqueezeLand: Representations

Alice in Stretch & SqueezeLand: Representations 14

August 12, 2012

Chapter Abstract

Alice in Stretch & SqueezeLand: Representations

Chapter Summary-01

Data can be described (embedded) in many different ways.

Each faithful (1 to 1) description is a representation.

When are two representations equivalent or not?

What representation labels are needed to distinguish among inequivalent representations?

Insights from Group Theory provide guidance.

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question 0

Introduction

Embeddings

Embeddings-

Embeddings

Embedding 04

Embedding

What do we do with Data?

Alice in Stretch & SqueezeLand: 14 Representations

Chapter

Question-01

Question o.

Introduction

01

Embeddings 01

Embeddings

Embeddings

Embedding 04

Embeddin 05

What do we do with Data?



Alice in Stretch & SqueezeLand: Representations

What do we do with Data?

Question-02

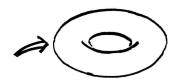


Alice in Stretch & SqueezeLand: Representations

Question-02

What do we do with Data?





Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question-02

Introduction 01

Embeddings

Embeddings 02

Embeddings 03

Embedding 04

05

What do we do with Data?



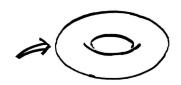




Table of Contents

Alice in Stretch & SqueezeLand: Representations

Introduction-

Outline

- Embeddings
- Whitney, Takens, Wu
- Tori: q=1
- Representation Labels
- Increasing Dimension
- Universal Embedding
- Representation Program
- **9** Tori: q > 1

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction-

Embeddings-01

Embeddings 02

Embedding 03

Embeddings 04

Embedding

What to do with Data

Step 1: Data \rightarrow Embedding

Step 2: Analyze Reconstructed Attractor

Step 3: What do you learn about:

The Data
The Embedding

???????

Embeddings 03

Embedding 04

Embed 05

Important Theorems

Whitney (1936): $\mathcal{M}^n \to R^N$: N generic functions - Embedding if $N \ge 2n + 1$.

Takens (1981) : $(\mathcal{M}^n, \dot{X} = F(X)) \to (R^N, Flow)$: One generic function at N measurement intervals. Embedding if $N \geq 2n+1$.

Wu (1958): All embeddings $\mathcal{M}^n \to R^N$ are isotopic for $N \geq 2n+1$ and n>1.

Embeddings and Representations

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Embeddings and Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a representation of an attractor.

Equivalence is by Isotopy.

Preference is for embeddings of lowest possible dimension.

Possible Inequivalence for $n \leq N \leq 2n$.

Chapter Summary-01

Question-01

Introduction

Embeddings

Embedding:

Embeddings-03

Embeddings 04

05

Geometry, Dynamics, Topology

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction-

Embeddings 01

Embeddings 02

Embedding 03

Embeddings-04

Embedo 05

What do you want to learn?

- Geometry (Fractals, ...): "Independent" of Embedding
- Dynamics (Lyapunovs, ...) "Independent" of Embedding, but beware of spurious LEs
- Topology: some indices depend on embedding, others (mechanism) do not.

Mechanism

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction

Embeddings

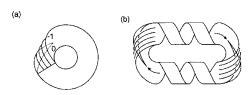
Embeddings

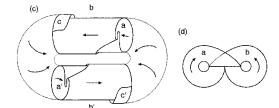
Embeddings

Embeddings

Embeddings-

Revealed by Branched Manifolds







Torus and Genus

Alice in Stretch & SqueezeLand: 14 Representations

Chapter

Question-01

Question 01

Introduction

Introduction-01

Embeddings

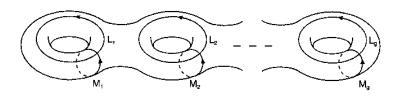
Embeddings 02

Embedding 03

Embedding 04

Embeddings

Classification of 3D Attractors



Program: $\mathcal{M}^3 \to R^3, R^4, R^5, R^6$

Representation Labels

Alice in Stretch & SqueezeLand: 14 Representations

Inequivalent Representations in R^3 : g = 1

Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity P

Global Torsion N

Knot Type KT

$\Gamma^{P,N,KT}(\mathcal{SA})$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Another Visualization

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

. . .

Introduction-

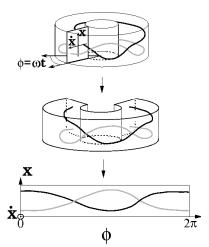
Embeddings

Embeddings-

Embeddings 03

Embeddings-

04 EmbeddingsCutting Open a Torus



Two Phase Spaces: R^3 and $D^2 \times S^1$

Alice in Stretch & SqueezeLand: 14 Representations

Chapter

Question-01

Question 01

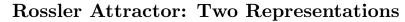
Introduction

Embeddings

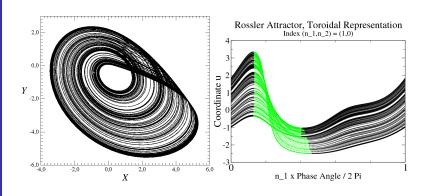
Embeddings 02

Embeddings 03

Embeddings 04







Other Diffeomorphic Attractors

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction

01

01

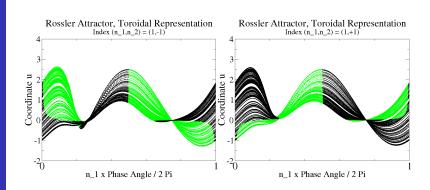
Embeddings

Embeddings

Embeddings-05

Rossler Attractor:

Two More Representations with $n = \pm 1$



Oriented Knot Type

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Knot Representations

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

Repere Mobile: $\mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)$

$$\frac{d}{ds} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

$$(X(t),Y(t)) \rightarrow \mathbf{X}(t) = \mathbf{K}(\theta) + X(t)\mathbf{n}(\theta) + Y(t)\mathbf{b}(\theta)$$

$$\frac{\theta}{2\pi} = \frac{t}{7}$$

Creating Isotopies

Alice in Stretch & SqueezeLand: Representations

Equivalent Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension = 3.

Are these obstructions removed by injections into higher dimensions: R^4, R^5, R^6 ?

Creating Isotopies

Alice in Stretch & SqueezeLand: Representations

Necessary Labels

	Parity	Knot Type	Global Torsion
R^3	Y, ±	Y	Y, <i>N</i>
R^4	-	-	Y , Z_2
R^5	-	-	-

There is one *Universal* representation in \mathbb{R}^N , N > 5. In \mathbb{R}^N the only topological invariant is *mechanism*.

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction-

Embeddings

Embeddings

Embeddings

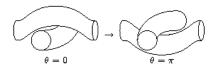
Embeddings 04

Embedding 05

Parity Isotopy in R^4

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{\text{Inject}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{pmatrix} \xrightarrow{\text{Isotopy}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \cos \theta \\ x^3 \sin \theta \end{pmatrix} \xrightarrow{\text{Project}} \begin{pmatrix} x^1 \\ x^2 \\ -x^3 \end{pmatrix}.$$

Knot Type Isotopy in \mathbb{R}^4



Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction

Embeddings

Embeddings

Embeddings

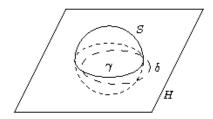
Embeddings 04

Embedding 05

Global Torsion Isotopy in R⁵

$$\begin{bmatrix} s \\ re^{i\phi} \end{bmatrix} \mapsto \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{bmatrix}$$

Continued Inequivalence in R^4



Representation Theory for Strange Attractors

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question-02

Introduction-01

Embeddings 01

Embedding

Embedding 03

Embedding 04

05

The General Program

- $\bullet \mathcal{M}^n \to R^n$
- Identify all representation labels
- $R^n \to R^{n+1}$: Which labels drop away?
- $\bullet \to n+2, n+3, \ldots, 2n$: Which labels drop away?
- Group Theory: Complete set of Reps separate points.
- Dynamical Systems: Complete set of Reps separate diffeomorphisms.

Representation Theory Redux

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Representation Theory for g > 1

Can we extend the representation theory for strange attractors "with a hole in the middle" (i.e., genus = 1) to higher-genus attractors?

Yes. The results are similar.

Begin as follows:

Chapter Summary-01

Question-01

Introduction

Embeddings

Embedding 02

Embedding 03

Embedding 04

05

Aufbau Princip for Bounding Tori

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question-02

Introduction 01

Embeddings 01

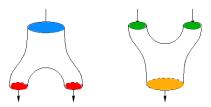
Embeddings 02

Embeddings 03

Embeddings 04

mbeddings-

Any bounding torus can be built up from equal numbers of stretching and squeezing units

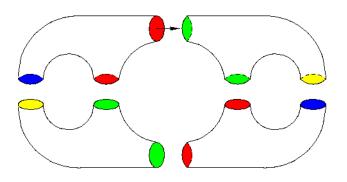


- Outputs to Inputs
- No Free Ends
- Colorless

Aufbau Princip for Bounding Tori

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Application: Lorenz Dynamics, g=3

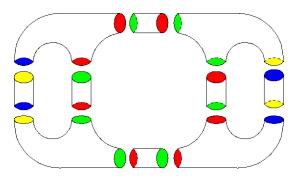


q-1 Pairs of "trinions"

Indices for Tori in \mathbb{R}^3

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Insert A Flow Tube at Each Input



 $3 \times (q-1)$ Local Torsion integers: Isotope in R^5 Isotope in \mathbb{R}^4 Parity:

Isotope in \mathbb{R}^4 Knot Type:



Embeddings

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction

01

01

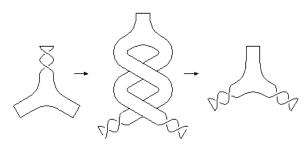
Embeddings 02

Embeddings

Embedding:

Embeddin 05

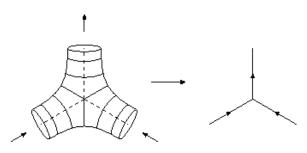
Embeddings



Embeddings

Alice in Stretch & SqueezeLand: Representations

Reduction to Networks



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Bounding Tori

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question-0

Introduction

Embeddings

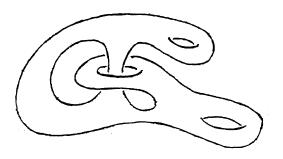
Embeddings

Embeddings

Embeddings 04

Embeddin

Extrinsic Embedding of Intrinsic Tori



A specific simple example.

Partial classification by links of homotopy group generators. Nightmare Numbers are Expected.

Creating Isotopies

Alice in Stretch & SqueezeLand: Representations

Equivalences by Injection Obstructions to Isotopy

Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus-g strange attractor become equivalent under isotopy.

Alice in Stretch & SqueezeLand: 14 Representations

Summary-01

Question-01

Introduction-

Embeddings-

mbeddings-

Embeddings

nbeddings

04

Embeddir

Representations

Alice in Stretch & SqueezeLand: 14 Representations

Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a representation of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

Chapter Summary-01

Question-01

Introduction

Embeddings 01

Embeddings 02

Embedding 03

Embedding: 04

05

Representations

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Introduction-01

Embeddings

Embeddings

Embeddings

Embeddings 04

Embedding

Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.

Representation Labels

Alice in Stretch & SqueezeLand: Representations

Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

> **Parity** Ρ Global Torsion

N

Knot Type KT

$\Gamma^{P,N,KT}(\mathcal{SA})$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.



Representation Labels

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Ouestion 0

Introduction

Embeddings

Embeddings 02

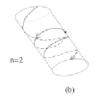
Embeddings

Embeddings 04

Embedding 05

Global Torsion & Parity







Inequivalence in \mathbb{R}^3

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question 02

Introduction 01

Embeddings 01

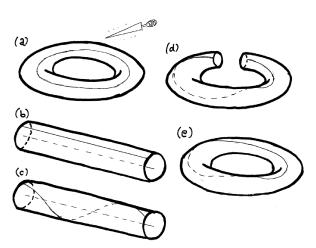
Embeddings 02

Embeddings 03

Embeddings

Embeddings

Inequivalence in R³



Creating Isotopies

Alice in Stretch & SqueezeLand: Representations

Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

Equivalences

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

Question-0

Introduction 01

Embeddings

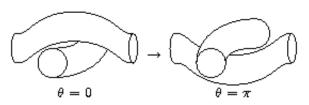
Embeddings 02

Embedding

Embedding 04

04 Embedding 05

Crossing Exchange in R^4



Parity reversal is also possible in \mathbb{R}^4 by isotopy.

Isotopies

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

. . .

Introduction

Embedding

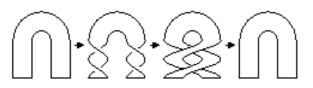
Embeddings

Embeddings

Embedding

Embedding 05

2 Twists = 1 Writhe = Identity



 $Z \longrightarrow Z_2$

Global Torsion \longrightarrow Binary Op

Creating Isotopies

Alice in Stretch & SqueezeLand: 14 Representations

Equivalences by Injection Obstructions to Isotopy

Summary-0:

Question-01

Introduction-

Embeddings-

Embedding

Embeddings 03

Embeddings 04

05

 R^3 o R^4 o R^5 Global Torsion Parity Knot Type

There is one *Universal* reducible representation in R^N , $N \geq 5$. In R^N the only topological invariant is *mechanism*.

La Fin

Alice in Stretch & SqueezeLand: 14 Representations

Merci Bien pour votre attention.

Summary-01

Question-01

Introduction

01

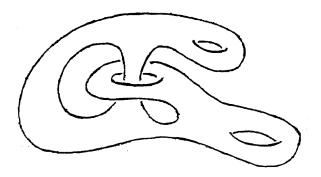
Embeddings 01

Embeddings 02

Embeddings 03

Embedding

04 Embeddings 05

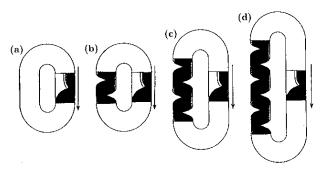


Determine Topological Invariants

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What Do We Learn?

- BM Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ...
- Independent of Embedding: Mechanism



Perestroikas of Strange Attractors

Alice in Stretch & SqueezeLand: 14 Representations

Chapter Summary-01

Question-01

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Introduction

Embeddings

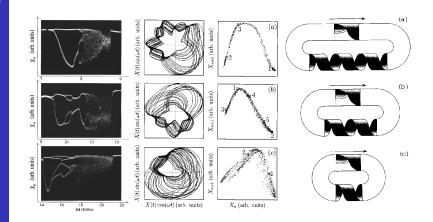
Embeddings

Embedding

Embeddings

Embeddings-

Evolution Under Parameter Change



Lefranc - Cargese

