Alice in Stretch & SqueezeLand:
14 Representations

August 12, 2012
Data can be described (embedded) in many different ways. Each faithful (1 to 1) description is a representation. When are two representations equivalent or not? What representation labels are needed to distinguish among inequivalent representations? Insights from Group Theory provide guidance.
What do we do with Data?
Data

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What to do with Data

Step 1: Data → Embedding

Step 2: Analyze Reconstructed Attractor

Step 3: What do you learn about:
   The Data
     The Embedding

????????
Background

Important Theorems

Whitney (1936): $\mathcal{M}^n \to \mathbb{R}^N$:
$N$ generic functions -
Embedding if $N \geq 2n + 1$.

Takens (1981) : $(\mathcal{M}^n, \dot{X} = F(X)) \to (\mathbb{R}^N, \text{Flow})$:
One generic function at $N$ measurement intervals.
Embedding if $N \geq 2n + 1$.

Wu (1958): All embeddings $\mathcal{M}^n \to \mathbb{R}^N$ are isotopic for
$N \geq 2n + 1$ and $n > 1$. 
Embeddings and Representations

An embedding creates a diffeomorphism between an (‘invisible’) dynamics in someone’s laboratory and a (‘visible’) attractor in somebody’s computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Preference is for embeddings of lowest possible dimension.

Possible Inequivalence for $n \leq N \leq 2n$. 
What do you want to learn?

- Geometry (Fractals, ...): “Independent” of Embedding
- Dynamics (Lyapunovs, ...) “Independent” of Embedding, but beware of spurious LEs
- Topology: some indices depend on embedding, others (*mechanism*) do not.
Revealed by Branched Manifolds
Torus and Genus

Classification of 3D Attractors

Program: \( \mathcal{M}^3 \to \mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5, \mathbb{R}^6 \)
Inequivalent Representations in $\mathbb{R}^3$: $g = 1$

Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

- Parity $P$
- Global Torsion $N$
- Knot Type $KT$

\[ \Gamma^{P,N,KT}(SA) \]

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.
Another Visualization

Cutting Open a Torus

$\phi = \omega t$
Two Phase Spaces: $\mathbb{R}^3$ and $D^2 \times S^1$

**Rossler Attractor: Two Representations**

$\mathbb{R}^3$  

$D^2 \times S^1$

Rossler Attractor, Toroidal Representation

Index $(n_1,n_2) = (1,0)$

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Alice in Stretch & SqueezeLand: 14 Representations
Other Diffeomorphic Attractors

Rossler Attractor:
Two More Representations with $n = \pm 1$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (1,-1)$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (1,+1)$
Oriented Knot Type

Knot Representations

\[ K(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = K(\theta + 2\pi) \]

Repere Mobile: \( t(\theta), n(\theta), b(\theta) \)

\[
\frac{d}{ds} \begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}
\]

\[(X(t), Y(t)) \rightarrow X(t) = K(\theta) + X(t)n(\theta) + Y(t)b(\theta)\]

\[
\frac{\theta}{2\pi} = \frac{t}{T}
\]
Creating Isotopies

Equivalent Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension \( = 3 \).

Are these obstructions removed by injections into higher dimensions: \( \mathbb{R}^4, \mathbb{R}^5, \mathbb{R}^6 \)?
Creating Isotopies

Necessary Labels

<table>
<thead>
<tr>
<th>$\mathbb{R}^3$</th>
<th>Parity</th>
<th>Knot Type</th>
<th>Global Torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}^4$</td>
<td>$Y, \pm$</td>
<td>$Y$</td>
<td>$Y, N$</td>
</tr>
<tr>
<td>$\mathbb{R}^5$</td>
<td>$-$</td>
<td>$-$</td>
<td>$Y, \mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

There is one \textit{Universal} representation in $\mathbb{R}^N$, $N \geq 5$. In $\mathbb{R}^N$ the only topological invariant is \textit{mechanism}. 
$R^3 \to R^4$

Parity Isotopy in $R^4$

\[
\begin{pmatrix}
    x^1 \\
    x^2 \\
    x^3
\end{pmatrix}
\xrightarrow{\text{Inject}}
\begin{pmatrix}
    x^1 \\
    x^2 \\
    x^3 \\
    0
\end{pmatrix}
\xrightarrow{\text{Isotopy}}
\begin{pmatrix}
    x^1 \\
    x^2 \\
    x^3 \cos \theta \\
    x^3 \sin \theta
\end{pmatrix}
\xrightarrow{\text{Project, } \theta=\pi}
\begin{pmatrix}
    x^1 \\
    x^2 \\
    -x^3
\end{pmatrix}.
\]

Knot Type Isotopy in $R^4$

\[
\theta = 0 \quad \rightarrow \quad \theta = \pi
\]
$R^3 \rightarrow R^5$

Global Torsion Isotopy in $R^5$

\[
\begin{bmatrix}
  s \\
  re^{i\phi} \\
  re^{i(\phi+s)} \\
\end{bmatrix} \mapsto \begin{bmatrix}
  s \\
  re^{i\phi} \\
  re^{i(\phi+s)} \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 \\
  0 & \cos \theta & \sin \theta \\
  0 & -\sin \theta & \cos \theta \\
\end{bmatrix} \begin{bmatrix}
  s \\
  re^{i\phi} \\
  re^{i(\phi+s)} \\
\end{bmatrix}
\]

Continued Inequivalence in $R^4$
The General Program

- $\mathcal{M}^n \to \mathbb{R}^n$
- Identify all representation labels
- $\mathbb{R}^n \to \mathbb{R}^{n+1}$: Which labels drop away?
- $\to n + 2, n + 3, \ldots, 2n$: Which labels drop away?

- Group Theory: Complete set of Reps separate points.
- Dynamical Systems: Complete set of Reps separate diffeomorphisms.
Can we extend the representation theory for strange attractors “with a hole in the middle” (i.e., genus = 1) to higher-genus attractors?

Yes. The results are similar.

Begin as follows:
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units

- Outputs to Inputs
- No Free Ends
- Colorless
Aufbau Princip for Bounding Tori

Application: Lorenz Dynamics, $g=3$

$g-1$ Pairs of “trinions”
Insert A Flow Tube at Each Input

3 \times (g - 1) \text{ Local Torsion integers: Isotope in } \mathbb{R}^5

Parity: Isotope in \mathbb{R}^4

Knot Type: Isotope in \mathbb{R}^4
Embeddings
Embeddings

Reduction to Networks

Equivalent to embedding a specific class of directed networks into $\mathbb{R}^3$
Extrinsic Embedding of Bounding Tori

Extrinsic Embedding of Intrinsic Tori

A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.
Creating Isotopies

### Equivalences by Injection

Obstructions to Isotopy

<table>
<thead>
<tr>
<th>Index</th>
<th>$R^3$</th>
<th>$R^4$</th>
<th>$R^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global Torsion</td>
<td>$\mathbb{Z} \otimes 3(g-1)$</td>
<td>$\mathbb{Z}_2 \otimes 2(g-1)$</td>
<td>-</td>
</tr>
<tr>
<td>Parity</td>
<td>$\mathbb{Z}_2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Knot Type</td>
<td>General KT.</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

In $R^5$ all representations (embeddings) of a genus-$g$ strange attractor become equivalent under isotopy.
An embedding creates a diffeomorphism between an (‘invisible’) dynamics in someone’s laboratory and a (‘visible’) attractor in somebody’s computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension.
Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.
Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

- Parity \( P \)
- Global Torsion \( N \)
- Knot Type \( KT \)

\[ \Gamma^{P,N,KT}(SA) \]

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.
Global Torsion & Parity

(a)

n=2

(b)

Parity=−1

(c)

Representation Labels
Inequivalence in $R^3$
Creating Isotopies

Equivalent Reducible Representations

Topological indices \((P,N,KT)\) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?
Equivalences

**Crossing Exchange in** $R^4$

Parity reversal is also possible in $R^4$ by isotopy.
Isotopies

2 Twists = 1 Writhe = Identity

$\mathbb{Z} \rightarrow \mathbb{Z}_2$

Global Torsion $\rightarrow$ Binary Op
Creating Isotopies

Equivalences by Injection
Obstructions to Isotopy

\[ \mathbb{R}^3 \rightarrow \mathbb{R}^4 \rightarrow \mathbb{R}^5 \]

Global Torsion
Parity
Knot Type
Global Torsion

There is one *Universal* reducible representation in \( \mathbb{R}^N, N \geq 5 \). In \( \mathbb{R}^N \) the only topological invariant is *mechanism*. 
Merci Bien pour votre attention.
Determine Topological Invariants

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don’t
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism
Evolution Under Parameter Change

Perestroikas of Strange Attractors

Lefranc - Cargese