Chapter Abstract

Periodic orbits exist in abundance in a strange attractor.

The problems are: to find them, to determine how they are organized among themselves.

Whatever mechanism exists to create the strange attractor, it simultaneously organizes all the unstable periodic orbits in the attractor in a unique way.

We can classify *mechanisms* by sets of *integers*.

There is an *Aufbau Principal* for building up strange attractors.
Chaos

Motion that is

- Deterministic: \( \frac{dx}{dt} = f(x) \)
- Recurrent
- Non Periodic
- Sensitive to Initial Conditions
Strange Attractor

The $\Omega$ limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor
UPOs: Skeletons of Strange Attractors
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UPOs Outline Strange Attractors

Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.
Organization of UPOs in $R^3$:

Gauss Linking Number

\[
LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}
\]

# Interpretations of LN $\simeq$ # Mathematicians in World
Linking Number of Two UPOs

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese
Mechanisms for Generating Chaos

Stretching and Folding

(a) squeeze
(b) stretch
(c) boundary layer
Mechanisms for Generating Chaos

Tearing and Squeezing
Motion of Blobs in Phase Space

Stretching — Squeezing

[Diagram showing stretching and squeezing processes]
Birman - Williams Projection

Identify $x$ and $y$ if

$$\lim_{t \to \infty} |x(t) - y(t)| \to 0$$
Fundamental Theorem

Birman - Williams Theorem

If:

Then:
Fundamental Theorem

Birman - Williams Theorem

If: Certain Assumptions

Then:
Fundamental Theorem

Birman - Williams Theorem

If: Certain Assumptions

Then: Specific Conclusions
Birman-Williams Theorem

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- on $\mathbb{R}^n$ is dissipative, $n=3$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0, \lambda_1 + \lambda_2 + \lambda_3 < 0$

- Generates a hyperbolic strange attractor $SA$

IMPORTANT: The underlined assumptions can be relaxed.
Birman-Williams Theorem

Conclusions, B-W Theorem

- The projection maps the strange attractor $SA$ onto a 2-dimensional branched manifold $BM$ and the flow $\Phi_t(x)$ on $SA$ to a semiflow $\Phi(x)_t$ on $BM$.

- UPOs of $\Phi_t(x)$ on $SA$ are in 1-1 correspondence with UPOs of $\Phi(x)_t$ on $BM$. Moreover, every link of UPOs of $(\Phi_t(x), SA)$ is isotopic to the correspond link of UPOs of $(\Phi(x)_t, BM)$.

Remark: “One of the few theorems useful to experimentalists.”
A Very Common Mechanism

Rössler:

Attractor

Branched Manifold
A Mechanism with Symmetry

Lorenz: Attractor Branched Manifold
Examples of Branched Manifolds

Inequivalent Branched Manifolds

(a)  

(b)  

(c)  

(d)
Template Holding All Knot Types
Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units

subject to the conditions:
- Outputs to Inputs
- No Free Ends
Rossler System

(a) Rössler Equations

\[
\frac{dx}{dt} = -y - z \nonumber \\
\frac{dy}{dt} = x + ay \nonumber \\
\frac{dz}{dt} = -bz + x(z - c) \nonumber 
\]

(b) Rossler System Graphs

(c) Rossler System Graphs

(d) Rossler System Graphs

(f) Jacobian Matrix

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & +1 \\
\end{pmatrix}
\]
Lorenz System

(a) Lorenz Equations

\[
\frac{dx}{dt} = -\alpha x + \sigma y \\
\frac{dy}{dt} = R x - y - xz \\
\frac{dz}{dt} = -\beta z + xy
\]

(b) Graph of Lorenz attractor over time

(c) Graph of Lorenz system in 3D space
Poincaré Smiles at Us in $R^3$

- Determine organization of UPOs $\Rightarrow$
- Determine branched manifold $\Rightarrow$
- Determine equivalence class of $SA$