Chaos: What Have We Learned?

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Abstract

Analysis of experimental data from many physical systems (lasers, chemical reactions, electrical circuits, vibrating strings) has lead to a deeper understanding of low-dimensional strange attractors and their perestroikas. They can now be classified. The classification is topological, with four levels of structure. Each is discrete. The signatures that identify these levels can be and have been extracted from experimental data. These advances have raised additional questions that require new mathematics for their resolution.

Outline

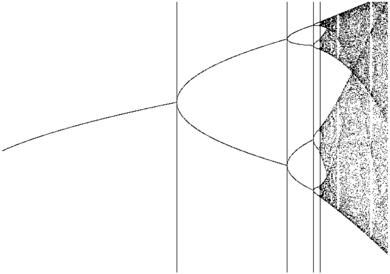
Background

J. R. Tredicce

Can you explain my data?

I dare you to explain my data!

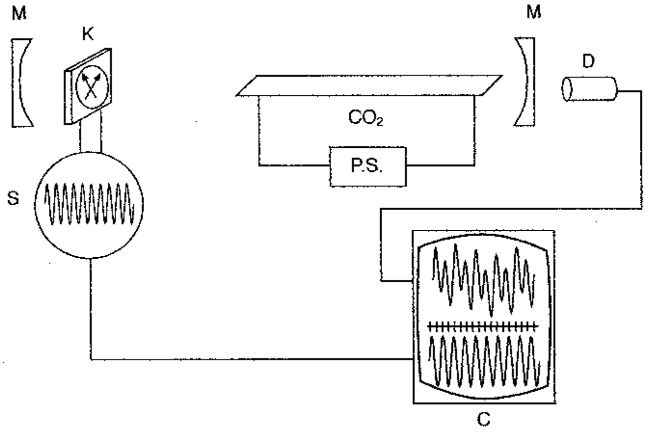
Where is Tredicce coming from?



Feigenbaum:

$$\alpha = \quad 4.66920\ 16091\ \ldots\ldots$$
$$\delta = -2.50290\ 78750\ \ldots\ldots$$

Laser with Modulated Losses Experimental Arrangement



Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The 4 Levels of Structure

- **Basis Sets of Orbits**
- **Branched Manifolds**
- **Bounding Tori**
- **Extrinsic Embeddings**

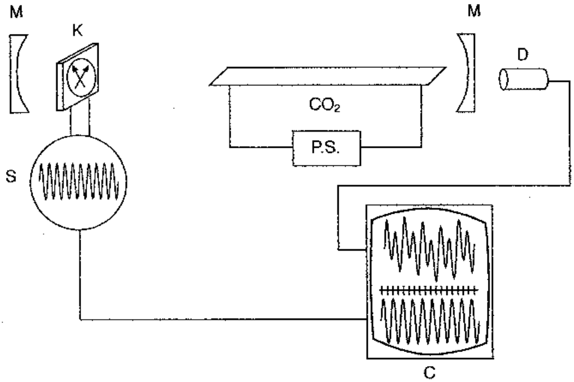
What Have We Learned?

1. Cover and Image Relations
2. Continuations: Analytical, Topological, Group
3. Cauchy Riemann & Clebsch-Gordonnery for Dynamical Systems
4. "Quantizing Chaos"
5. Representation Theory for Dynamical Systems

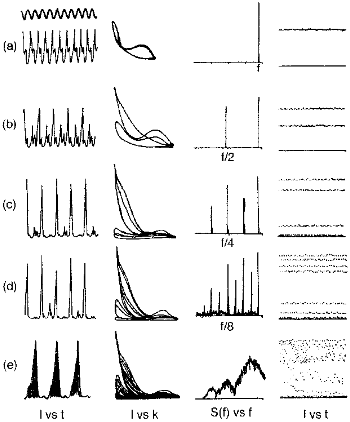
What Do We Need to Learn?

1. Higher Dimensions
2. Invariants
3. Mechanisms

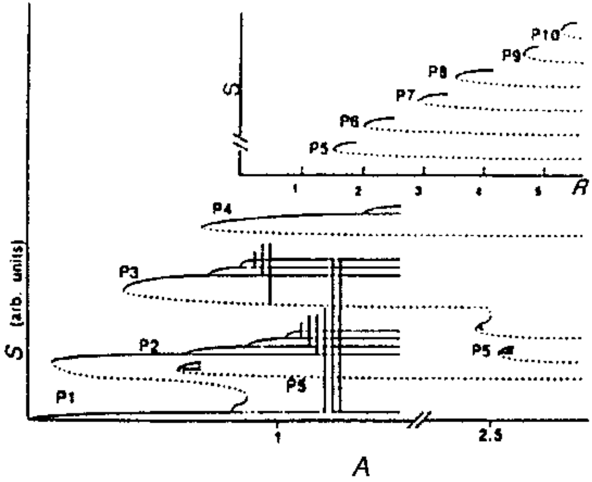
Laser Experimental Arrangement



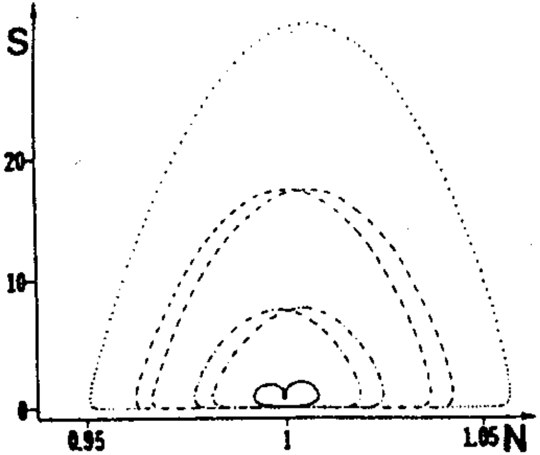
Oscilloscope Traces



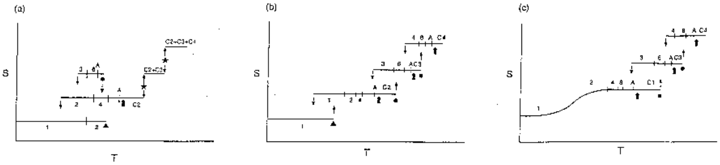
Bifurcation Schematics



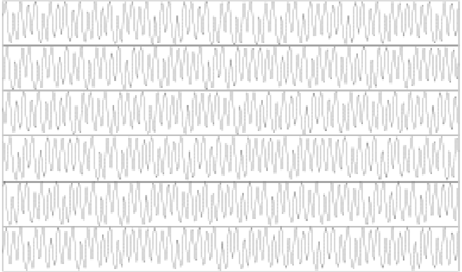
Coexisting Basins of Attraction

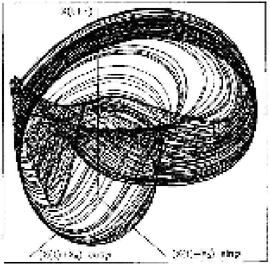


Bifurcation Perestroikas



Experimental Data: LSA





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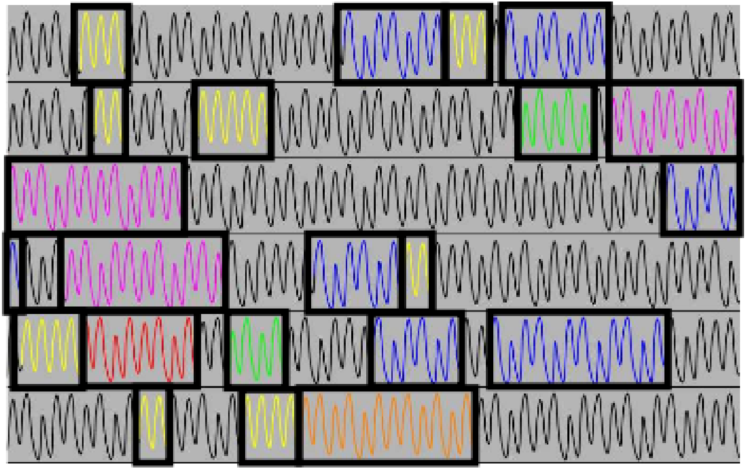
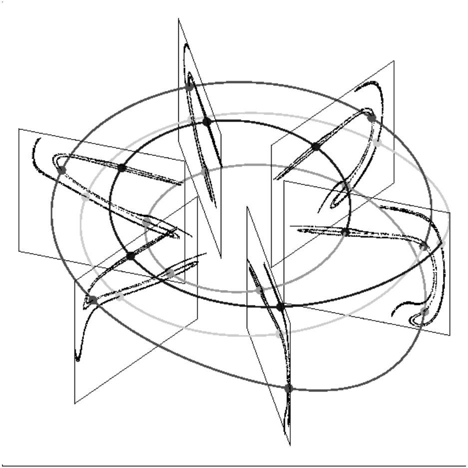
Real Data

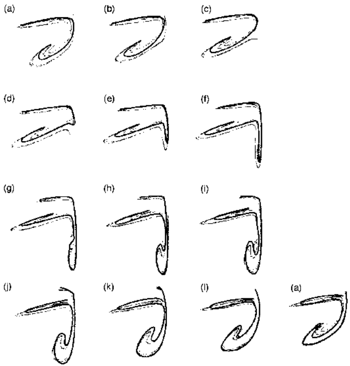
Experimental Data: LSA



Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Time Evolution

Rotating the Poincaré Section around the axis of the torus

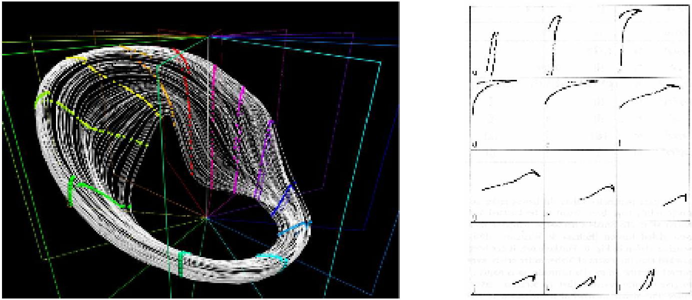


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

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Chaos

Motion that is

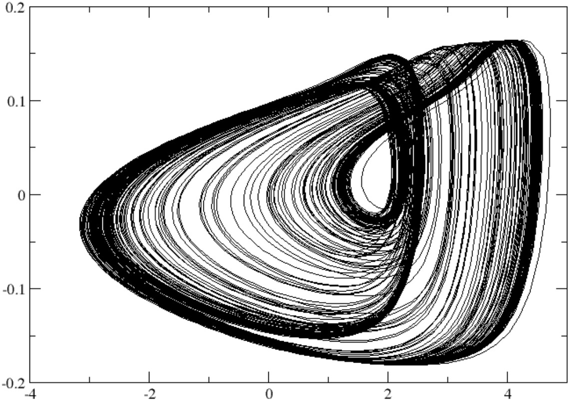
- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

Strange Attractor

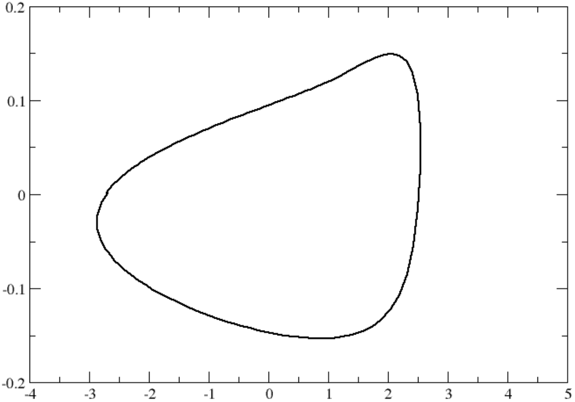
The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

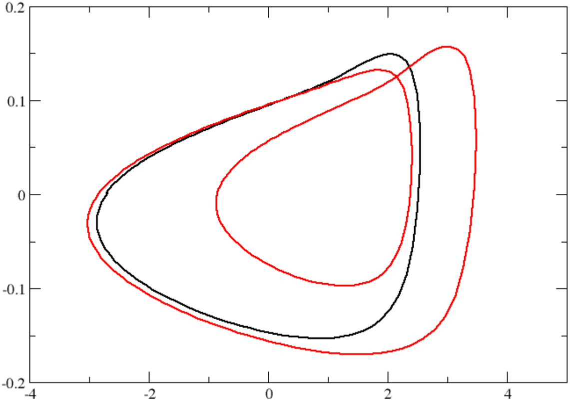
UPOs: Skeletons of Strange Attractors



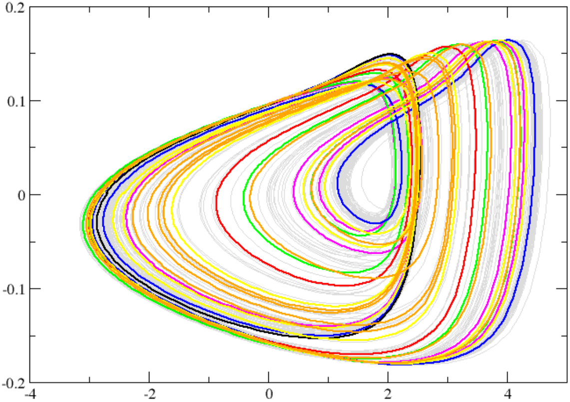
UPOs: Skeletons of Strange Attractors



UPOs: Skeletons of Strange Attractors



UPOs: Skeletons of Strange Attractors



UPOs Outline Strange Attractors

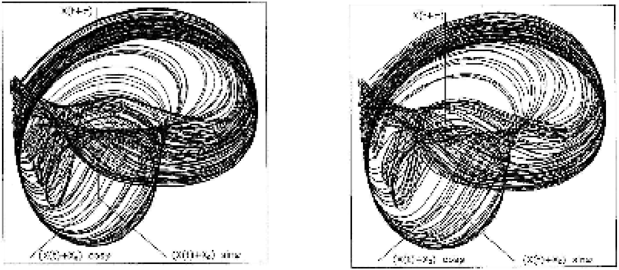


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

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Organization of UPOs in R^3: Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

Linking Number of Two UPOs

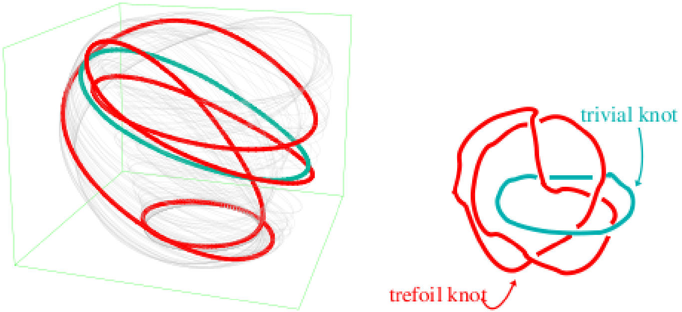

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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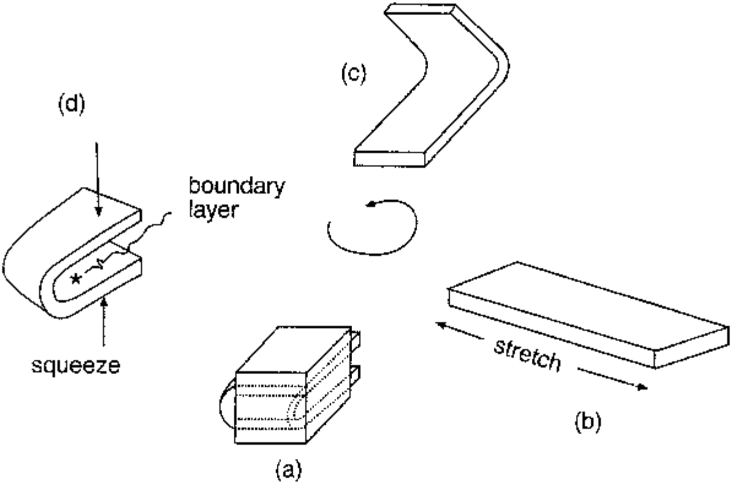
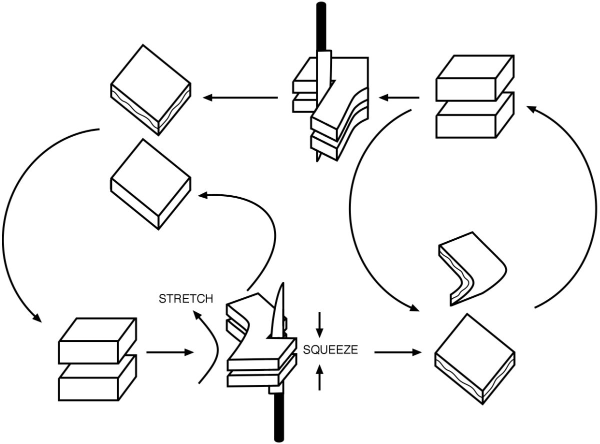
Stretching and Folding

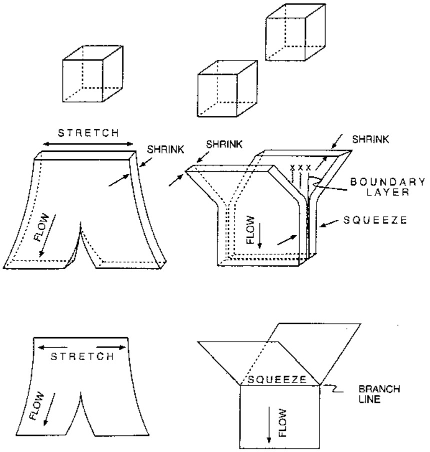


Tearing and Squeezing



Motion of Blobs in Phase Space

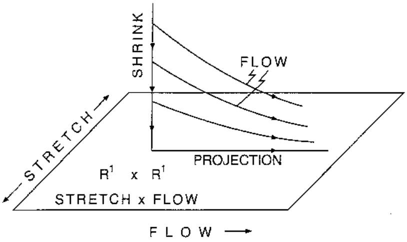
Stretching — Squeezing



Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then:

Fundamental Theorem

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.
- **Generates a** **<u>hyperbolic</u>** **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

• The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.

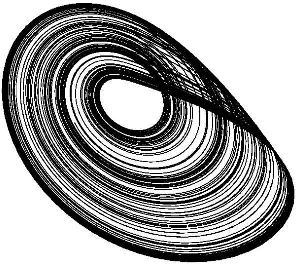
• UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

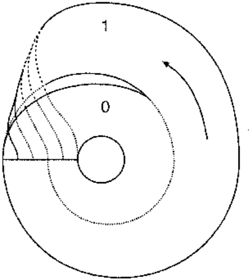
Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

Rössler:

Attractor **Branched Manifold**

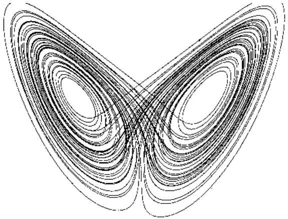


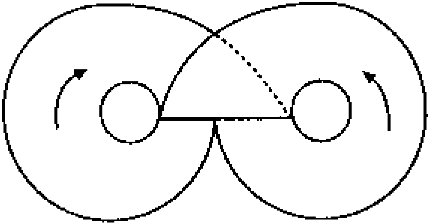


A Mechanism with Symmetry

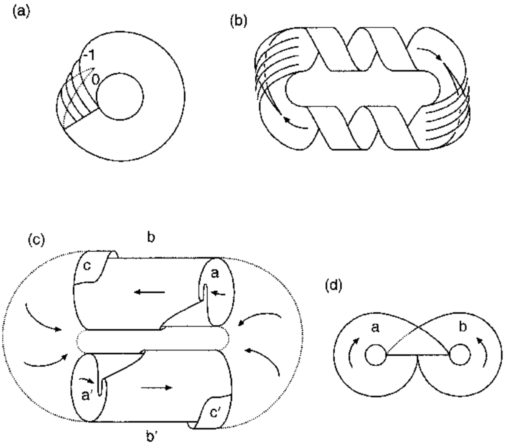
Lorenz:

Attractor **Branched Manifold**



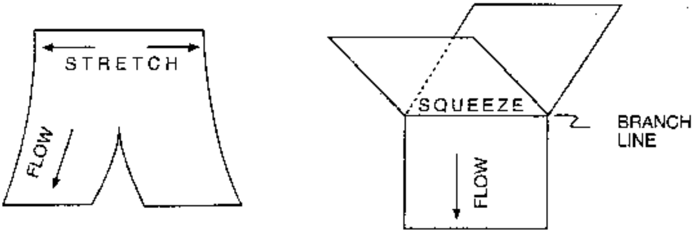


Inequivalent Branched Manifolds



Aufbau Princip for Branched Manifolds

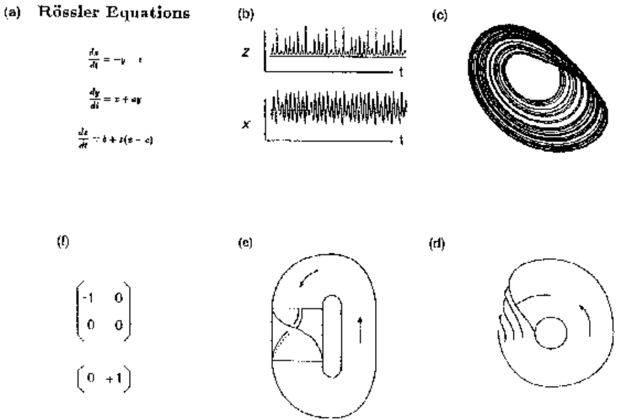
Any branched manifold can be built up from stretching and squeezing units



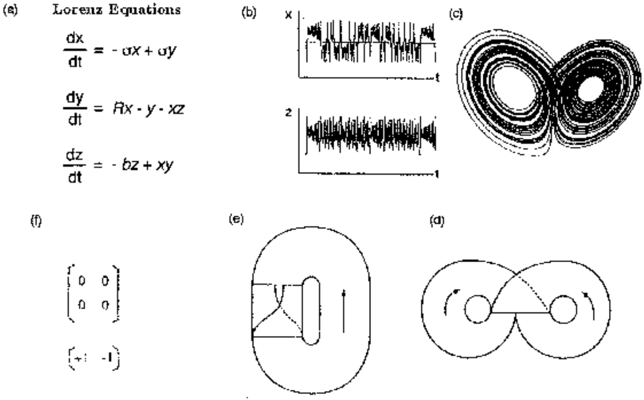
subject to the conditions:

- Outputs to Inputs
- No Free Ends

Rossler System



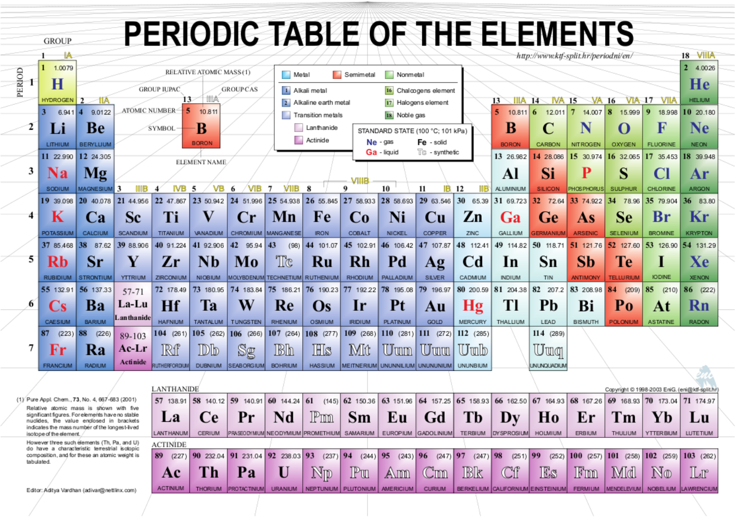
Lorenz System



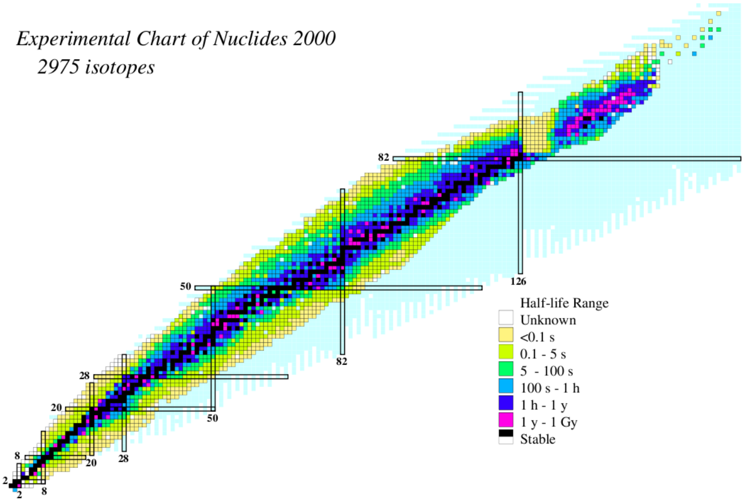
Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

We Like to be Organized



We Like to be Organized



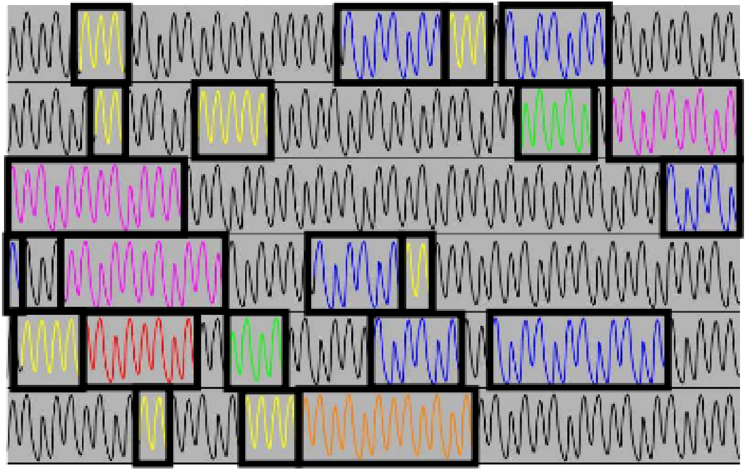
Topological Analysis Program

- Locate Periodic Orbits
- Create an Embedding
- Determine Topological Invariants (LN)
- Identify a Branched Manifold
- Verify the Branched Manifold

Additional Steps

- Model the Dynamics
- Validate the Model

Method of Close Returns



Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

Locate UPOs

An Embedding and Periodic Orbits

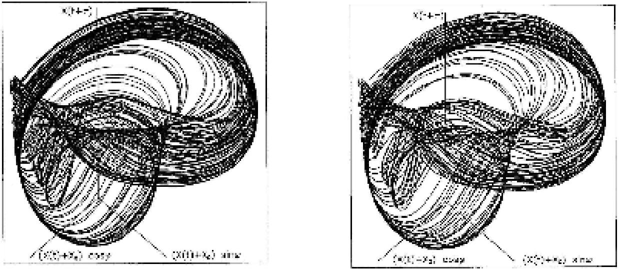


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Linking Number of Orbit Pairs

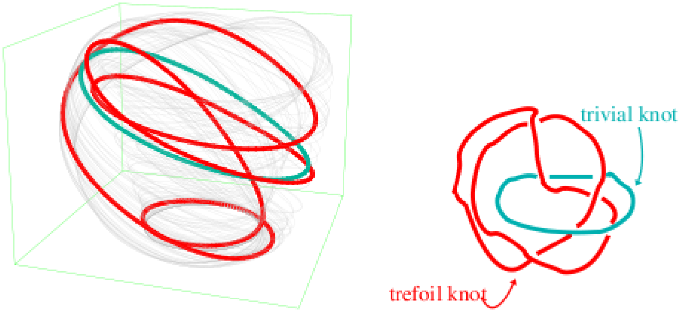


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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Compute Table of Experimental LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

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Introduction-01
Introduction-02
Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01
Experimental-02

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1s$	$1f$	2_1	$3f$	$3s$	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Guess Branched Manifold

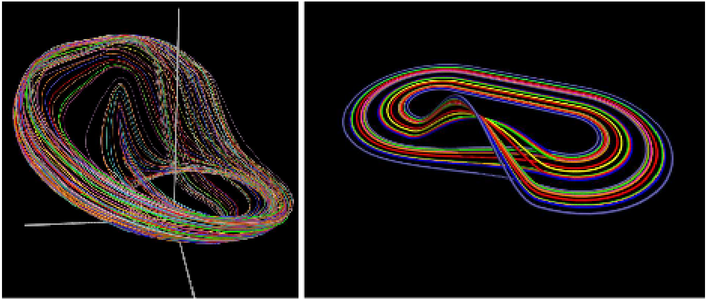

Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

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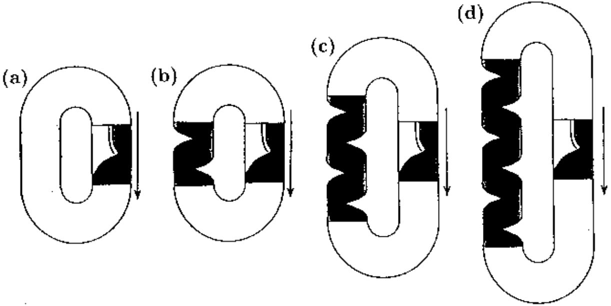
Identification & ‘Confirmation’

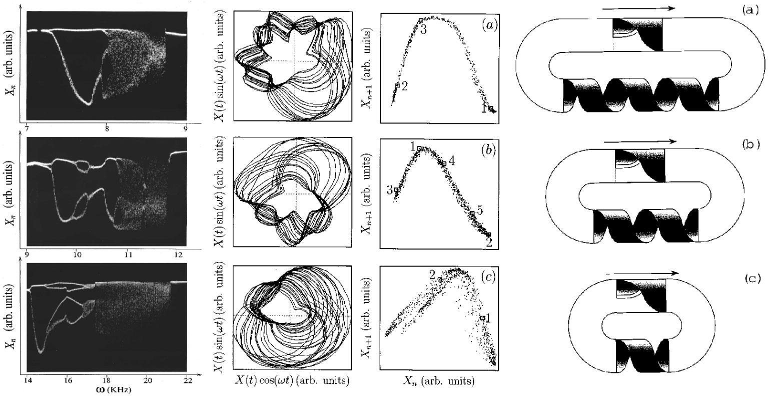
- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change



Evolution Under Parameter Change

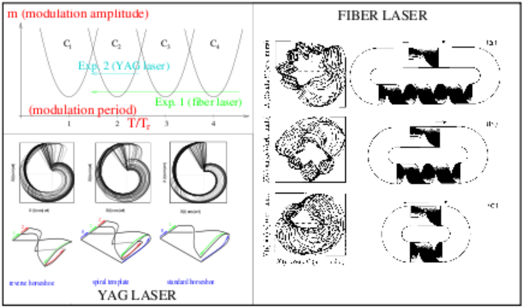


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Analysis of Nonstationary Data

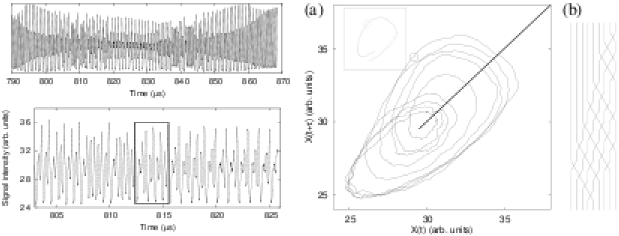


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

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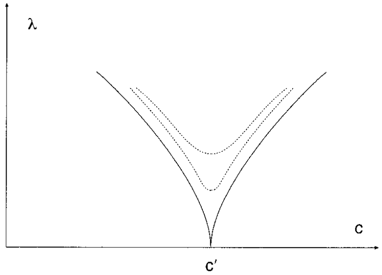
Model the Dynamics

A hodgepodge of methods exist: # Methods $\simeq$ # Physicists

Validate the Model

Needed: Nonlinear analog of χ^2 test. OPPORTUNITY: Tests that depend on entrainment/synchronization.



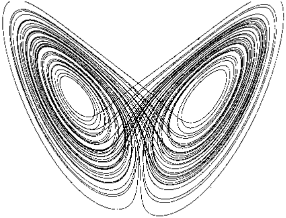
Compare with Original Objectives

Construct a simple, algorithmic procedure for:

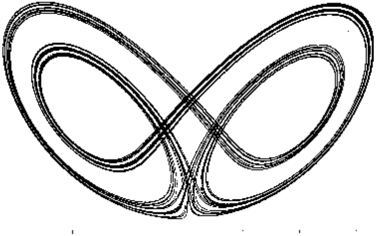
- Classifying strange attractors
- Extracting classification information

from experimental signals.

There Are Some Missing Orbits

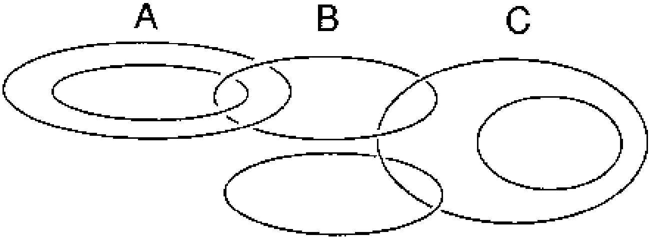


Lorenz



Shimizu-Morioka

Orbit Forcing

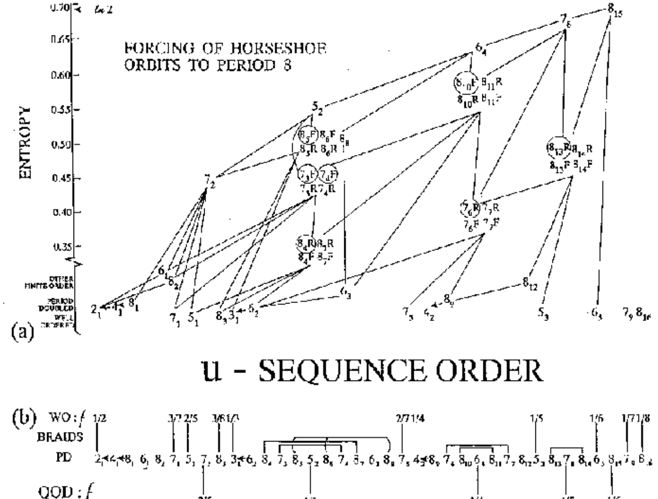


$A \Rightarrow B$ $B \Rightarrow C$

$A \Rightarrow C$

An Ongoing Problem

Forcing Diagram - Horseshoe



Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

Constraints on Branched Manifolds

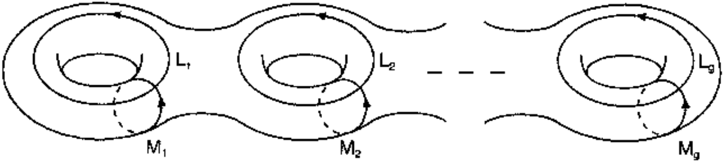
“Inflate” a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Surface Singularities

Flow field: three eigenvalues: +, 0, –

Vector field “perpendicular” to surface

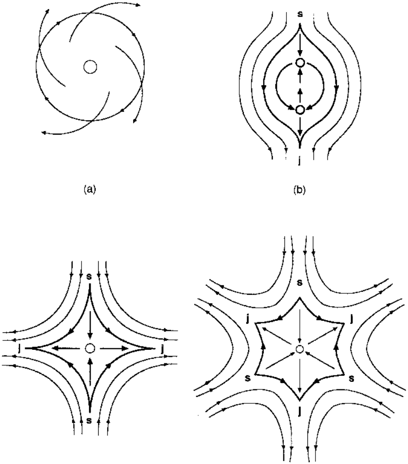
Eigenvalues on surface at fixed point: +, –

All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

 # fixed points on surface = index = 2g - 2

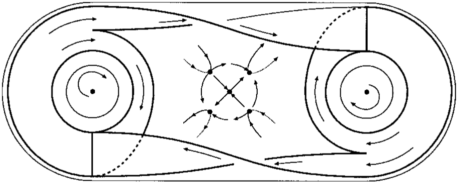
Flow Near a Singularity



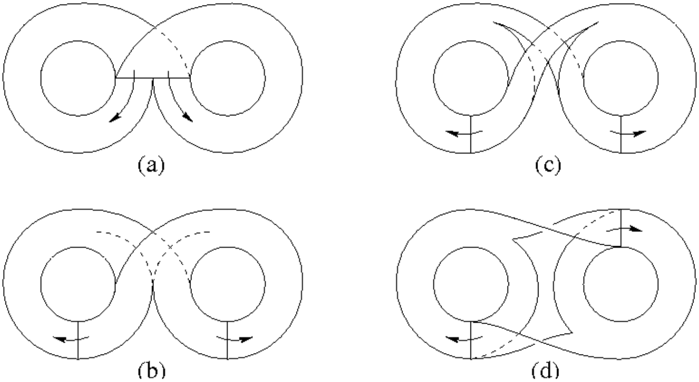
(a) (b)

(c) (d)

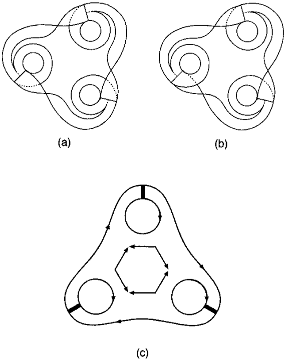
Torus Bounding Lorenz-like Flows



Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.



Bounding Tori contain all known Strange Attractors

Tab.1. All known strange attractors of dimension $d_L < 3$ are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g-1$ Orbit
Rossler, Duffing, Burke and Shaw	A_1	1
Various Lasers, Gateau Roule	A_1	1
Neuron with Subthreshold Oscillations	A_1	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	A_2	$(12)^2$
Multispiral attractors	A_n	$(12^{n-1})^2$
C_n Covers of Rossler	C_n	1^n
C_2 Cover of Lorenz$^{(a)}$	C_4	1^4
C_2 Cover of Lorenz$^{(b)}$	A_3	$(122)^2$
C_n Cover of Lorenz$^{(a)}$	C_{2n}	1^{2n}
C_n Cover of Lorenz$^{(b)}$	P_{n+1}	$(1n)^n$
$2 \rightarrow 1$ Image of Fig. 8 Branched Manifold	A_3	$(122)^2$
Fig. 8 Branched Manifold	P_5	$(14)^4$

$^{(a)}$ Rotation axis through origin.
$^{(b)}$ Rotation axis through one focus.

Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 $\times$ g-1 matrices

One is cyclic g-1 $\times$ g-1 matrix

Other represents union of cycles

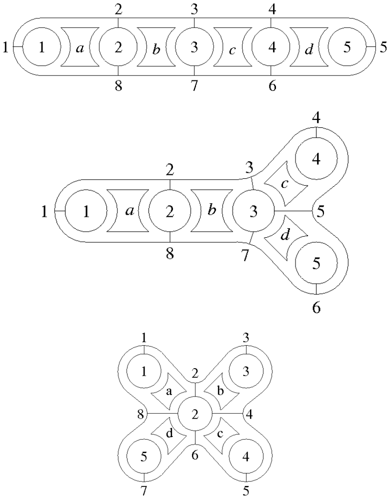
Labeling via (permutation) group theory

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

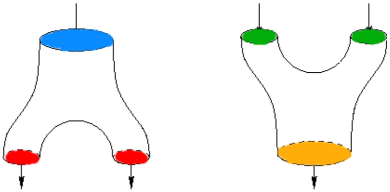
g	m	$(p_1, p_2, \ldots p_m)$	$n_1 n_2 \ldots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

Some Genus-9 Bounding Tori



Aufbau Princip for Bounding Tori

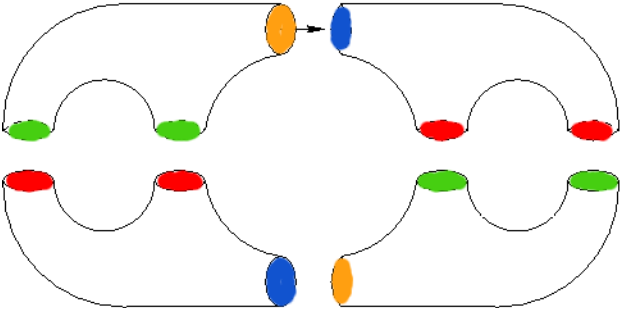
Any bounding torus can be built up from equal numbers of stretching and squeezing units



Rules

- Outputs to Inputs
- Colorless
- No Free Ends

Application: Lorenz Dynamics, g=3



Construction of Poincaré Section

P. S. = Union 

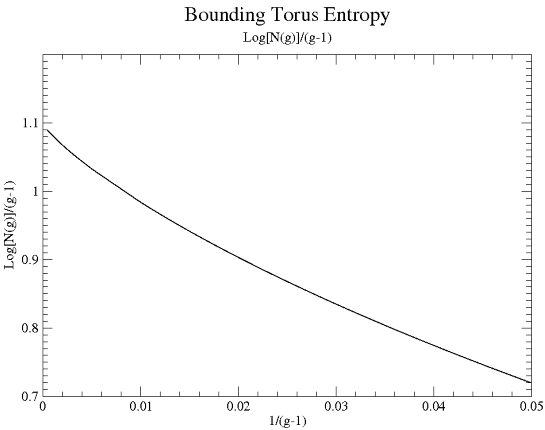
Components = g-1

The Growth is Exponential

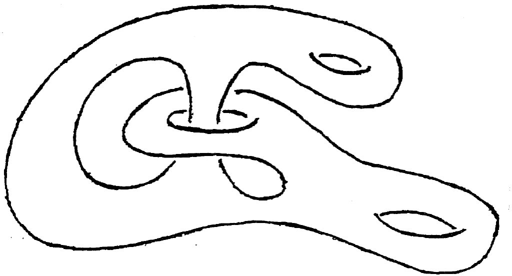
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

The Growth is Exponential
The Entropy is log 3



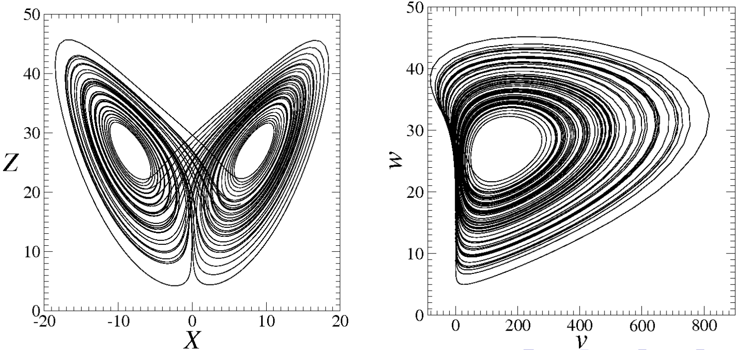
Extrinsic Embedding of Intrinsic Tori



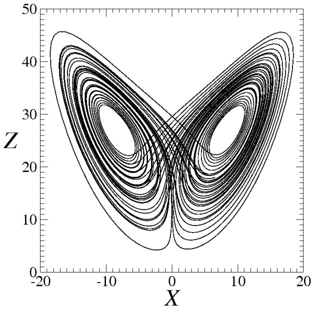
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

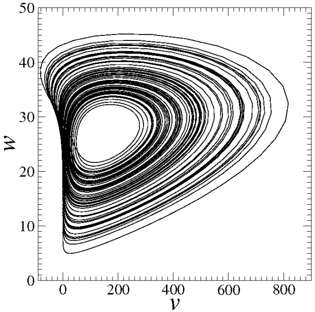
Modding Out a Rotation Symmetry

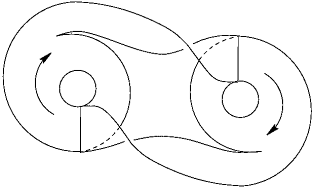
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$

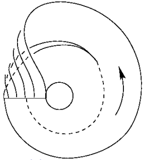


Lorenz Attractor and Its Image



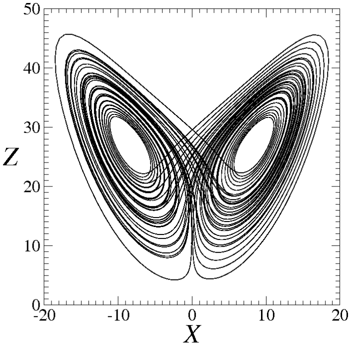


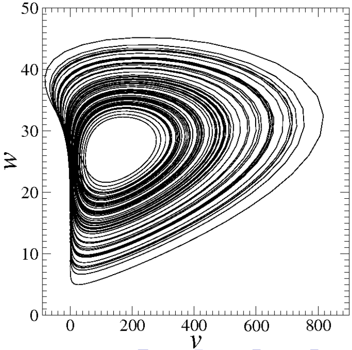




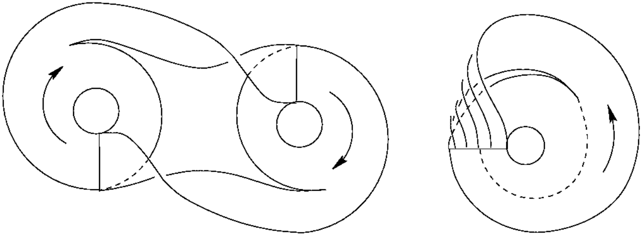
Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$

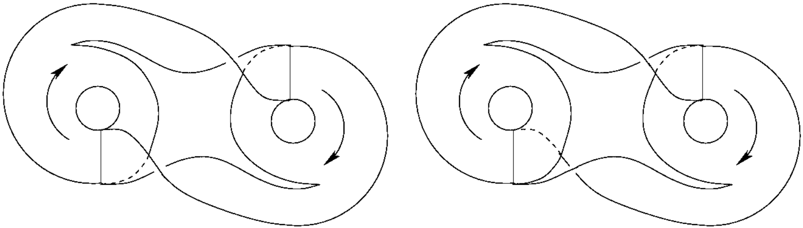




Cover-Image Branched Manifolds

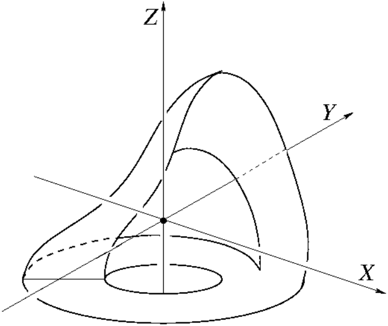


Two Two-fold Lifts Different Symmetry



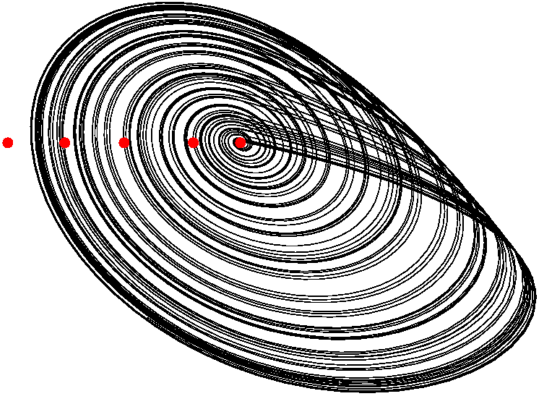
Rotation Symmetry **Inversion Symmetry**

Topological Index: Choose Group
Choose Rotation Axis (Singular Set)

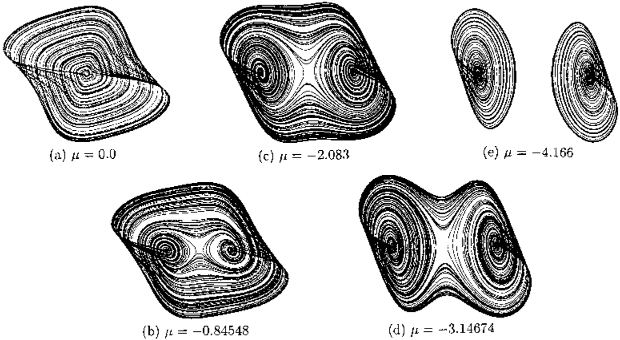


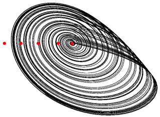
Locate the Singular Set wrt Image

Different Rotation Axes Produce Different (Nonisotopic) Lifts

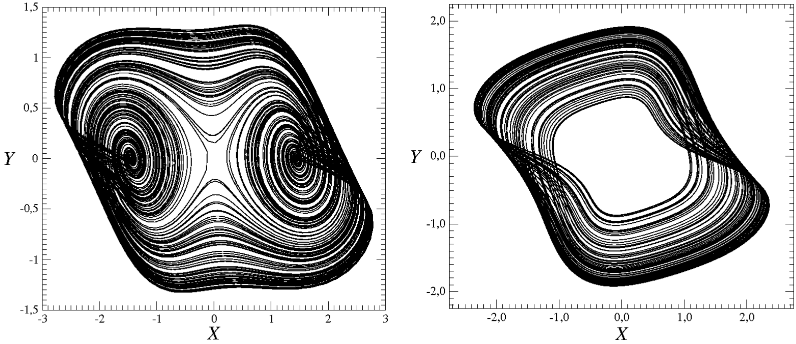


Nonisotopic Locally Diffeomorphic Lifts

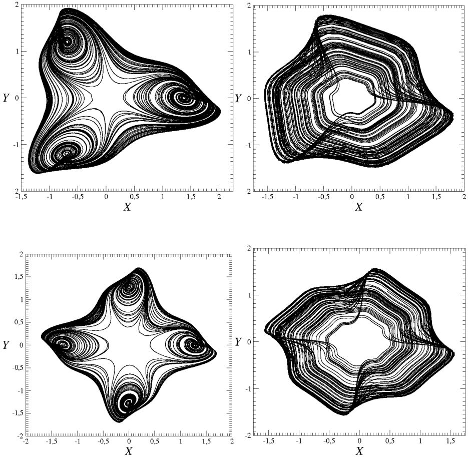




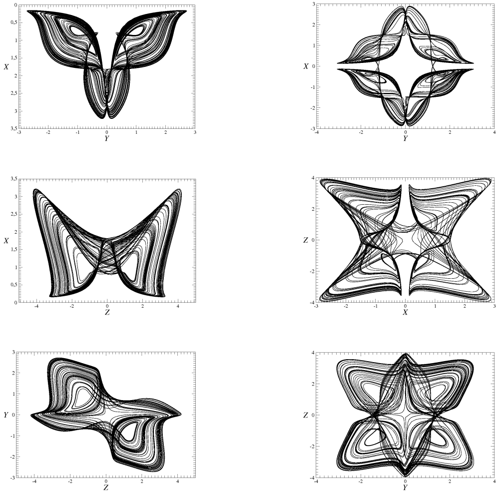
Two Two-fold Covers Same Symmetry



Three-fold, Four-fold Covers



Two Inequivalent Lifts with V_4 Symmetry



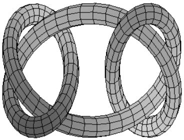
Algorithm

- Construct Invariant Polynomials, Syzygies, Radicals
- Construct Singular Sets
- Determine Topological Indices
- Construct Spectrum of Structurally Stable Covers
- Structurally Unstable Covers Interpolate

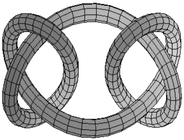
Symmetries Due to Symmetry

- Schur's Lemmas & Equivariant Dynamics
- Cauchy Riemann Symmetries
- Clebsch-Gordon Symmetries
- Continuations
    - Analytic Continuation
    - Topological Continuation
    - Group Continuation

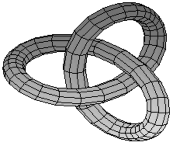
Covers of a Trefoil Torus



Granny Knot



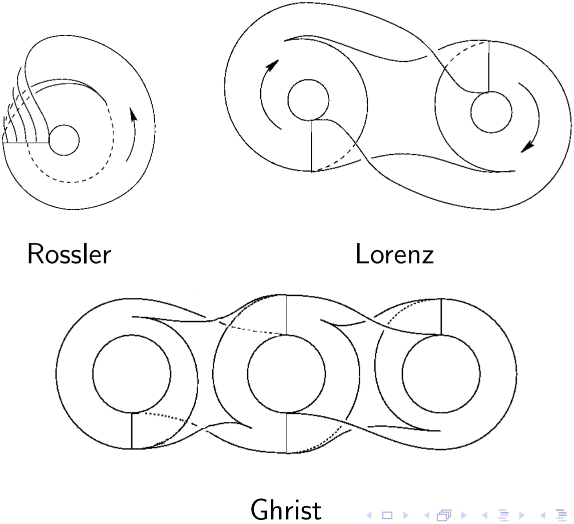
Square Knot



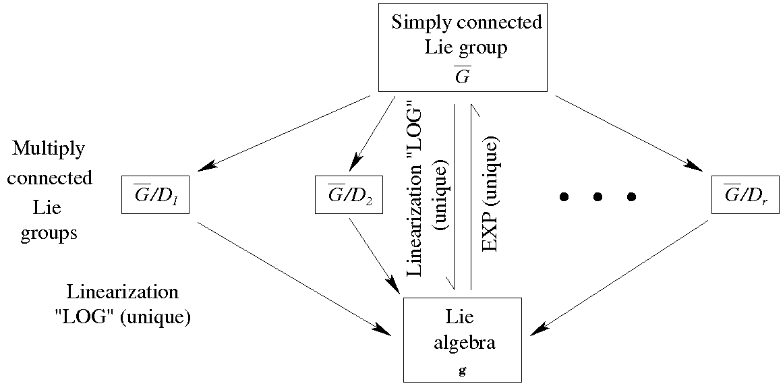
Trefoil Knot

You Can Cover a Cover = Lift a Lift

Covers of Covers of Covers



Cartan's Theorem for Lie Groups



Locally Diffeomorphic Covers of $\underline{D}$

$$
\begin{tikzcd}[column sep=small]
D_{1,T_1}/G_1 \arrow[drrr] & D_{1,T_2}/G_1 \arrow[drr] & \bullet\bullet\bullet & D_{2,T_1}/G_2 \arrow[d] & D_{2,T_2}/G_2 \arrow[dl] & \bullet\bullet\bullet & \bullet\bullet\bullet \\
 & & & \underline{D} & & &
\end{tikzcd}
$$

$\underline{D}$: Universal Image Dynamical System

Rotating the Attractor

$$\frac{d}{dt}\left[\begin{array}{c} X \\ Y \end{array}\right] = \left[\begin{array}{c} F_1(X,Y) \\ F_2(X,Y) \end{array}\right] + \left[\begin{array}{c} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{array}\right]$$

$$\left[\begin{array}{c} u(t) \\ v(t) \end{array}\right] = \left[\begin{array}{cc} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{array}\right] \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right]$$

$$\frac{d}{dt}\left[\begin{array}{c} u \\ v \end{array}\right] = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \left[\begin{array}{c} -v \\ +u \end{array}\right]$$

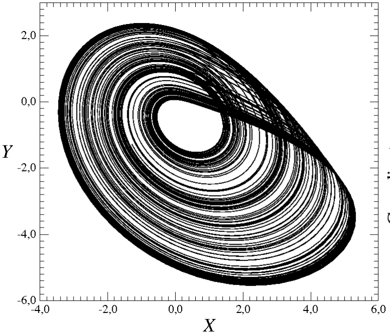
$\Omega = n\ \omega_d$ — Global Diffeomorphisms

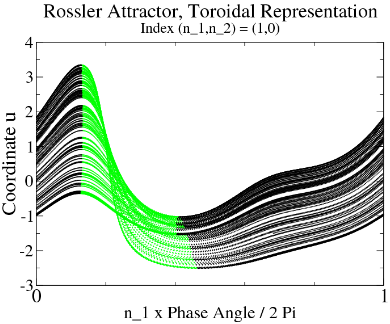
$q\ \Omega = p\ \omega_d$ — Local Diffeomorphisms (p-fold covers)

Two Phase Spaces: R^3 and $D^2 \times S^1$

Rossler Attractor: Two Representations

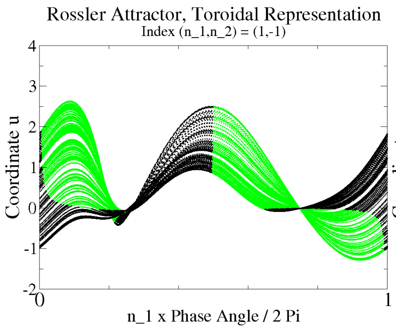
R^3 $\qquad$ $D^2 \times S^1$

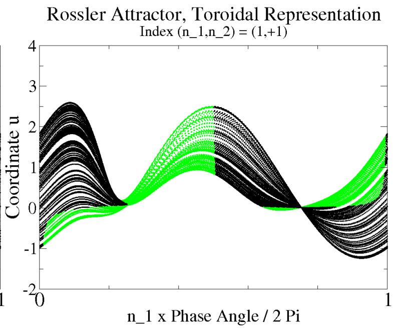




Rossler Attractor:

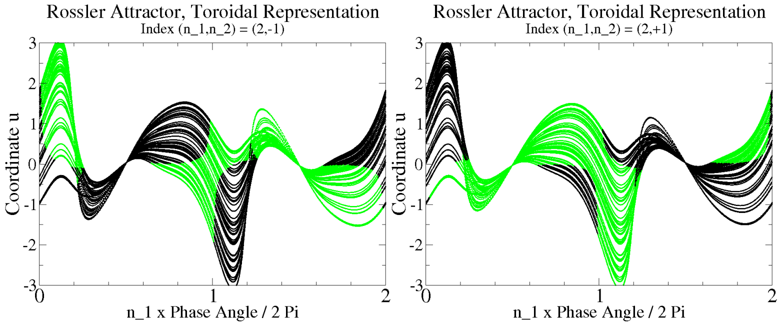
Two More Representations with $n = \pm 1$





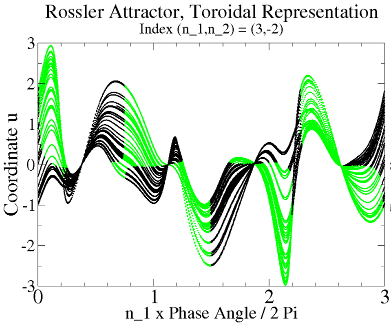
Rossler Attractor:

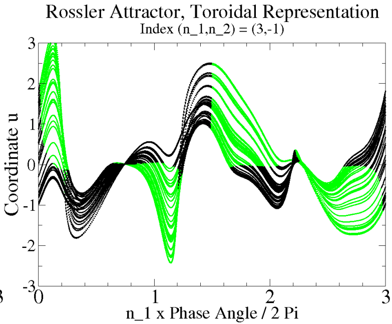
Two Two-Fold Covers with $p/q = \pm 1/2$



Rossler Attractor:

Two Three-Fold Covers with $p/q = -2/3, -1/3$

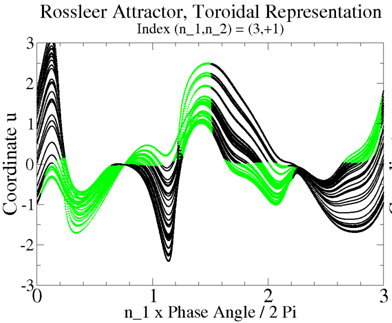


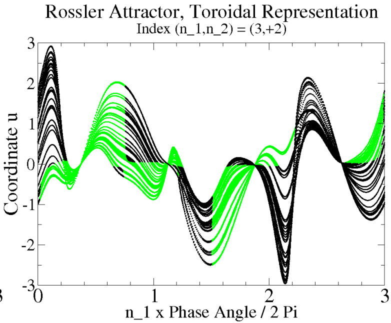


Subharmonic, Locally Diffeomorphic Attractors

Rossler Attractor:

And Even More Covers (with $p/q = +1/3, +2/3$)





Angular Momentum and Energy

$$L(0) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau X dY - Y dX$$

$$K(0) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) dt$$

$$L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle$$

$$= L(0) + \Omega \langle R^2 \rangle$$

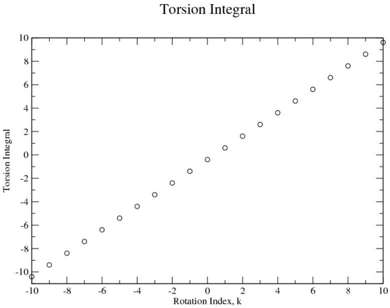
$$K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \rangle$$

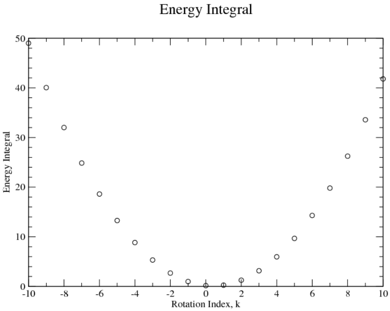
$$= K(0) + \Omega L(0) + \frac{1}{2}\Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau (X^2 + Y^2) dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau (u^2 + v^2) dt$$

Energy and Angular Momentum

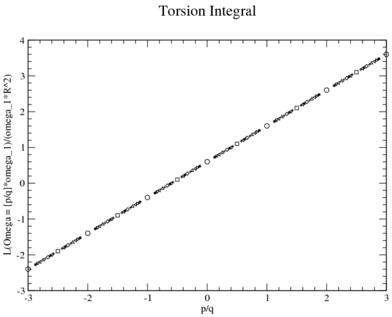
Diffeomorphic, Quantum Number n

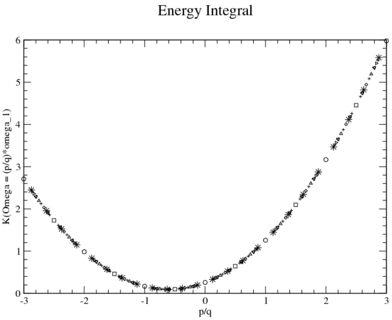




Energy and Angular Momentum

Subharmonics, Quantum Numbers p/q





Embeddings

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is of Minimum Dimension

Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

Equivalences by Injection

Obstructions to Isotopy

R^3	$\rightarrow$	R^4	$\rightarrow$	R^5
Global Torsion Parity Knot Type		Global Torsion (mod 2)		

There is one *Universal* reducible representation in R^N, $N \geq 5$. In R^N the only topological invariant is *mechanism*.

Summary

1 Question Answered $\Rightarrow$ 2 Questions Raised

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

What Have We Learned?

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits

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2. Branched Manifolds

Four Levels of Structure

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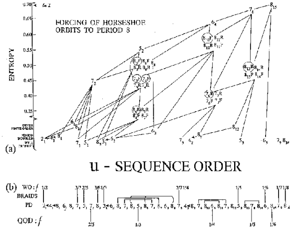
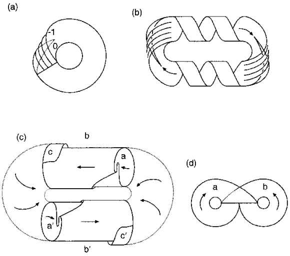
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

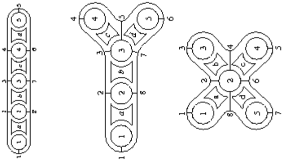
The Classification Theory has 4 Levels of Structure

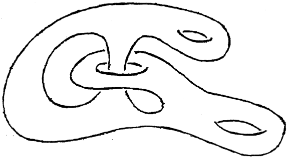
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g-1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuation
- Cauchy-Riemann Symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors

Unanswered Questions

HELP:

- Robust topological invariants for R^N, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \rightarrow D^2$, $n \times D^2 \rightarrow n \times D^2$ (e.g., Lorenz), $D^N \rightarrow D^N$, $N > 2$
- A new Representation theory
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Catastrophe Theory $\leftrightarrow$ Nonlinear Dynamics Connection