A Tale of Two Maps

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BigView Logistic Ma

A Tale of Two Maps

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June 3, 2014

Abstract

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> BigView Logistic Ma

- The logistic map has been used to enrich our understanding of a large class of highly dissipative dynamical systems, especially those contained in a genus-one torus.
- A different unimodal map can be used to enrich our understanding of highly dissipative dynamical systems contained in tori of genus g>1.
- The two maps are dual in a precise topological sense. The two maps are described and their properties and predictions compared.

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BigView Logistic Map

Outline

- 1 Tori and Poincare Surfaces of Section
- One-dimensional Return Maps for Dissipative Dynamics
- Concavity and Convexity
- Period-One Orbits
- Period-Two Orbits
- 6 Bifurcation Diagrams
- Caustics and Anticaustics
- Windows and Explosions
- Renormalization
- Monotonicity: Increasing and Decreasing
- Topological Entropy

Overview-01

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Overview-01

What We Did

- Studied maps with 2 branches
- 2 L & R
- Separated by a Singularity
- Models for Tearing Mechanism
- 6 Looked for "Universality"
- Searched for Scaling

Overview-02

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Overview-02

What We Found

- **1** Simple form: $x' = a |x|^k$
- 2 $k=2 \simeq$ folding; $k=\frac{1}{2} \simeq$ tearing

For 0 < k < 1:

- Localized global attractor
- Either chaos or stable period 1 fixed point
- Orbits of periods 1 and 2 organize systematics
- 4 Explosions
- Opening Prime and Compound orbits
- Output
 Local and Global focal points

Rössler Attractor

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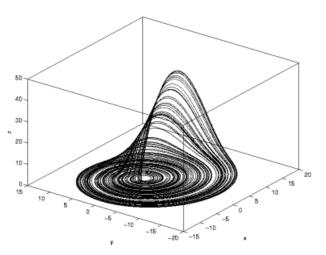
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Rössler Attractor

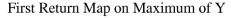


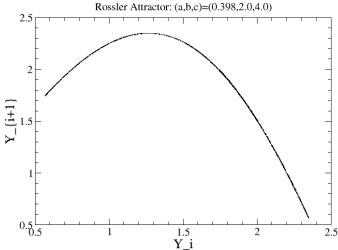
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Lorenz Attractor

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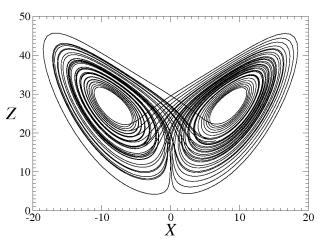
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Lorenz Attractor



Lorenz Attractor

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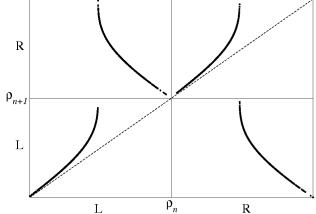
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Return Map for Lorenz Attractor

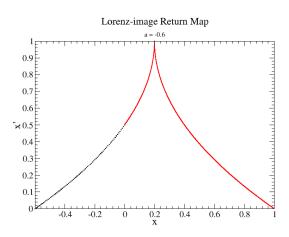


Lorenz Attractor

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Lorenz-03

Image of Lorenz Return Map



Side by Side-01

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Comparison: Two Maps

Logistic Map

a.k.a. Fold Map

$$x' = f(x; a) = a - (|x|)^2$$

Piecewise Concave

Knife Map

a.k.a. Cusp Map

$$y' = f(y; b) = b - (|y|)^{1/2}$$

Piecewise Convex

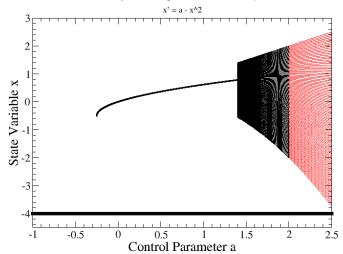
BigView: Logistic Map

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Logistic Map







BigView: Knife Map

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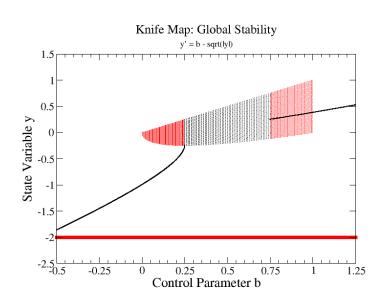
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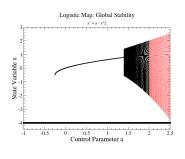


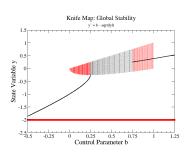


Comparison-02

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Stability Regions





Black = stable

Red = unstable

Comparison-03

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Lorenz Map

The Lorenz return Map (Fig. 4 in Lorenz 1963) for $(R,\sigma,b)=(28.0,10.0,8/3)$ scales to

$$z' = b - z^k$$

$$b \simeq 0.32 \pm$$

$$k \simeq 0.43 \pm$$

Logistic-01

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Return Map - Rössler Attractor Basin Boundaries

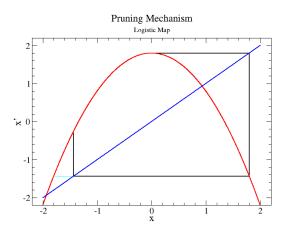
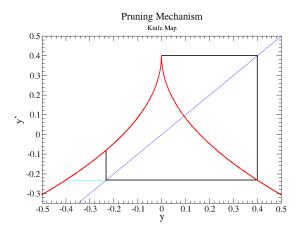


Image Lorenz-01

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Return Map - Lorenz Image **Basin Boundaries**



Logistic-04

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Logistic Map for several values of a

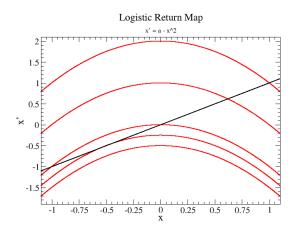
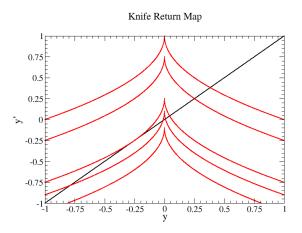


Image Lorenz-04

A Tale of Two Maps

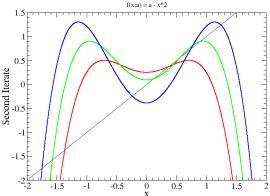
Knife Return Map for several values of b



Orbit Search-01

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Logistic Map, Second Iterate $f(x:a) = a - x^2$



Second Return Map

Orbit Search-02

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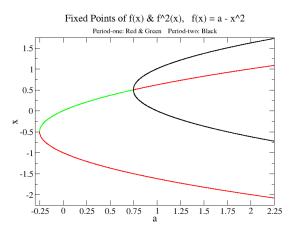
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BigView Logistic Map

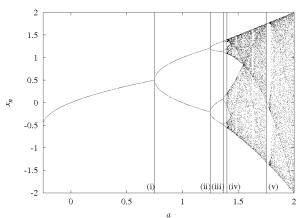
Period 1 & 2 Orbits - Logistic



Bifurcation-01

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Bifurcation Diagram



Notice: caustics and focal points

Bifurcation-02

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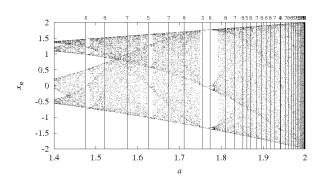
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.. Blow Up with Caustics



Logistic Caustic 1

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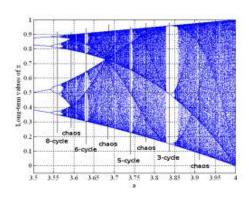
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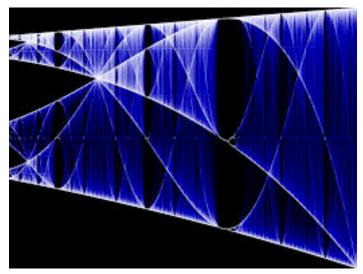
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Logistic Caustic 2

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Caustics-01

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Caustics

Historically, caustics are regions burned by an anomalously high concentration of sunlight. Dark regions, or high density regions, appear in the bifurcation diagram of the logistic map. They occur at forward images of the critical point, which has zero slope. Their zero-crossings provide information about the parameter values at which orbits become superstable.

For the dual map $y' = b - \sqrt{|y|}$ with a vertical slope, forward images of the critical point are not accompanied by caustics rather, anticaustics. Zero crossings of anticaustics provide information about the parameter values at which orbit explosions occur.

Caustics-01a

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Side by Side-01

Caustic / Anticaustic Density near Critical Point

$$z' = f(z,c) = c - |z|^p$$

$$\frac{dz'}{dz} = \frac{df}{dz} = -p|z|^{p-1}$$

$$\rho(z') = \frac{dz}{dz'}(z') = \frac{1}{p}(c - |z'|)^{(1/p)-1}$$

For p=2 (fold map) $\rho(z')$ exhibits a van Hove singularity.

For
$$p < 1$$
 (cusp map) $\rho(z') \to 0$ as $c - |z'| \to 0$.

Bifurcation-03

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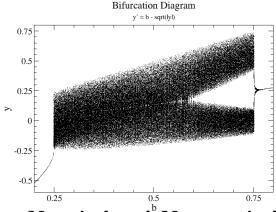
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Logistic iviap

Cusp Map - Bifurcation Diagram



No windows! No caustics!

Bifurcation-03a

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Scaling: Lorenz Map to Cusp Map

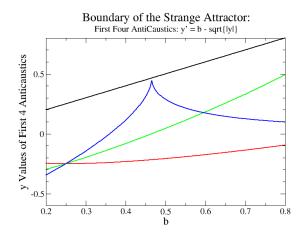
In the parameter range $26 \le R \le 46$ with $(\sigma, b) = (10.0, 8/3)$ the image of the Lorenz return map scales to the canonical form

$$z' = a - |z|^p$$

where a slowly increases from 0.32 to 0.38 and p slowly decreases from 0.46 to 0.41.

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Boundaries of the Strange Attractor



Bifurcation-05

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Lorenz-0

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BigView

Boundaries of Strange Attractors

- Logistic Map:
- The first two caustics bound the strange attractor when both even and odd period orbits are present.
- The third and fourth caustics provide additional boundaries when only even period orbits are present.
- Lorenz Map:
- The first two anticaustics bound the strange attractor when both even and odd period orbits are present.
- The third and fourth anticaustics provide additional boundaries when only even period orbits are present.

Orbit Search-03

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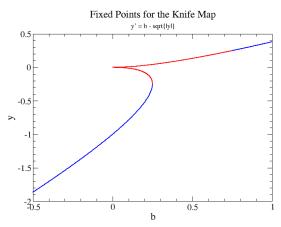
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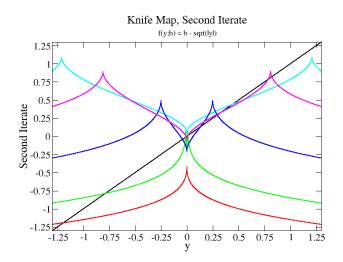
Fixed Points (Cusp Map)



Orbit Search-04

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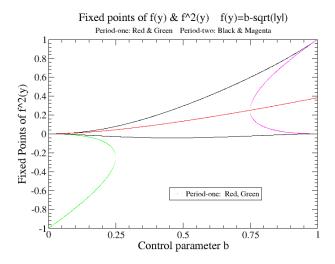
Second Iterates - Cusp Map



Skeleton-01

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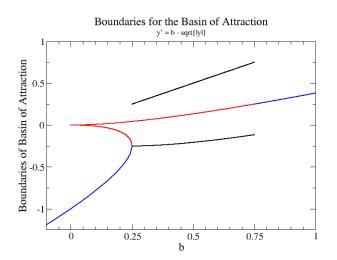
Period-One & Period-Two Orbits



Skeleton-02

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Attractor boundary (Knife)



Skeleton-03

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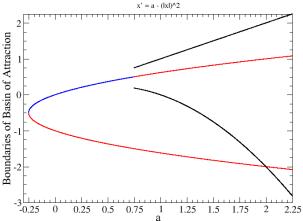
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Attractor Boundaries - Logistic





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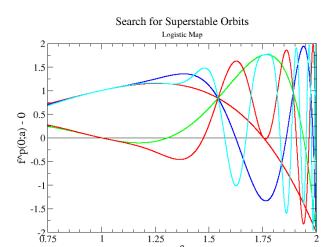
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Hunt for Saddle-Node Bifurcations Caustic Crossings



Orbit Search-05a

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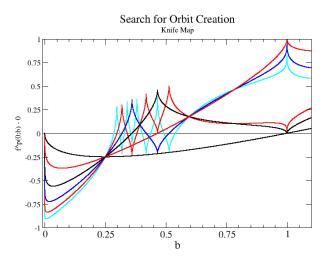
Hunt for Saddle-Node Bifurcations

Caustic Fingerprints

- ullet Zero crossings of the $p^{
 m th}$ caustic identify locations of superstable period-p orbits.
- Caustic "focal point" identifies end of the period-halving bifurcations.

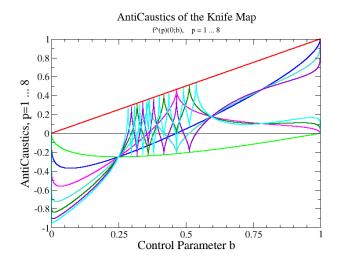
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Hunt for Singular SNBs



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Anti Caustic Crossings



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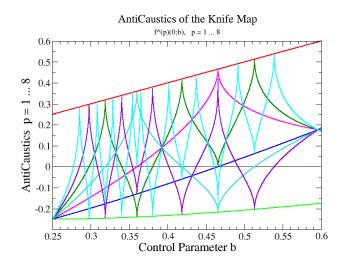
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BigView Logistic Map

Anti Caustic Crossings: Expansion



Orbit Search-08a

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Hunt for Explosions

Caustic Fingerprints

- ullet Zero crossings of the $p^{ ext{th}}$ anticaustic identify locations of period-p orbit explosions.
- At an explosion two *prime* orbits $K\sigma_2\sigma_3\cdots\sigma_n$ of period pare created, $\sigma_i \in \{0, 1\}$.
- All possible compound orbits based on the two prime orbits are simultaneously created.
- Anticaustic focal point at (b = 0.595743, y = 0.176100)identifies end of the period-halving bifurcations as bdecreases from b=1.
- Anticaustic focal point at (b = 1/4, y = -1/4) bounds the region 0 < b < 1/4 where trajectories consisting of all possible symbol sequences exist and no bifurcations occur.

Caustics-02

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BigView Logistic Map

Caustic & Anticaustic Crossings

For the logistic map there is one point at which caustics focus, at (x=0.839286, a=1.543689). To the left of this point the caustics are separated into those with p even (lower) and p odd (upper). This crossing marks the point at which the last noisy period-halving bifurcation occurs. To the left there are only even period windows. Odd period windows begin immediately to the right of the caustic focal point.

For the dual map there are two anticaustic focal points. The one at the right at $(b_2=0.595743,y=0.176100)$ marks the point at which only unstable even period orbits can be found (to its right). The one on the left at $(b_1=1/4,y=-1/4)$ separates the control parameter region in which trajectories of all possible periods and symbol sequences exist $(b < b_1)$ from the region $b_1 \le b \le 1$ in which orbits are systematically removed by explosions

Rite of Passage-01

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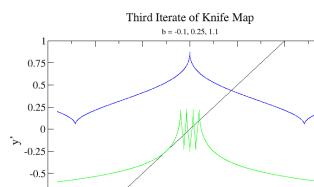
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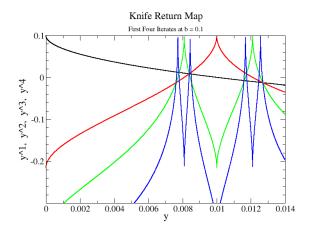
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BigView Logistic Map

Knife Map Iterates



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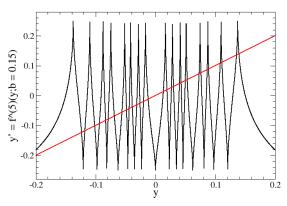
Lorenz-0

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Structural Stability: $0 < b < \frac{1}{4}$

Knife Map, fifth iterate at b=0.15



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Table: Values $M^{(p)}$ of y where the pth iterate $f^{(p)}(y;b)$ has maxima. These locations are determined by a simple recursion relation (last line) where the indices $s_p = \pm 1$ are incoherent.

p	Number Max.	Coordinate Values
1	1	0
2	2	$\pm b^2$
3	4	$\pm (b \pm b^2)^2$
	• • •	• • •
p+1	2^p	$M^{(p+1)} = s_p(b + M^{(p)})^2$

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Lorenz-0

BigView Logistic Ma As $p\to\infty$, with all $s_j=+1$, the abscissa of the rightmost point goes to a limit. The quadratic equation for this limit gives:

$$y_r(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b}\right)$$

At $b=\frac{1}{4}$ the bounding box is a rectangle — beyond that the diagonal fails to intersect all the zig - zags. Orbits begin to get pruned away in singular saddle node bifurcations.

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Extent of No-Explosion Range

- All local maxima (near y = 0) occur at $y_{loc, Max} = b$.
- All local minima (near y=0) occur at $y_{loc, min} = b \sqrt{b}$.
- As long as the rightmost peak occurs at $y_r(b) < b$ there are $2^p \pm 1$ fixed points for $f^p(y;b)$ near y=0.
- Singular bifurcations begin when

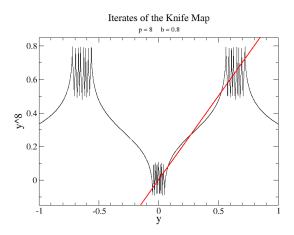
$$y_r(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b}\right) > b$$

• No explosions: $0 < b < \frac{1}{4}$.

Rite of Passage-02

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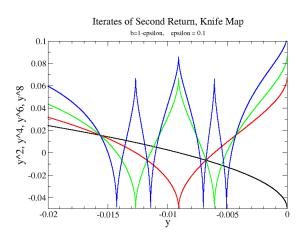
End Play - Near b=1



Rite of Passage-03

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Iterates Near b=1



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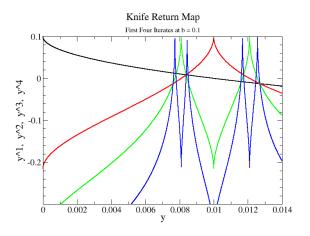
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Note Scaling Relations



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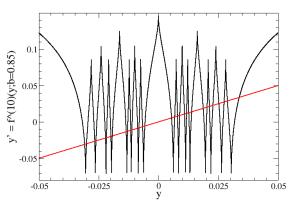
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Structural Stability: $\frac{3}{4} < b < 1$

Knife Map: 10th iterate near y=0



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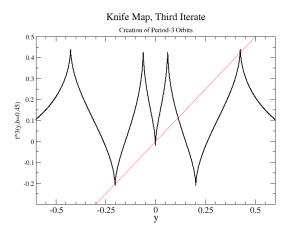
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Period Three Singular SNB



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BigView Logistic Ma Local expression near y=0 for the period-three explosion:

$$h(y;b) = f^{(3)}(y;b) = b - \sqrt{|b - \sqrt{|b - \sqrt{|y|}|}}$$

$$h(b_3 + \epsilon; y) \rightarrow \left(b_3 - \sqrt{\sqrt{b_3} - b_3}\right) +$$

$$\left(1 + \frac{2\sqrt{b_3} - 1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\epsilon + \left(\frac{1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\sqrt{|y|}$$

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Lorenz-0

BigView Logistic Ma Renormalization for the period-three explosion.

$$y' = h(y; b_3 + \epsilon) \rightarrow \Delta(b - b_3) + \alpha \sqrt{|y|} =$$

$$1.286974759(b - b_3) + 0.7869747590\sqrt{|y|}$$

$$z' = (\Delta/\alpha^2)(b_3 - b) - \sqrt{|z|}$$

A Tale of Two Maps

Renormalization Algorithm: K10*

- Write down the symbol sequence for the primary period-porbit: $K10* = K\sigma_1\sigma_2\cdots\sigma_{n-1}$.
- Make the identification $\sigma = 0 \rightarrow s = -1$. $\sigma = +1 \rightarrow s = +1$,
- **3** Construct $f^{(p)}(b;y) \rightarrow$

$$b - \sqrt{s_{p-1}(b - \cdots \sqrt{s_2(b - \sqrt{s_1(b - \sqrt{y})})} \cdots)}$$

4 Taylor expand this function to terms linear in b and \sqrt{y} and determine the value of b for which the constant term vanishes.

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Equations: K10*

For the saddle node pair $5_2 = K1001$ this algorithm gives

$$b - \sqrt{(+1)(b - \sqrt{(-1)(b - \sqrt{(-1)(b - \sqrt{y})})})}$$

The constant term vanishes for b = 0.418656, and for this value of b

$$y' = \Delta(b - b_{5_2}) + \alpha\sqrt{|y|} = -3.231180\Delta b - 1.983690\sqrt{|y|}$$

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BigView Logistic Map Results: K10* to Period 6

$$y' = \Delta(b - b_c) + \alpha \sqrt{|y|}$$
 $y', y \simeq 0$

Orbit	Symbolics	b_c	Δ	α
$\overline{}_{3_1}$	K10	0.465571	1.286974	0.786974
4_2	K100	0.360157	2.624703	1.180563
5_3	K1000	0.318897	4.647225	1.664335
5_2	K1001	0.418656	-3.231180	-1.983690
5_1	K1011	0.513175	2.628970	1.509712
6_5	K10000	0.297846	7.481728	2.233184
6_4	K10001	0.340328	-8.535145	-3.639587
6_3	K10011	0.380540	7.596535	3.574548

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BigView Logistic Map Renormalization for the final period-two explosion.

$$f^{(2)}(1-\epsilon,y) \simeq -\frac{\epsilon}{2} + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)\sqrt{|y|}$$
 (1)

Important Markers

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Breakpoints

Table: Important parameter values for global stability and unstable periodic orbit behavior.

Global Stability	Unstable Orbits
	0.0
1/4	1/4
	0.5957439420
3/4	
	0.7825988587
	1.0

Important Markers

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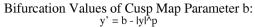
Parameter Ranges

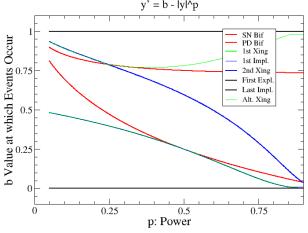
Left	Right	What goes on
$-\infty$	0	Stable period one orbit
0	1/4	Explosion at $b=0$ creates all orbits
		based on symbols 0, 1. No further
		bifurcations in this range
1/4	0.575974	Explosions remove all odd- and
		most even-period orbits
0.575943	0.782598	Explosions remove even period orbits
0.782598	1	No further bifurcations in this range
1	$+\infty$	Stable period one orbit.
		Explosion at $b=1$ removes remaining
		even-period orbits.

Generic Cusp Map $y' = b - |y|^p$

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Generic Cusp Map $y' = b - |y|^p$

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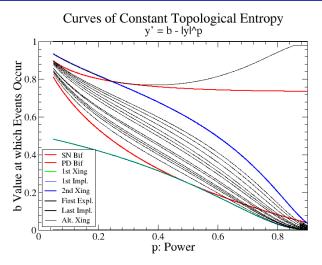
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U Sequence

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Logistic Map

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)^a

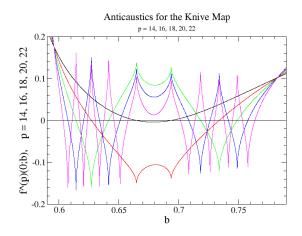
Name	Bifurcation	Name	$_{s}$ Bifurcation	Name	Bifurcation
0	$1_1[s_1]$	00101 01	$7_3[s_7^3]$	0001 11	$6_4[s_6^3]$
· 01	$2_1[s_1 \times 2^1]$	001010 01	$8_5[s_8^4]$	000111 11	$8_{11}[s_8^9]$
0111	$4_1[s_1 imes 2^2]$	001 01	$5_2[s_5^2]$	00011 11	$*7_{7}[s_{7}^{7}]$
01010111	$8_1[s_1 \times 2^3]$	001110 01	$8_6[s_8^5]$	000110 1	$8_{12}[s_8^{10}]$
0111^{0}_{1}	$6_1[s_6^1]$	00111 01	$7_4[s_7^4]$	000 11	$5_3[s_5^3]$
011111 ⁶ 1	$8_2[s_8^1]$	001111 91	$8_7[s_8^6]$	000010 01	$8_{13}[s_8^{11}]$
01111^{0}_{1}	$7_1[s_7^1]$	0011 01	$6_3[s_6^2]$	00001 01	$7_8[s_7^8]$
$011^{\frac{5}{1}}$ 1	$5_1[s_5^1]$	001101 1	$8_8[s_8^7]$	000011 11	$8_{14}[s_8^{12}]$
$01101^{0}1$	$7_2[s_7^2]$	00110 1	$7_{5}[s_{7}^{5}]$	0000 11	$6_5[s_6^4]$
011011 ⁶ 1	$8_3[s_8^2]$	00 1	$4_2[s_4^1]$	000001 11	$8_{15}[s_8^{13}]$
0 1	$3_1[s_3]$	00010011	$8_9[s_4^1 imes 2^1]$	00000 01	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	00010 01	$7_{6}[s_{7}^{6}]$	000000 11	$8_{16}[s_8^{14}]$
001011 01	$8_4[s_8^3]$	000101 11	$8_{10}[s_8^8]$	_	. • •

^aThe notation P_i refers to the *i*th bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the *i*th saddle-node bifurcation of period P is denoted s_P^i , and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i .

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Symbol Exchange Near Endplay



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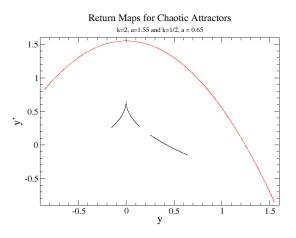
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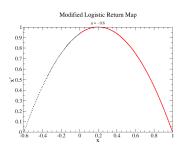
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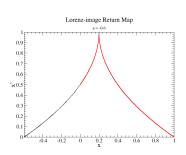
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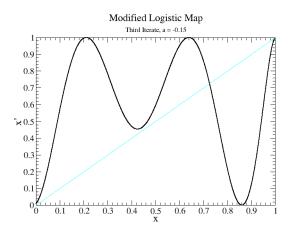
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Map Comparisons





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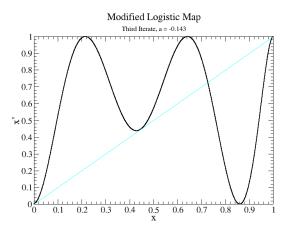
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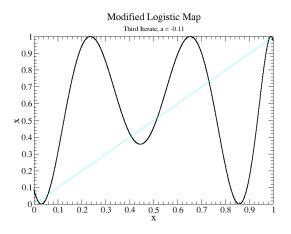
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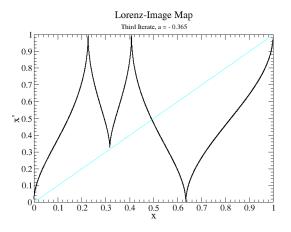
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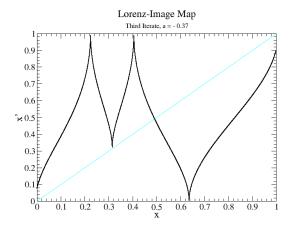
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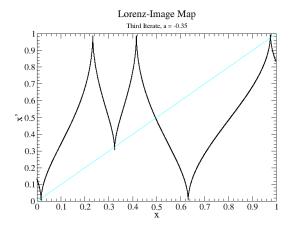
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Comparison: Logistic and Knife

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Scaling

- Logistic: SNB Period 3 = scaled version SNB of M.
- Renormalization theory applies.
- U Sequence
- Knife: S-SNB Period 3 = scaled version S-SNB of K.
- Renormalization theory applies.
- U^{-1} Sequence

Comparison: Logistic and Knife Cover Flows

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Topological Organization

The knife and logistic maps are suspensions of flows. Corresponding orbits (identical names) in each suspension are organized identically.

Identical lifts of the logistic and knife maps lead to identical covering orbit organization.

The mysteries of orbit organization in flows with g>1 are the same for stretch-and-fold and for tear-and-squeeze mechanicms.

U Sequence

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BigView Logistic Map

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)^a

Name	Bifurcation	Name	*Bifurcation	Name	Bifurcation
0	$1_1[s_1]$	00101 01	$7_3[s_7^3]$	0001 11	$6_4[s_6^3]$
· oî	$2_1[s_1 \times 2^1]$	001010 01	$8_5[s_8^4]$	000111 11	$8_{11}[s_8^9]$
0111	$4_1[s_1 imes 2^2]$	001 01	$5_2[s_5^2]$	00011 n	$*7_{7}[s_{7}^{7}]$
01010111	$8_1[s_1 \times 2^3]$	001110 01	$8_6[s_8^5]$	000110 1	$8_{12}[s_8^{10}]$
$0111_{1}^{0}1$	$6_1[s_6^1]$	00111 01	$7_4[s_7^4]$	000 1	$5_3[s_5^3]$
011111 ⁶ 1	$8_2[s_8^1]$	001111 91	$8_7[s_8^6]$	000010 11	$8_{13}[s_8^{11}]$
$01111^{\frac{1}{0}}1$	$7_1[s_7^1]$	0011 01	$6_3[s_6^2]$	00001 1	$7_8[s_7^8]$
$011^{\frac{5}{1}}1$	$5_1[s_5^1]$	001101 1	$8_8[s_8^7]$	000011 11	$8_{14}[s_8^{12}]$
$01101^{\frac{6}{1}}1$	$7_2[s_7^2]$	00110 1	$7_{5}[s_{7}^{5}]$	0000 11	$6_5[s_6^4]$
011011 1	$8_3[s_8^2]$	00 01	$4_2[s_4^1]$	000001 1	$8_{15}[s_8^{13}]$
0 01	$3_1[s_3]$	00010011	$8_9[s_4^1 imes 2^1]$	00000 01	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	00010 11	$7_6[s_7^6]$	000000 11	$8_{16}[s_8^{14}]$
$001011_{1}^{0}1$	$8_4[s_8^3]$	000101 11	$8_{10}[s_8^8]$	_	

^aThe notation P_i refers to the *i*th bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the *i*th saddle-node bifurcation of period P is denoted s_P^i , and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i .

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BigView Logistic Map Table : Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right) $^{\rm a}$

Name

Bifurcation

Name

Bifurcation

$0 \\ 1$	$1_1[s_1]$	$00101_{\ 1}^{\ 0}1$	$7_3[s_7^3]$	$0001_{\ 1}^{\ 0}1$
01	$2_1[s_1 \times 2^1]$	$001010_{1}^{0}1$	$8_5[s_8^4]$	$000111_{1}^{0}1$
0111	$4_1[s_1 \times 2^2]$	$001_{1}^{0}1$	$5_2[s_5^2]$	$00011_{1}^{0}1$
01010111	$8_1[s_1 \times 2^3]$	$001110_{1}^{0}1$	$8_6[s_8^5]$	$000110_{1}^{0}1$
$0111_{\ 1}^{\ 0}1$	$6_1[s_6^1]$	$00111_{1}^{0}1$	$7_4[s_7^4]$	$000_{1}^{0}1$
$0111111 { 0 \atop 1}1$	$8_2[s_8^1]$	$001111_{1}^{0}1$	$8_7[s_8^6]$	$000010_{\ 1}^{\ 0}1$
$01111_{1}^{0}1$	$7_1[s_7^1]$	$0011_{1}^{0}1$	$6_3[s_6^2]$	$00001_{1}^{0}1$
$011_{1}^{0}1$	$5_1[s_5^1]$	$001101_{1}^{0}1$	$8_8[s_8^7]$	$000011_{1}^{0}1$
$01101_{\ 1}^{\ 0}1$	$7_2[s_7^2]$	$00110_{1}^{0}1$	$7_5[s_7^5]$	$0000_{1}^{0}1$
$011011_{\ 1}^{\ 0}1$	$8_3[s_8^2]$	$00_{1}^{0}1$	$4_2[s_4^1]$	$000001_{1}^{0}1$
0.01	$3_1[s_2]$	00010011	$8_0[s^1 \times 2^1]$	= 000€0 £

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Return Map Approximations

The Rossler return map is well approximated by the following maps:

$$x' = \lambda x (1 - x)$$

$$x' = a - x^2$$

$$x' = 1 - \mu x^2$$

$$x' = 1 - \left| \frac{x - m}{w} \right|^2$$

Image Lorenz-02

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Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

$$y' = b - |y|^{1/2}$$

$$y' = 1 - \mu |y|^{1/2}$$

$$y' = 1 - \left| \frac{y - m}{w} \right|^{1/2}$$

Endplay-02

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Symbol Exchange Near Endplay

- ullet Symbols 0, 1 created at b=0
- New orbit, (11), created at $b=\frac{3}{4}$
- ullet Symbol pair 11 -, replaced by (11) as b o 1
- Implosions begin at b=0.5957..., end at midpoint
- Explosions begin at midpoint, end at b = 0.7825...
- Implosions and explosions symmetrically matched