Entropy of Bounding Tori

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Branched manifolds that describe strange attractors in R^3 can be enclosed in, and are organized by, canonical bounding tori. Tori of genus g are labeled by a symbol sequence, or "periodic orbit", of period $g-1$. We show that the number of distinct canonical bounding tori grows exponentially like $N(g) \sim e^{\lambda(g-1)}$, with $e^{\lambda} = 3$, so that the "bounding tori entropy" is log 3.

I. INTRODUCTION

Low dimensional strange attractors — those with Lyapunov dimension $d_L < 3$ — can be discretely classified. A doubly discrete classification has been described in [1]. This classification depends ultimately on the existence and rigid organization of an infinity of unstable periodic orbits in a strange attractor [2, 3]. At the lowest level this classification depends on a basis set of orbits. This is a set of orbits with positive topological entropy whose presence forces the existence of all the other unstable periodic orbits in the attractor [4–6]. The basis set of orbits for any attractor is discrete, and up to any finite period the basis set of orbits is finite.

At the second level of this organizational hierarchy for strange attractors are branched manifolds [1, 2, 7–9]. These are obtained from the flow that generates a strange attractor by projecting the flow down along the stable direction. The unstable periodic orbits that exist in the strange attractor exist in 1-1 correspondence with the periodic orbits on the branched manifold, with possibly a small number of exceptions. Information about branched manifolds can be extracted from experimental data [10].

Recently a third level of discreteness in the description and classification of low dimensional strange attractors has been introduced [11, 12]. Branched manifolds can be enclosed in bounding tori. These serve to organize branched manifolds in the same way that branched manifolds organize the periodic orbits in a strange attractor. A bounding torus provides a canonical form for any flow in R^3 that generates a strange attractor. An algorithm for transforming a flow to its canonical form is given in [12]. The bounding tori that enclose every strange attractor that has been studied in R^3 have been described in [11, 12].

Bounding tori are described first by their genus, $g \geq 1$. However, genus alone does not uniquely identify a bounding torus when $g > 4$, and in fact the number of distinct bounding tori of genus g, $N(g)$, grows rapidly with g. It was proposed in [12] that the growth might be exponential, so that an entropy-like limit of the type $\lim_{g\to\infty} \log[N(g)]/g$ might exist, in analogy with the limiting definition of topological entropy for periodic orbits in a strange attractor.

The purpose of the present work is to show that this limit exists and to evaluate it. We show that

$$\lim_{g\to\infty} \frac{\log[N(g)]}{g-1} = \log(3) \qquad (1)$$

so that an entropy of log(3) can be associated with the growth in the number of bounding tori with genus g in R^3.

II. BACKGROUND

A bounding torus of genus $g = 8$ is shown in Fig. 1. This represents a projection of a two dimensional surface in R^3 down onto a plane. The projection consists of the outer boundary of a disk and g interior disks. The interior disks are of two types: n_c circles and n_p even-sided polygons. The flow on the outer boundary is unidirectional; the flow on the n_c interior circles is also unidirectional, and in the same direction as the flow on the exterior boundary. All singularities of the flow lie on the n_p interior polygons: a polygon with $2n$ sides ($n > 1$) has $2n$ singularities, one at each vertex. The genus of the bounding torus is the total number of interior holes: $g = n_c + n_p$. The total number of singularities on the bounding torus (all at the vertices of the interior polygons) is $2(g-1)$ [11, 12].

For the bounding torus shown in Fig. 1 there are $n_c = 5$ interior uniflow circles labeled $A \to E$ and three interior polygons labeled a, b, c. The global Poincaré section of any flow bounded by this torus has $g-1 = 7$ disconnected components [11, 12]. These are shown as line segments in Fig. 1 and labeled $1 \to 7$, sequentially in the direction of the flow along the exterior boundary.

There are several ways that bounding tori can be uniquely identified. The labeling algorithms are described in Eq. (2).

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & \dots \\
 & & & & & & & & \\
A & B & C & B & D & B & E & A & \dots \\
\quad a & \quad b & \quad b & \quad c & \quad c & \quad a & \quad a & & \dots \\
1 & 3 & 1 & 3 & 1 & 3 & 1 & 1 & \dots \quad (2,4,6)
\end{array}
\tag{2}
$$

The first row lists the components of the global Poincaré section in the order they are encountered traversing the exterior boundary of the projection.

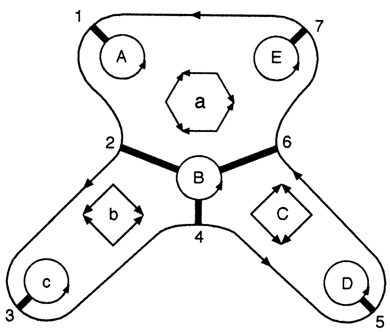

FIG. 1: A canonical bounding torus with genus 8. This is partly described by Young partition (3,2,2).

Below each number i is the capital letter (A, B, C, D, E) that identifies the uniflow circle to which the i^{th} component of the surface of section is attached. The sequence $(ABCBDBE)$ that is encountered is shown in the second row of Eq. (2). In moving from component i to component $i+1$ a hole with singularities is encountered. The sequence $(abbccaa)$ that is encountered is shown in the third row of Eq. (2). There is a 1-1 correspondence between the bounding torus and each of the two letter sequences ($ABCBDBE$ and $abbccaa$), up to the usual symmetries (relabeling the holes, changing the starting point). In fact, these two descriptions of a bounding torus are dual to each other. Both sequence strings are in fact infinite, but of finite period $g-1=7$. The last string of integers in Eq. (2) indicates that there is a period-3 orbit around hole B and period-1 orbits around the holes A, C, D, E. A permutation group representation of this bounding torus in terms of permutation group generating cycles is (2,4,6)(1)(3)(5)(7) or more simply (2,4,6). This representation in terms of generating cycles can be used algorithmically to construct the transition matrix for this bounding torus [11, 12].

Part of the degeneracy associated with enumerating bounding tori of genus g can be lifted by introducing Young partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n_p})$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_p} \geq 2$ [11, 12]. Each internal polygon with $2\lambda_i$ edges and singularities is visited exactly λ_i times in a tour around the exterior boundary. The partition associated with the torus has n_p rows, one for each interior polygon. For the bounding torus shown in Fig. 1, $\lambda = (3, 2, 2)$. All allowed bounding tori that can be associated with this partition are obtained by distributing the $g-1=7$ letters aaa, bb, and cc on the perimeter of a circle subject to the single condition that no interleaving occurs ($..a.b.b.a..$ is allowed but $..a.b.a.b..$ is not).

The number of bounding tori of genus g can be determined by

1. listing all allowed Young partitions;

2. counting the number of allowed letter distributions (up to cyclic permutation) for each Young partition.

The sequence of polygon encounters can be replaced by a sequence of three symbols: (,), and *. The opening and closing parentheses stand for the first and last occurrence, respectively, of a given letter, intermediate occurrences being indicated by a $*$. The noninterleaving property implies that each $*$ belongs to the innermost pair of parentheses between which it is embedded. Thus $aaa \to (*)$, $aaabb \to (*)()$, $aabba \to (*())$, and $baaab \to ((*))$. This construction guarantees that at each position of the sequence the cumulative number of opening parentheses is not less than the cumulative number of closing parentheses, counting from the left.

The complete set of bounding tori of genus g is obtained by constructing all three-symbol sequences that satisfy the requirements (a) that the total number of opening parentheses be equal to that of closing parentheses, (b) that the cumulative number of opening parentheses be always not less than that of closing parentheses, (c) that a $*$ can only appear if the number of opening parentheses preceding it is larger than the number of closing parentheses. Finally, (d) sequences that are related by a cyclic permutation are equivalent. Cyclic permutations are more easily described in terms of letter sequences than three-symbol sequences. Thus $aaabb \mapsto baaab$ translates to $(*)() \mapsto ((*))$.

This algorithm for describing the complete set of bounding tori of genus-g is described more fully in Sect. IV.

III. UPPER BOUND ON TORAL ENTROPY

An upper bound on toral entropy is log(3). This upper bound on $N(g)$ is obtained by noting that a word of length $g-1$ can be formed with the three-symbol alphabet (, *,) in 3^{g-1} ways. This bound ignores the requirements (a)-(d) specified above.

A more refined upper bound is constructed by noting that requirements (a)-(c) of Sec. II are in 1-1 correspondence with the properties satisfied by the coupled states of spin-1 particles. This is established by observing that coupling $k-1$ spins $s=1$ with total spin S_{k-1} to a

TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

single spin $s = 1$ is isomorphic with the three-symbol coupling problem under the association: (increases the spin $[S_k = S_{k-1} + 1]$; * preserves the spin $[S_k = S_{k-1}]$; and) decreases the spin $[S_k = S_{k-1} - 1]$, subject to the condition that $S_{k-1} = 0 \Rightarrow S_k = 1$. Requirement (a) corresponds to the specific case $S_{g-1} = S_{Tot} = 0$. The number of ways that $g-1$ particles of spin $s = 1$ can be combined to total spin $S_{Tot} = 0$ is $\mathcal{N}(g-1, s = 1, S_{Tot} = 0) = f(g-1)$, where

$$f(n) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{i} \binom{n+1-i}{i} \frac{n-3i+1}{n-i+1} \qquad (3)$$

The limit can be taken using Stirling's approximation to give an upper bound on the toral entropy

$$\lim_{g\to\infty} \frac{\log[\mathcal{N}(g-1, s=1, S_{Tot}=0)]}{g-1} = \log(3) \qquad (4)$$

IV. LOWER BOUND AND EXACT RESULTS

The algorithm for building (and counting) the complete set of inequivalent three-symbol sequences of length $g-1$ that respect requirements (a)-(d) proceeds as follows: An overcomplete list is generated from the complete set of sequences of length $g-2$ by applying to each one of them the following operations: (1) Inserting a $*$ at each legal position (i.e., lengthening a cycle). (2) Replacing a $*$ by the sequence () (i.e., embedding a two-cycle). (3) Replacing a $*$ by the sequence)((i.e, splitting a cycle into two cycles). In fact, operation (3) is only capable of generating sequences that have not already been generated by operations (1) and (2) if applied to a length $g-2$ sequence with a maximum ($\lfloor \frac{g-2}{2} \rfloor$) number of cycles.

The list thus created contains repetitions that have to be eliminated. Furthermore, sequences on the list that are equivalent by cyclic permutations to other sequences need to be discarded. This algorithm was implemented by Maple and Fortran codes and used to compute $N(g)$ for g up to 20. These results are reported in Table I. Values for $N(g)$ were computed by hand up to $g = 11$ to validate the algorithm and the coding of it.

TABLE II: Comparison of exact results for $N(g)$ with analytic result for prime numbers.

$g-1$	Exact	From (5)
3	1	1
5	2	2
7	6	6
9	28	26 $\frac{2}{3}$
11	145	145
13	870	870
15	5549	5540 $\frac{11}{15}$
17	36824	36824
19	252927	252927

In the case that $g-1 = pr$ is prime a simple closed-form expression for $N(g)$ can be constructed. It is deduced from the expression for the number of ways that a total spin $S = 0$ can be computed from pr spins $s = 1$. The number of ways that n spins $s = 1$ can be combined to a total spin $S = 0$ is given in (3). The number $f(n)$ includes spin coupling patterns that correspond to letter sequences that are related by cyclic permutations. These must be removed to relate the spin coupling problem to the bounding torus problem, that is, to satisfy requirement (d) of Sec. II. One of the sequences (a^{pr}) is already cyclically invariant and all the others have periodicity $pr = g-1$. The relation between $N(g)$ and $f(n)$ for $n = g-1$ prime is

$$N(g) = 1 + \frac{f(g-1) - 1}{g-1} \qquad (5)$$

When $g-1$ is not a prime some of the spin coupling patterns correspond to periodicity lower than $g-1$, so that the above expression for $N(g)$ is actually a lower bound for nonprime cases. The extension of eq. (5) to the case $g-1$ nonprime requires the determination of the number of ways of forming distinct sequences with the various periodicities that correspond to the factors of $g-1$. For example, the cyclic permutations of $aaabb$ are all distinct while those of $(aab)^3$ ($9 = 3 \times 3$) are not. The simpler problem involving just (and) [or coupling of spin $\frac{1}{2}$ particles] has been solved, and involves complicated number-theoretic functions such as the Euler totient function [13]. Expression (5) was used to compute $N(g)$ for odd $g-1$. Results are presented in Table II. This table shows that the values computed for prime values of $g-1$ are equal to those computed by hand ($g-1 \leq 11$) and by the Fortran algorithm, while the values of $N(g)$ for $g-1$ not prime (9, 15) are slightly below the exact values by a relative fraction that decreases as g increases.

We have computed $\log[N(g)]/(g-1)$ for prime values of $g-1$ and plotted this ratio as a function of $1/(g-1)$ for primes below 2000. The results are presented in Fig. 2. The lower bound can be computed analytically in the limit $g-1 \to \infty$ using Stirling's approximation and is equal to log(3). Since the upper bound is also log(3), the entropy of bounding tori is log(3).

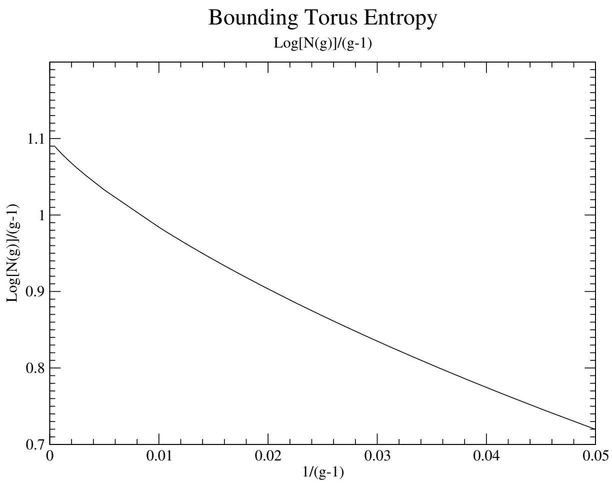

FIG. 2: The ratio $\log[N(g)]/(g-1)$ converges to log(3).

V. CONCLUSION

Topological entropy describes how the number of unstable periodic orbits of period p in a strange attractor grows exponentially with period p. We have shown that the number of inequivalent bounding tori of genus g in R^n $(n=3)$ grows exponentially with $g-1$. The limit $\log[N(g)]/(g-1)$ exists as $g \to \infty$ and is $\log(n)$, with $n=3$.

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