Alice in Stretch & SqueezeLand
The Marvels of Topology and Chaos

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Abstract

Suppose you have data from a physical system that is behaving chaotically. What do you do? How do you analyze these data? What should you look for? What is the mechanism that generates chaos?

For a large class of systems an algorithm now exists for addressing each of these questions successively and successfully. We will go through the steps of this algorithm, showing how each works using experimental data and pointing out the connection with topology. In the process we will develop a classification scheme for strange attractors.
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1. Overview
2. Experimental Challenge
3. Topology of Orbits
4. Topological Analysis Program
5. Basis Sets of Orbits
6. Bounding Tori
7. Covers and Images
8. Quantizing Chaos
9. Representation Theory of Strange Attractors
10. Summary
Can you explain my data?

I dare you to explain my data!

J. R. Tredicce
Where is Tredicce coming from?

Feigenbaum:

\[ \alpha = 4.66920 16091 \ldots \]

\[ \delta = -2.50290 78750 \ldots \]
Laser with Modulated Losses
Experimental Arrangement
Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
Result

There is now a classification theory.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to $\mathbb{R}^3$ only — for now
The 4 Levels of Structure

- Basis Sets of Orbits
- Branched Manifolds
- Bounding Tori
- Extrinsic Embeddings
What Have We Learned?

1. Cover and Image Relations
2. Continuations: Analytical, Topological, Group
3. Cauchy Riemann & Clebsch-Gordonnery for Dynamical Systems
4. “Quantizing Chaos”
5. Representation Theory for Dynamical Systems

What Do We Need to Learn?

1. Higher Dimensions
2. Invariants
3. Mechanisms
Experimental Schematic

Laser Experimental Arrangement

![Experimental Schematic Diagram](Image)
Experimental Motivation

Oscilloscope Traces

(a) [Diagram of oscillations]
(b) [Diagram of oscillations]
(c) [Diagram of oscillations]
(d) [Diagram of oscillations]
(e) [Diagram of oscillations]

I vs t  I vs k  S(f) vs f  I vs t
Results, Single Experiment

Bifurcation Schematics

The Marvels of Topology and Chaos

Robert Gilmore

Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01
Some Attractors

Coexisting Basins of Attraction
Many Experiments

Bifurcation Perestroikas

(a) Bifurcation Diagram

(b) Bifurcation Diagram

(c) Bifurcation Diagram
Experimental Data: LSA

Lefranc - Cargese
Experimental Data: LSA
Real Data

Experimental Data: LSA
Mechanism

Stretching & Squeezing in a Torus
Time Evolution

Rotating the Poincaré Section around the axis of the torus

(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

(m) (n) (o)
Rotating the Poincaré Section around the axis of the torus

**Figure 2.** Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.
Another Visualization

Cutting Open a Torus

\[ \phi = \omega t \]
Satisfying Boundary Conditions

Global Torsion

(a)

(b)
A Chemical Experiment

The Belousov-Zhabotinskii Reaction
The Lasers in Zaragosa

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

<table>
<thead>
<tr>
<th>Species</th>
<th>Horseshoe</th>
<th>Reverse horseshoe</th>
<th>Out-to-in spiral</th>
<th>In-to-out spiral</th>
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<td>(0 2 1)</td>
<td>(1 2 0)</td>
<td>(0 2 1) or (1 2 0)</td>
<td>(2 1 0)</td>
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<td>Sketch of the folding process</td>
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<td><img src="image" alt="Sketch" /></td>
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<td><img src="image" alt="Sketch" /></td>
</tr>
</tbody>
</table>

Modulation frequency normalized to the natural frequency

m=0.93
m=0.78
m=0.73
TABLE 2 – Linking numbers between the UPOs extracted from the time series corresponding to pump modulation frequency $f=4.25$ KHz and modulation index $m=0.73$

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Close Returns Plot

\[ |x_i - x_{i+p}| < \epsilon \quad \text{pixel} \rightarrow \text{black} \]
First Embedding Attempt: $x, \dot{x}, \ddot{x}$
Second Embedding Attempt: $\int x, x, \dot{x}$

Nonstationary!
Third embedding attempt: \( \int x e^{-\frac{t'}{\tau}}, x, \dot{x} \)
Orbits to Organization

Once you have an embedding:

- Find a Poincaré Section
- Construct a First Return Map on the Section
- Introduce a Symbolic Encoding
- Encode all Unstable Periodic Orbits
- Find their Linking Numbers
Return Maps

Two Symbols Suffice! 0 and 1
Embedded Periodic Orbits

Some Named Low-Period Orbits

[Diagrams of low-period orbits]
### Some Extracted and Reconstructed Periodic Orbits

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<thead>
<tr>
<th>Orbit</th>
<th>Name</th>
<th>Symbolics</th>
<th>Local Torsion</th>
<th>Self-Linking</th>
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Table of Experimental Linking Numbers

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<td>16</td>
<td>21</td>
<td>24</td>
<td>21</td>
</tr>
</tbody>
</table>

*All indices are negative.*
Testing the Result

(a), (c) $y_1^m$ compared with $y_1^d$. (b), (d) $y_3^m$ compared with $y_3^d$. 
Chaos

Motion that is

- Deterministic: \( \frac{dx}{dt} = f(x) \)
- Recurrent
- Non Periodic
- Sensitive to Initial Conditions
Strange Attractor

The $\Omega$ limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor
UPOs: Skeletons of Strange Attractors
UPOs: Skeletons of Strange Attractors
UPOs: Skeletons of Strange Attractors
UPOs: Skeletons of Strange Attractors

The Marvels of Topology and Chaos

Robert Gilmore

Introduction-01
Introduction-02
Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01
Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.
Organization of UPOs in $\mathbb{R}^3$:

Gauss Linking Number

\[
LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}
\]

# Interpretations of LN $\sim$ # Mathematicians in World
Linking Number of Two UPOs

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese
Mechanisms for Generating Chaos

Stretching and Folding
Mechanisms for Generating Chaos

Tearing and Squeezing
Motion of Blobs in Phase Space

Stretching — Squeezing

[Diagram showing stretching and squeezing processes in phase space]

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Birman - Williams Projection

Identify \(x\) and \(y\) if

\[
\lim_{t \to \infty} |x(t) - y(t)| \to 0
\]
Fundamental Theorem

Birman - Williams Theorem

If:

Then:
Fundamental Theorem

Birman - Williams Theorem

If: Certain Assumptions

Then:
Birman - Williams Theorem

If: Certain Assumptions

Then: Specific Conclusions
Assumptions, B-W Theorem

A flow $\Phi_t(x)$

• on $\mathbb{R}^n$ is dissipative, $n=3$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.

• Generates a hyperbolic strange attractor $SA$

IMPORTANT: The underlined assumptions can be relaxed.
Conclusions, B-W Theorem

- The projection maps the strange attractor \( SA \) onto a 2-dimensional branched manifold \( BM \) and the flow \( \Phi_t(x) \) on \( SA \) to a semiflow \( \overline{\Phi}(x)_t \) on \( BM \).

- UPOs of \( \Phi_t(x) \) on \( SA \) are in 1-1 correspondence with UPOs of \( \overline{\Phi}(x)_t \) on \( BM \). Moreover, every link of UPOs of \( (\Phi_t(x), SA) \) is isotopic to the correspond link of UPOs of \( (\overline{\Phi}(x)_t, BM) \).

Remark: “One of the few theorems useful to experimentalists.”
A Very Common Mechanism

Rössler:

Attractor   Branched Manifold
A Mechanism with Symmetry

Lorenz:

Attractor

Branched Manifold
Examples of Branched Manifolds

Inequivalent Branched Manifolds

(a)

(b)

(c)

(d)
Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units

subject to the conditions:

- Outputs to Inputs
- No Free Ends
Dynamics and Topology

Rossler System

(a) Rossler Equations

\[ \begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*} \]

(b) Graphs of z and x vs. t

(f) Matrix

\[
\begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & -1
\end{pmatrix}
\]
Lorenz System

Lorenz Equations

\[
\begin{align*}
\frac{dx}{dt} &= -\sigma x + \sigma y \\
\frac{dy}{dt} &= R x - y - xz \\
\frac{dz}{dt} &= -bz + xy
\end{align*}
\]

(d)

(b)
Dynamics and Topology

Poincaré Smiles at Us in $R^3$

- Determine organization of UPOs $\Rightarrow$
- Determine branched manifold $\Rightarrow$
- Determine equivalence class of $SA$
We Like to be Organized

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Experimental-01
We Like to be Organized

Experimental Chart of Nuclides 2000
2975 isotopes
Topological Analysis Program

Locate Periodic Orbits
Create an Embedding
Determine Topological Invariants (LN)
Identify a Branched Manifold
Verify the Branched Manifold

Model the Dynamics
Validate the Model
Method of Close Returns
Method of Close Returns

\[ |x_i - x_{i+p}| < \epsilon, \quad \text{pixel} \rightarrow \text{black} \]
Embeddings

This is a tricky business. There are many problems ... 

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding
Locate UPOs

Periodic Orbits Outline the Attractor

Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese
Determine Topological Invariants

**Linking Number of Orbit Pairs**

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese
### Compute Table of Expt’l LN

**Table 7.2** Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data

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*a All indices are negative.*
Determine Topological Invariants

Compare w. LN From Various $BM$

Table 2.1  Linking numbers for orbits to period five in Smale horseshoe dynamics.

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Determine Topological Invariants

Guess Branched Manifold

Figure 7. “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese
Determine Topological Invariants

Identification & ‘Confirmation’

- $BM$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion
Determine Topological Invariants

**What Do We Learn?**

- $BM$ Depends on Embedding
- Some things depend on embedding, some don’t
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism

![Diagram showing different shapes and embeddings](image)
Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global tension increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].
Perestroikas of Strange Attractors

Evolution Under Parameter Change

Lefranc - Cargese
Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese
Model the Dynamics

A hodgepodge of methods exist: \# Methods \(\sim\) \# Physicists

Validate the Model

Needed: Nonlinear analog of \(\chi^2\) test. OPPORTUNITY: Tests that depend on entrainment/synchronization.
Our Hope → Now a Result

Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
Determinism

How to predict the future from the past
Some Prediction Results

Tightly binned predictions suggest determinism
Orbits Can be “Pruned”

There Are Some Missing Orbits

Lorenz

Shimizu-Morioka
Linking Numbers, Relative Rotation Rates, Braids

Orbit Forcing

A \Rightarrow B \quad B \Rightarrow C

A \Rightarrow C
Forcing Diagram - Horseshoe

An Ongoing Problem
An Ongoing Problem

Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required
Nerve Data

Is This Predictable or Not?
Some Variables are Strongly Correlated
The Attractor can be Projected in Many Ways

$y_4 - y_5$ Plane

$\frac{y_4}{dt} - y_5$ Plane
First Return Maps at Different Temperatures

The Return Map

“Drifts” with Temperature

\[ T = 12^\circ \text{C} \quad T = 13.5^\circ \text{C} \quad T = 16.5^\circ \text{C} \]
Scroll Templates

Outside to Inside

Inside to Outside

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Some Periodic Orbits

(a): $4f$  
(b): $4r$  
(c): $5f$  
(d): $(4f, 5f)$
Steps in Constructing Scroll Template

Twist  Relax  Return
A Simple Two-Parameter Model of Chaotic Nerves

\[ \Phi = \text{Drift} \quad \lambda = \text{Stretch} \]
Perestroikas of Branched Manifolds

Constraints

Branched manifolds largely constrain the ‘perestroikas” that forcing diagrams can undergo.

Is there some mechanism/structure that constrains the types of perestroikas that branched manifolds can undergo?
constraints on branched manifolds

“inflated” a strange attractor
union of $\epsilon$ ball around each point
boundary is surface of bounded 3d manifold
torus that bounds strange attractor
Torus, Longitudes, Meridians

Tori are identified by genus $g$ and dressed with a surface flow induced from that creating the strange attractor.
Surface Singularities

**Flow field:** three eigenvalues: +, 0, –

**Vector field** “perpendicular” to surface

**Eigenvalues on surface at fixed point:** +, –

**All singularities are regular saddles**

\[ \sum_{s.p.} (-1)^{\text{index}} = \chi(S) = 2 - 2g \]

# fixed points on surface = index = 2g - 2

**Singularities organize the surface flow dressing the torus**
Flow Near a Singularity
Some Bounding Tori

Torus Bounding Lorenz-like Flows
Twisting the Lorenz Attractor

(a)

(b)

(c)

(d)
Two possible branched manifolds in the torus with $g=4$. 

Constraints Provided by Bounding Tori
Poincaré section is disjoint union of $g-1$ disks.

Transition matrix sum of two $g-1 \times g-1$ matrices.

Both are $g-1 \times g-1$ permutation matrices.

They identify mappings of Poincaré sections to P’sections.

Bounding tori labeled by (permutation) group theory.
Some Bounding Tori

Bounding Tori of Low Genus

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TABLE I: Enumeration of canonical forms up to genus 9.
Exponential Growth

The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus, \( g \).

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Motivation

Some Genus-9 Bounding Tori
These units ("pants, trinions") surround the stretching and squeezing units of branched manifolds.
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units

- Outputs to Inputs
- No Free Ends
- Colorless
Construction of Poincaré Section

$$P. \ S. = \text{Union}$$

$$\# \text{ Components} = g-1$$
Aufbau Princip for Bounding Tori

Application: Lorenz Dynamics, g=3
Representation Theory for \( g > 1 \)

Can we extend the representation theory for strange attractors “with a hole in the middle” (i.e., genus = 1) to higher-genus attractors?

Yes. The results are similar.

Begin as follows:
Application: Lorenz Dynamics, $g=3$
Embeddings
Preparations for Embedding tori into $R^3$
Extrinsic Embedding of Bounding Tori

Extrinsic Embedding of Intrinsic Tori

A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.
Creating Isotopies

Equivalences by Injection
Obstructions to Isotopy

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<td>Gen. KT.</td>
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In $R^5$ all representations (embeddings) of a genus-$g$ strange attractor become equivalent under isotopy.
Modding Out a Rotation Symmetry

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
=
\begin{pmatrix}
\text{Re} (X + iY)^2 \\
\text{Im} (X + iY)^2 \\
Z
\end{pmatrix}
\]
Lorenz Attractor and Its Image
Lifting an Attractor: Cover-Image Relations

Creating a Cover with Symmetry

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} \leftarrow \begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix}
\text{Re} (X + iY)^2 \\
\text{Im} (X + iY)^2 \\
Z
\end{pmatrix}
\]
Cover-Image Related Branched Manifolds

Cover-Image Branched Manifolds
Covering Branched Manifolds

Two Two-fold Lifts
Different Symmetry

Rotation Symmetry

Inversion Symmetry
Topological Indices

Topological Index: Choose Group
Choose Rotation Axis (Singular Set)
Locate the Singular Set wrt Image

Different Rotation Axes Produce Different (Nonisotopic) Lifts
Nonisotopic Locally Diffeomorphic Lifts

(a) $\mu = 0.0$
(b) $\mu = -0.84548$
(c) $\mu = -2.083$
(d) $\mu = -3.14674$
(e) $\mu = -4.166$
Indices (0,1) and (1,1)

Two Two-fold Covers
Same Symmetry
Indices (0,1) and (1,1)

Three-fold, Four-fold Covers

![Graphs showing three-fold and four-fold covers with X and Y axes labeled.](image)
Two Inequivalent Lifts with $V_4$ Symmetry
Algorithm

- Construct Invariant Polynomials, Syzygies, Radicals
- Construct Singular Sets
- Determine Topological Indices
- Construct Spectrum of Structurally Stable Covers
- Structurally Unstable Covers Interpolate
Surprising New Findings

Symmetries Due to Symmetry

- Schur’s Lemmas & Equivariant Dynamics
- Cauchy Riemann Symmetries
- Clebsch-Gordon Symmetries
- Continuations
  - Analytic Continuation
  - Topological Continuation
  - Group Continuation
Covers of a Trefoil Torus

Granny Knot

Square Knot

Trefoil Knot
You Can Cover a Cover = Lift a Lift

Covers of Covers of Covers of Covers

Rossler

Lorenz

Ghrist
EveryKnot Lives Here
Isomorphisms and Diffeomorphisms

Local Stuff

Groups:
Local Isomorphisms
Cartan’s Theorem

Dynamical Systems:
Local Diffeomorphisms
?? Anytthing Useful ???
Cartan’s Theorem for Lie Groups

Simply connected Lie group $\overline{G}$

Multiply connected Lie groups

Linearization "LOG" (unique)

$\overline{G}/D_1$

$\overline{G}/D_2$

$\overline{G}/D_r$

EXP (unique)

Lie algebra $\mathfrak{g}$

Universal Covering Group
Locally Diffeomorphic Covers of $D$

$D_1, T_1/G_1$  $D_1, T_2/G_1$  $\cdots$  $D_2, T_1/G_2$  $D_2, T_2/G_2$  $\cdots$  $\cdots$

$D$: Universal Image Dynamical System
Useful Analogs

Local Isomorphisms & Diffeomorphisms
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups

Local Isomorphisms
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups

Local Isomorphisms
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms

Local Diffeos

Multiply connected Lie groups

Linearization "LOG" (unique)

Simply connected Lie group $\hat{G}$

$G/D_{1}$

$G/D_{2}$

$G/D_{r}$

Lie algebra $\mathfrak{g}$

Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms

Local Diffeos

Multiply connected Lie groups

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$G/D_{r}$

Lie algebra $\mathfrak{g}$
Useful Analogs

Local Isomorphisms & Diffeomorphisms

Lie Groups

Local Isomorphisms

Dynamical Systems

Local Diffeos

\[
\begin{aligned}
D_{G_1} & \quad D_{G_2} & \quad D_{G_3} & \quad D_{G_4} \\
\end{aligned}
\]
Creating New Attractors

Rotating the Attractor

\[ \frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X, Y) \\ F_2(X, Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix} \]

\[ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \]

\[ \frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = RF(R^{-1}u) + Rt + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix} \]

\[ \Omega = n \omega_d \quad q \Omega = p \omega_d \]

Global Diffeomorphisms

Local Diffeomorphisms

(p-fold covers)
Another Visualization

Cutting Open a Torus

\[ \phi = \omega t \]

\[ x \]

\[ \dot{x} \]

2\pi
Satisfying Boundary Conditions

Global Torsion

(a)
Two Phase Spaces: $R^3$ and $D^2 \times S^1$

**Rossler Attractor: Two Representations**

$R^3$

$D^2 \times S^1$

![Rossler Attractor, Toroidal Representation](Image)
Rossler Attractor:

Two More Representations with $n = \pm 1$
Subharmonic, Locally Diffeomorphic Attracers

Rossler Attractor:

Two Two-Fold Covers with $p/q = \pm 1/2$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (2,-1)$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (2,+1)$
Rossler Attractor: Two Three-Fold Covers with $p/q = -2/3, -1/3$
Subharmonic, Locally Diffeomorphic Attractors

Rossler Attractor:

And Even More Covers (with $p/q = +1/3, +2/3$)

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (3,+1)$

Rossler Attractor, Toroidal Representation
Index $(n_1,n_2) = (3,+2)$
New Measures

Angular Momentum and Energy

\[ L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau X dY - Y dX \]

\[ K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) dt \]

\[ L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle \]

\[ = L(0) + \Omega \langle R^2 \rangle \]

\[ K(\Omega) = \langle \frac{1}{2} (\dot{u}^2 + \dot{v}^2) \rangle \]

\[ = K(0) + \Omega L(0) + \frac{1}{2} \Omega^2 \langle R^2 \rangle \]

\[ \langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (u^2 + v^2) dt \]
New Measures, Diffeomorphic Attractors

Energy and Angular Momentum

Diffeomorphic, Quantum Number $n$

---

Torsion Integral

![Graph of Torsion Integral vs Rotation Index $k$]

Energy Integral

![Graph of Energy Integral vs Rotation Index $k$]
New Measures, Subharmonic Covering Attractors

Energy and Angular Momentum
Subharmonics, Quantum Numbers p/q

Torsion Integral

Energy Integral
Representations

An embedding creates a diffeomorphism between an (‘invisible’) dynamics in someone’s laboratory and a (‘visible’) attractor in somebody’s computer.

Embeddings provide a representation of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension
Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.
Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

- Parity $P$
- Global Torsion $N$
- Knot Type $KT$

$$\Gamma^{P,N,KT}(SA)$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.
Global Torsion & Parity

(a)

(b) n=2

(c) Parity=−1
Inequivalence in $R^3$
Creating Isotopies

Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?
Parity reversal is also possible in $R^4$ by isotopy.
Isotopies

2 Twists = 1 Writhe = Identity

Global Torsion → Binary Op
Creating Isotopies

Equivalences by Injection

Obstructions to Isotopy

\[ R^3 \rightarrow R^4 \rightarrow R^5 \]

Global Torsion
Parity
Knot Type

Global Torsion

There is one *Universal* reducible representation in \( R^N, N \geq 5 \). In \( R^N \) the only topological invariant is *mechanism*. 
What We Did
What We Found
Rössler Attractor
Basis Set of Orbits

Rössler Attractor - Return Map
Basis Set of Orbits

Lorenz Attractor
Basis Set of Orbits

Return Map for Lorenz Attractor

\[ \begin{align*}
\mathcal{L} & \rightarrow \mathcal{R} \\
\mathcal{R} & \rightarrow \mathcal{L} \\
\rho_{n+1} & \rightarrow \rho_n \\
\end{align*} \]
Basis Set of Orbits

Image of Lorenz Return Map

To be supplied
Stability Regions

Logistic Map: Global Stability
\[ x' = ax^2 \]

Knife Map: Global Stability
\[ y' = b \cdot \sqrt{y} \]
Logistic Map: Global Stability

\( x' = a - x^2 \)
BigView: Knife Map

Knife Map: Global Stability

\[ y' = b - \sqrt{y} \]
Stability Regions

Logistic Map: Global Stability
\[ x^{'} = a \cdot x^2 \]

Knife Map: Global Stability
\[ y^{'} = b \cdot \sqrt{1+y^2} \]
Return Map - Rössler Attractor

Pruning Mechanism

Logistic Map
Return Map - Lorenz Image
Return Map Approximations

The Rossler return map is well approximated by the following maps:

\[
\begin{align*}
x' &= \lambda x (1 - x) \\
x' &= a - x^2 \\
x' &= 1 - \mu x^2 \\
x' &= 1 - \left| \frac{x - m}{w} \right|^2
\end{align*}
\]
Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

\[ y' = b - |y|^{1/2} \]
\[ y' = 1 - \mu|y|^{1/2} \]
\[ y' = 1 - \left| \frac{y-m}{w} \right|^{1/2} \]
Comparison:

**Logistic Map**

\[ x' = f(x; a) = a - (|x|)^2 \]

**Knife Map**

\[ y' = f(y; b) = b - (|y|)^{1/2} \]
... for several values of $a$

Logistic Return Map

$x' = a - x^2$
Knife Return Maps

Knife Return Map

Image Lorenz-04
Second Return Map

Logistic Map, Second Iterate

\[ f(x; a) = a - x^2 \]
Period 1 & 2 Orbits - Logistic

Fixed Points of $f(x) & f^2(x)$, $f(x) = a - x^2$

Period-one: Red & Green    Period-two: Black
.. Blow Up .... with Caustics
Knife Map - Bifurcation Diagram

No windows! No caustics!
Fixed Points (Knife)

Fixed Points for the Knife Map

\[ y' = b - \sqrt{|y|} \]
Second Iterates - Knife Map

Knife Map, Second Iterate

\[ f(y; b) = b - \sqrt{y} \]
Period-One & Period-Two Orbits

Fixed points of $f(y)$ & $f^2(y)$  \( f(y)=b-\sqrt{y/y} \)

Period-one: Red & Green  Period-two: Black, Cyan & Magenta
Attractor boundary (Knife)

Boundaries for the Basin of Attraction

\[ y' = b - \sqrt{|y|} \]
Attractor Boundaries - Logistic

\[ x' = a - (lx)^2 \]
Rite of Passage-01

Forcing Diagram - Horseshoe

Third Iterate of Knife Map

$b = -0.1, 0.25, 1.1$
Forcing Diagram - Horseshoe

Knife Return Map
First Four Iterates at $b = 0.1$
Table: Values $M^{(p)}$ of $y$ where the $p$th iterate $f^{(p)}(y; b)$ has maxima. These locations are determined by a simple recursion relation (last line) where the indices $s_p = \pm 1$ are incoherent.

<table>
<thead>
<tr>
<th>$p$</th>
<th>Number Max.</th>
<th>Coordinate Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$\pm b^2$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$\pm (b \pm b^2)^2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$p+1$</td>
<td>$2^p$</td>
<td>$M^{(p+1)} = s_p(b + M^{(p)})^2$</td>
</tr>
</tbody>
</table>
As $p \to \infty$, with all $s_j = +1$, the abscissa of the rightmost point goes to a limit. The quadratic equation for this limit gives:

$$y(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b}\right)$$

At $b = \frac{1}{4}$ the bounding box is a square — beyond that the diagonal fails to intersect all the zig-zags. Orbits begin to get pruned away in singular saddle node bifurcations.
Structural Stability: \(0 < b < \frac{1}{4}\)

Knife Map, fifth iterate at \(b=0.15\)
End Play - Near $b = 1$

Iterates of the Knife Map

$p = 8 \quad b = 0.8$
Iterates Near $b = 1$

Iterates of Second Return, Knife Map

$b=1\text{-}\epsilon$, $\epsilon = 0.1$
Note Scaling Relations

Knife Return Map

First Four Iterates at $b = 0.1$
Structural Stability: $\frac{3}{4} < b < 1$

Knife Map: 10th iterate near $y=0$
Hunt for Saddle-Node Bifurcations
Caustic Crossings

Search for Superstable Orbits
Logistic Map

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Overview-06
Overview-07
Experimental-01
Hunt for Singular SNBs
Anti Caustic Crossings

AntiCaustics of the Knife Map

\( f^p(0; b), \quad p = 1 \ldots 8 \)
Anti Caustic Crossings: Expansion
Period Three Singular SNB

Knife Map, Third Iterate

Creation of Period-3 Orbits
Renormalization-02

Local expression near $y = 0$ for the period-three explosion:

$$h(y; b) = f^{(3)}(y; b) = b - \sqrt{|b - \sqrt{|b - \sqrt{|y|}}|}$$

$$h(b_3 + \epsilon; y) \rightarrow \left( b_3 - \sqrt{\sqrt{b_3 - b_3}} \right) +$$

$$\left( 1 + \frac{2\sqrt{b_3} - 1}{4\sqrt{\sqrt{b_3 - b_3} - b_3\sqrt{b_3}}} \right) \epsilon + \left( \frac{1}{4\sqrt{\sqrt{b_3 - b_3} - b_3\sqrt{b_3}}} \right) \sqrt{|y|}$$
Renormalization for the period-three explosion.

\[ y' = h(y; b_3 + \epsilon) \rightarrow \Delta(b - b_3) + \alpha \sqrt{|y|} = 1.286974759(b - b_3) + 0.7869747590 \sqrt{|y|} \]

\[ z' = (\Delta/\alpha^2)(b_3 - b) - \sqrt{|z|} \]
Renormalization Algorithm: K10*

1. Write down the symbol sequence for the primary period-\( p \) orbit: \( K10^* = K\sigma_1\sigma_2\cdots\sigma_{p-1} \).

2. Make the identification
\[
\sigma = +1 \rightarrow s = +1, \quad \sigma = 0 \rightarrow s = -1.
\]

3. Construct \( f^{(p)}(b; y) \) →
\[
b - \sqrt{s_{p-1}(b - \cdots \sqrt{s_2(b - \sqrt{s_1(b - \sqrt{y})})}) \cdots}
\]

4. Taylor expand this function to terms linear in \( b \) and \( \sqrt{y} \) and determine the value of \( b \) for which the constant term vanishes.
Equations: K10*

For the saddle node pair $5_2 = K1001$ this algorithm gives

$$b - \sqrt{(+1)(b - \sqrt{(-1)(b - \sqrt{(-1)(b - \sqrt{(+1)(b - \sqrt{y})})})})}$$

The constant term vanishes for $b = 0.418656$, and for this value of $b$

$$y' = \Delta(b - b_{5_2}) + \alpha \sqrt{|y|} = -3.231180\Delta b - 1.983690\sqrt{|y|}$$
Results: K10* to Period 6

\[ y' = \Delta(b - b_c) + \alpha \sqrt{|y|} \quad y', y \approx 0 \]

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Symbolics</th>
<th>( b_c )</th>
<th>( \Delta )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3_1</td>
<td>( K10 )</td>
<td>0.465571</td>
<td>1.286974</td>
<td>0.786974</td>
</tr>
<tr>
<td>4_2</td>
<td>( K100 )</td>
<td>0.360157</td>
<td>2.624703</td>
<td>1.180563</td>
</tr>
<tr>
<td>5_3</td>
<td>( K1000 )</td>
<td>0.318897</td>
<td>4.647225</td>
<td>1.664335</td>
</tr>
<tr>
<td>5_2</td>
<td>( K1001 )</td>
<td>0.418656</td>
<td>-3.231180</td>
<td>-1.983690</td>
</tr>
<tr>
<td>5_1</td>
<td>( K1011 )</td>
<td>0.513175</td>
<td>2.628970</td>
<td>1.509712</td>
</tr>
<tr>
<td>6_5</td>
<td>( K10000 )</td>
<td>0.297846</td>
<td>7.481728</td>
<td>2.233184</td>
</tr>
<tr>
<td>6_4</td>
<td>( K10001 )</td>
<td>0.340328</td>
<td>-8.535145</td>
<td>-3.639587</td>
</tr>
<tr>
<td>6_3</td>
<td>( K10011 )</td>
<td>0.380540</td>
<td>7.596535</td>
<td>3.574548</td>
</tr>
</tbody>
</table>
Renormalization for the final period-two explosion.

\[ f^{(2)}(1 - \epsilon, y) \simeq -\frac{\epsilon}{2} + \left(\frac{1}{2} + \frac{\epsilon}{4}\right) \sqrt{|y|} \]  \hspace{1cm} (1)
Orbit Search-05

Hunt for Saddle-Node Bifurcations

Search for Superstable Orbits
Logistic Map
Hunt for S. Saddle-Node Bifurcations

AntiCaustics of the Knife Map

\[ f^p(p; b), \quad p = 1 \ldots 8 \]
**Table:** Important parameter values for global stability and unstable periodic orbit behavior.

<table>
<thead>
<tr>
<th>Global Stability</th>
<th>Unstable Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>3/4</td>
<td>0.5957439420</td>
</tr>
<tr>
<td>3/4</td>
<td>0.7825988587</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>
Table 2.1  Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)\(^a\)

<table>
<thead>
<tr>
<th>Name</th>
<th>Bifurcation</th>
<th>Name</th>
<th>Bifurcation</th>
<th>Name</th>
<th>Bifurcation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0[1]</td>
<td>1[1][s_1]</td>
<td>00101</td>
<td>0[1]</td>
<td>7[1][s_1]</td>
<td>0001</td>
</tr>
<tr>
<td>0[1]</td>
<td>2[1][s_1 \times 2^1]</td>
<td>001010</td>
<td>0[1]</td>
<td>8[1][s_1]</td>
<td>000111</td>
</tr>
<tr>
<td>0111</td>
<td>4[1][s_1 \times 2^2]</td>
<td>0011</td>
<td>0[1]</td>
<td>5[1][s_1]</td>
<td>000111</td>
</tr>
<tr>
<td>01010111</td>
<td>8[1][s_1 \times 2^3]</td>
<td>001101</td>
<td>0[1]</td>
<td>6[1][s_1]</td>
<td>0001</td>
</tr>
<tr>
<td>011101</td>
<td>6[1][s_1]</td>
<td>001110</td>
<td>0[1]</td>
<td>7[1][s_1]</td>
<td>00010011</td>
</tr>
<tr>
<td>011111</td>
<td>8[1][s_1]</td>
<td>001101</td>
<td>0[1]</td>
<td>5[1][s_1]</td>
<td>000111</td>
</tr>
<tr>
<td>0111101</td>
<td>7[1][s_1]</td>
<td>0011</td>
<td>0[1]</td>
<td>6[1][s_1]</td>
<td>000111</td>
</tr>
<tr>
<td>01101</td>
<td>7[1][s_1]</td>
<td>001101</td>
<td>0[1]</td>
<td>6[1][s_1]</td>
<td>0001101</td>
</tr>
<tr>
<td>0110111</td>
<td>8[1][s_1]</td>
<td>001101</td>
<td>0[1]</td>
<td>6[1][s_1]</td>
<td>000101</td>
</tr>
<tr>
<td>00101</td>
<td>6[1][s_1 \times 2^1]</td>
<td>001001</td>
<td>0[1]</td>
<td>7[1][s_1]</td>
<td>00010011</td>
</tr>
<tr>
<td>0010111</td>
<td>8[1][s_1]</td>
<td>000101</td>
<td>0[1]</td>
<td>8[1][s_1]</td>
<td>0000001</td>
</tr>
</tbody>
</table>

\(^a\)The notation \(P_i\) refers to the \(i\)th bifurcation of period \(P\). We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the \(i\)th saddle-node bifurcation of period \(P\) is denoted \(s^i_P\), and \(s^i_P \times 2^k\) is the orbit of period \(P \times 2^k\) belonging to the period-doubling cascade originating from \(s^i_P\).
Symbol Exchange Near Endplay

Anticaustics for the Knife Map

\[ p = 14, 16, 18, 20, 22 \]
Symbol Exchange Near Endplay

- Symbols 0, 1 created at $b = 0$
- New orbit, (11), created at $b = \frac{3}{4}$
- Symbol pair - 11 -, replaced by - (11) - as $b \to 1$
- Implosions begin at $b = 0.5957...$, end at midpoint.
- Explosions begin at midpoint, end at $b = 0.7825..$
- Implosions and explosions symmetrically matched
Forcing Diagram - Horseshoe

Return Maps for Chaotic Attractors

$k=2, a=1.55$ and $k=1/2, a = 0.65$
Rossler 03

Return Map Approximations

The Rossler return map is well approximated by the following maps:

\[ x' = \lambda x(1 - x) \]
\[ x' = a - x^2 \]
\[ x' = 1 - \mu x^2 \]
\[ x' = 1 - \left| \frac{x - m}{w} \right|^2 \]
The image of the Lorenz return map is well approximated by the following maps:

\[ y' = b - |y|^{1/2} \]

\[ y' = 1 - \mu |y|^{1/2} \]

\[ y' = 1 - \left| \frac{y - m}{w} \right|^{1/2} \]
Basis Set of Orbits

Class of Lopsided Maps

\[ x' = f(x; k, a) = 1 - \left| \frac{x - m}{w} \right|^k \]

1. Zero crossings at \( x = +1 \) and \( x = a, -1 \leq a \leq 0 \)
2. Maximum at \( m = \frac{1+a}{2} \)
3. Half-width \( w = \frac{1-a}{2} \)
4. \( m + w = 1 \)
Forcing Diagram - Horseshoe

Return Maps for Chaotic Attractors

$k=2, a=1.55$ and $k=1/2, a = 0.65$
Basis Set of Orbits

Forcing Diagram - Horseshoe

Return Maps for Strange Attractors

\[ k=2, b=-0.290322581, \quad k=1/2, b=-0.612451550 \]
Basis Set of Orbits

Map Comparisons

Modified Logistic Return Map

Lorenz-image Return Map

$\alpha = 0.6$
Forcing Diagram - Horseshoe

Transformation Between Control Parameter Values

\[ y' = a - by^{1/2} \quad y' = 1 - (x - m)/\omega y^{1/2} \]
Basis Set of Orbits

Forcing Diagram - Horseshoe

Superstable Orbits for Logistic Map

\[
f^{\ell}(m) - m
\]

\[
-1 \quad -0.9 \quad -0.8 \quad -0.7 \quad -0.6 \quad -0.5 \quad -0.4 \quad -0.3 \quad -0.2 \quad -0.1 \quad 0
\]

Alice in Stretch & SqueezeLand
The Marvels of Topology and Chaos

Robert Gilmore

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Forcing Diagram - Horseshoe

Homoclinic Orbits, Lorenz-Image Map

Basis Set of Orbits
Basis Set of Orbits

Forcing Diagram - Horseshoe

Modified Logistic Map
Third Iterate, $a = -0.15$
Basis Set of Orbits

Forcing Diagram - Horseshoe

Modified Logistic Map
Third Iterate, \( a = -0.143 \)

![Logistic Map Diagram](image)
Forcing Diagram - Horseshoe

Modified Logistic Map
Third Iterate, \( a = -0.11 \)
Basis Set of Orbits

Forcing Diagram - Horseshoe

Lorenz-Image Map
Third Iterate, a = -0.365

The Marvels of Topology and Chaos
Alice in Stretch & SqueezeLand
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Basis Set of Orbits

Forcing Diagram - Horseshoe

Lorenz-Image Map
Third iterate, $a = 0.37$

![Lorenz-Image Map](Image 88x13 to 320x192)
Basis Set of Orbits

Forcing Diagram - Horseshoe

Lorenz-Image Map
Third Iterate, \( a = -0.35 \)
Comparison: Logistic and Knife

Scaling

- Logistic: SNB Period 3 = scaled version SNB of M.
- Renormalization theory applies.
- U Sequence

- Knife: S-SNB Period 3 = scaled version S-SNB of K.
- Renormalization theory applies.
- $U^{-1}$ Sequence
The Road Ahead

Summary

1 Question Answered ⇒ 2 Questions Raised

We must be on the right track!
Our Hope

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
Our Result

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to $R^3$ only — for now
Four Levels of Structure

The Classification Theory has

4 Levels of Structure
Four Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
Four Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings
Four Levels of Structure

- Alice in Stretch & SqueezeLand
- The Marvels of Topology and Chaos

Robert Gilmore

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- Overview-02
- Overview-03
- Overview-04
- Overview-05
- Overview-06
- Overview-07
- Experimental-01
Poetic Organization

LINKS OF PERIODIC ORBITS
organize
BOUNDING TORI
organize
BRANCHED MANIFOLDS
organize
LINKS OF PERIODIC ORBITS
There is a Representation Theory for Strange Attractors

There is a complete set of representation labels for strange attractors of any genus $g$.

The labels are complete and discrete.

Representations can become equivalent when immersed in higher dimension.

All representations (embeddings) of a 3-dimensional strange attractor become isotopic (equivalent) in $R^5$.

The *Universal Representation* of an attractor in $R^5$ identifies mechanism. No embedding artifacts are left.

The topological index in $R^5$ that identifies mechanism remains to be discovered.
Answered Questions

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g - 1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan’s Theorem for Lie Groups
- Topological Continuation – Group Continuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
Unanswered Questions

We hope to find:

- Robust topological invariants for $R^N$, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of $\chi^2$ test for NLD
- Better forcing results: Smale horseshoe, $D^2 \to D^2$, $n \times D^2 \to n \times D^2$ (e.g., Lorenz), $D^N \to D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy
Thanks

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Alice in Stretch & SqueezeLand
The Marvels of Topology and Chaos
Robert Gilmore
Introduction-01
Introduction-02
Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01
Folding - Squeezing - Global torsion

Basic Stretch - Fold - Roll Template

Javier Used and Juan Carlos Martin, Multiple topological structures of chaotic attractors ruling the emission of a driven laser, Phys. Rev. E82, 016218 (2010).
The “S” Folding Mechanism

FIG. 2. (Color online) Scheme of a template with three branches and an S folding process.
TABLE I. (Color online) Folding processes characteristic of the different species of templates treated in this work.

<table>
<thead>
<tr>
<th>Species</th>
<th>Horseshoe</th>
<th>Reverse horseshoe</th>
<th>Out-to-in spiral</th>
<th>In-to-out spinal</th>
<th>Staple</th>
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Spectrum of Behaviors in Resonance Regions

Modulation frequency normalized to the natural frequency

m=0.93
m=0.78
m=0.73
Constraints:

No

Yes
Poincaré Sections & Periodic Orbits

\[ X(t+\tau) \text{ (arb. units)} \]

\[ X(t) \text{ (arb. units)} \]
“Generating” Partition

Symbol Sets - Periodic Orbits
20 Equally-Spaced Planes
## Table of Linking Numbers

### Linking Numbers for Certain Orbits

TABLE II. Linking numbers between the UPOs extracted from the time series corresponding to pump modulation frequency $f = 4.25$ KHz and modulation index $m = 0.73$.

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